Space-time duality for fractional diffusion

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Abstract

Zolotarev proved a duality result that relates stable densities with different indices. In this talk, we show how Zolotarev duality leads to some interesting results on fractional diffusion. Fractional diffusion equations employ fractional derivatives in place of the usual integer order derivatives. They govern scaling limits of random walk models, with power law jumps leading to fractional derivatives in space, and power law waiting times between the jumps leading to fractional derivatives in time. The limit process is a stable Lévy motion that models the jumps, subordinated to an inverse stable process that models the waiting times. Using duality, we relate the density of a spectrally negative stable process with index $1 < \alpha < 2$ to the density of the hitting time of a stable subordinator with index $1/\alpha$, and thereby unify some recent results in the literature. These results provide a concrete interpretation of Zolotarev duality in terms of the fractional diffusion model. They also illuminate a current controversy in hydrology, regarding the appropriate use of space and time fractional derivatives to model contaminant transport in river flows.

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Fractional derivatives: An old idea gets new life

Fractional derivatives $\mathbb{D}_\alpha f(x)$ for any $\alpha > 0$ were invented by Leibnitz soon after the more familiar integer derivatives.

Some derivative formulas extended to the fractional case:

$$\mathbb{D}_\alpha \left[e^{\lambda x}\right] = \lambda^\alpha e^{\lambda x}$$

$$\mathbb{D}_\alpha [\sin x] = \sin\left(x + \frac{\pi}{2}\alpha\right)$$

$$\mathbb{D}_\alpha [x^p] = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}$$

Fractional derivatives and transforms

If the Laplace transform of $f(t)$ is defined for $0 < \alpha \leq 1$ by

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt$$

then $\mathbb{D}_\alpha f(t)$ has Laplace transform $s^\alpha \tilde{f}(s) - s^{\alpha-1} f(0)$.

If the Fourier transform of $f(x)$ is defined for $k \in \mathbb{R}$ by

$$\hat{f}(k) = \int_{-\infty}^\infty e^{-ikx} f(x) dx$$

then $\mathbb{D}_\alpha f(x)$ has Fourier transform $(ik)^\alpha \hat{f}(k)$.

Here $(ik)^\alpha = |k|^\alpha \,\mathrm{sign}(k) e^{i\alpha\pi/2}$.

Probability and transforms

If the random variable X has density $f(x)$ so that

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

then $f(x)$ has Fourier transform

$$\hat{f}(k) = \int_{-\infty}^{\infty} \left(1 - ikx + \frac{1}{2!}(ikx)^2 + \cdots\right) f(x)dx$$

$$= 1 - ik\mu_1 - \frac{1}{2}k^2\mu_2 + \cdots$$

where the pth moment

$$\mu_p = \int_{-\infty}^{\infty} x^p f(x)dx$$

Central limit theorem

If $\mu_1 = 0$ and $\mu_2 = 2$ then $\widehat{f}(k) = 1 - k^2 + \cdots$

The IID sum $S_n = X_1 + \cdots + X_n$ has FT $\widehat{f}(k)^n$ and the normalized sum $S_n/\sqrt{n}$ has FT

$$\begin{aligned}\widehat{f}(k/\sqrt{n})^n &= \left(1 - (k/\sqrt{n})^2 + \cdots\right)^n \\ &= \left(1 - \frac{k^2}{n} + \cdots\right)^n \\ &\to e^{-k^2} \equiv \widehat{g}(k) \quad \text{as } n \to \infty.\end{aligned}$$

Inverting the Fourier transform reveals a Gaussian density

$$g(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2/4}$$

Brownian motion

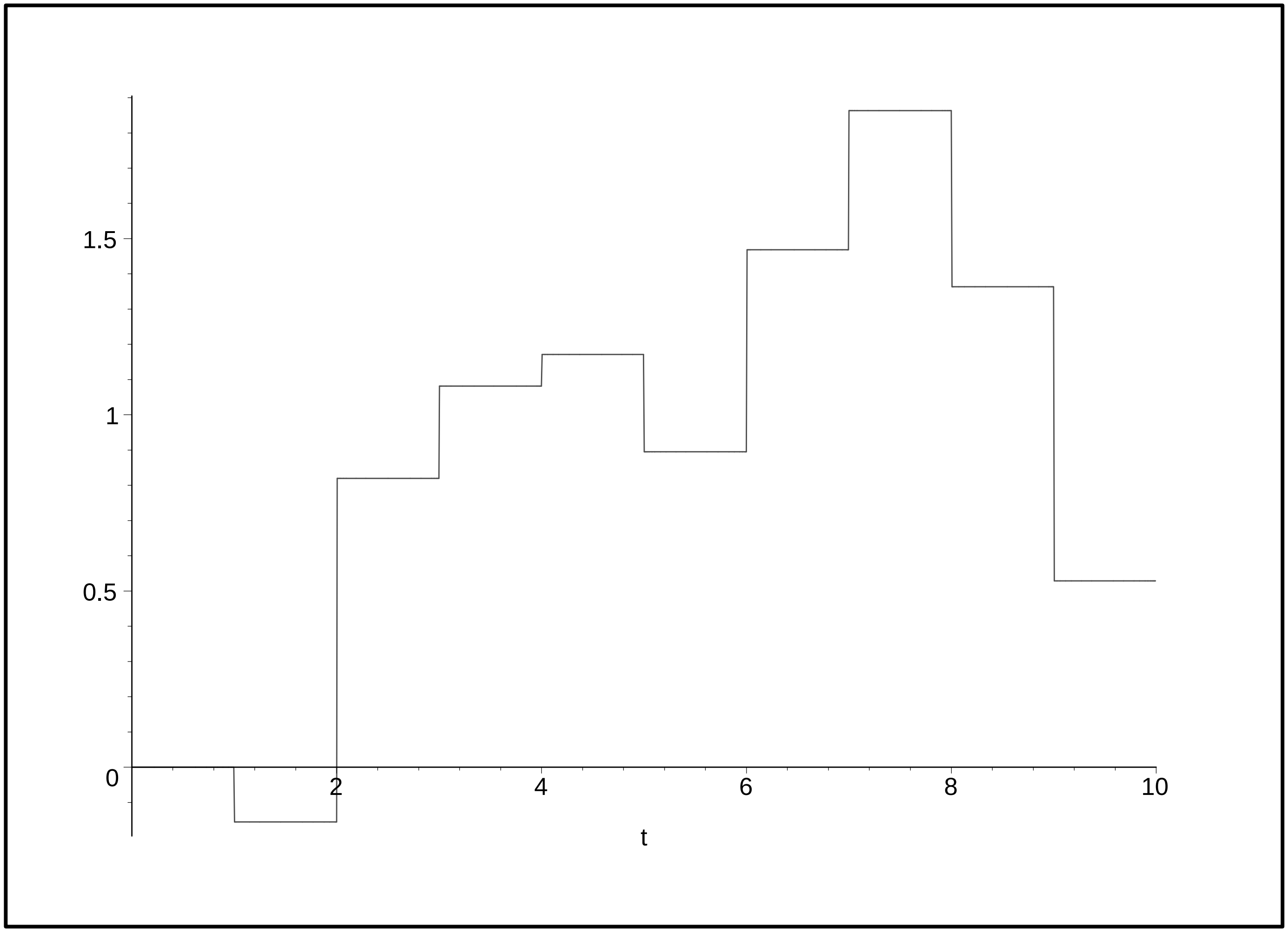
If X_n represents a particle jump at time n then $S_n = X_1 + \cdots + X_n$ is the location of the particle at time n. Expanding the time scale by a factor of $c > 0$ and taking limits as $c \to \infty$ shows that $c^{-1/2} S_{[ct]} \Rightarrow A_t$ since

$$\hat{f}(c^{-1/2}k)^{[ct]} = \left(1 - \frac{k^2}{c} + \cdots\right)^{[ct]} \\ \to e^{-k^2 t} \equiv \hat{c}(k,t)$$

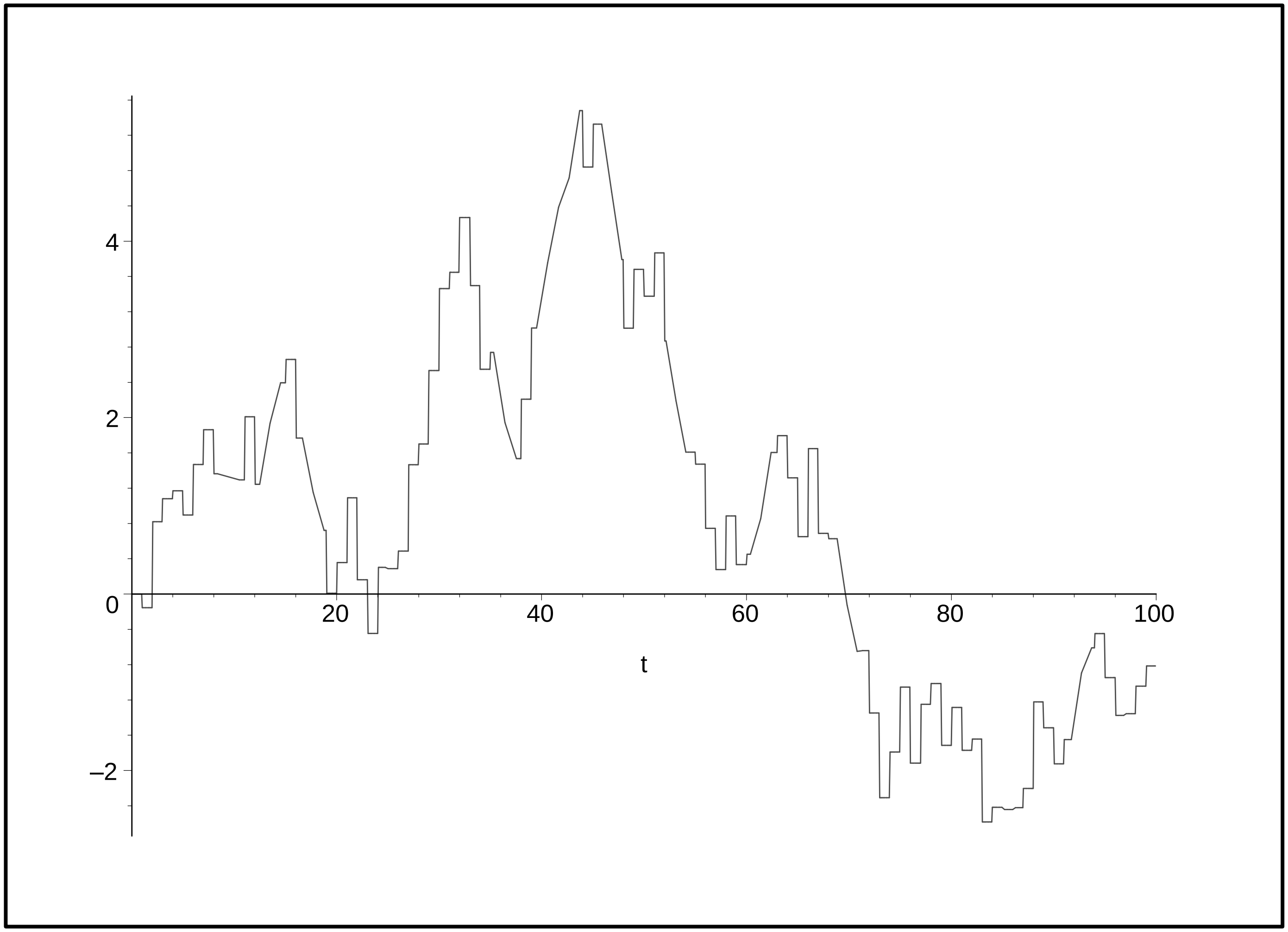
for all $t > 0$. Inverting the FT shows that the density of the limiting Brownian motion process A_t is Gaussian

$$c(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}.$$

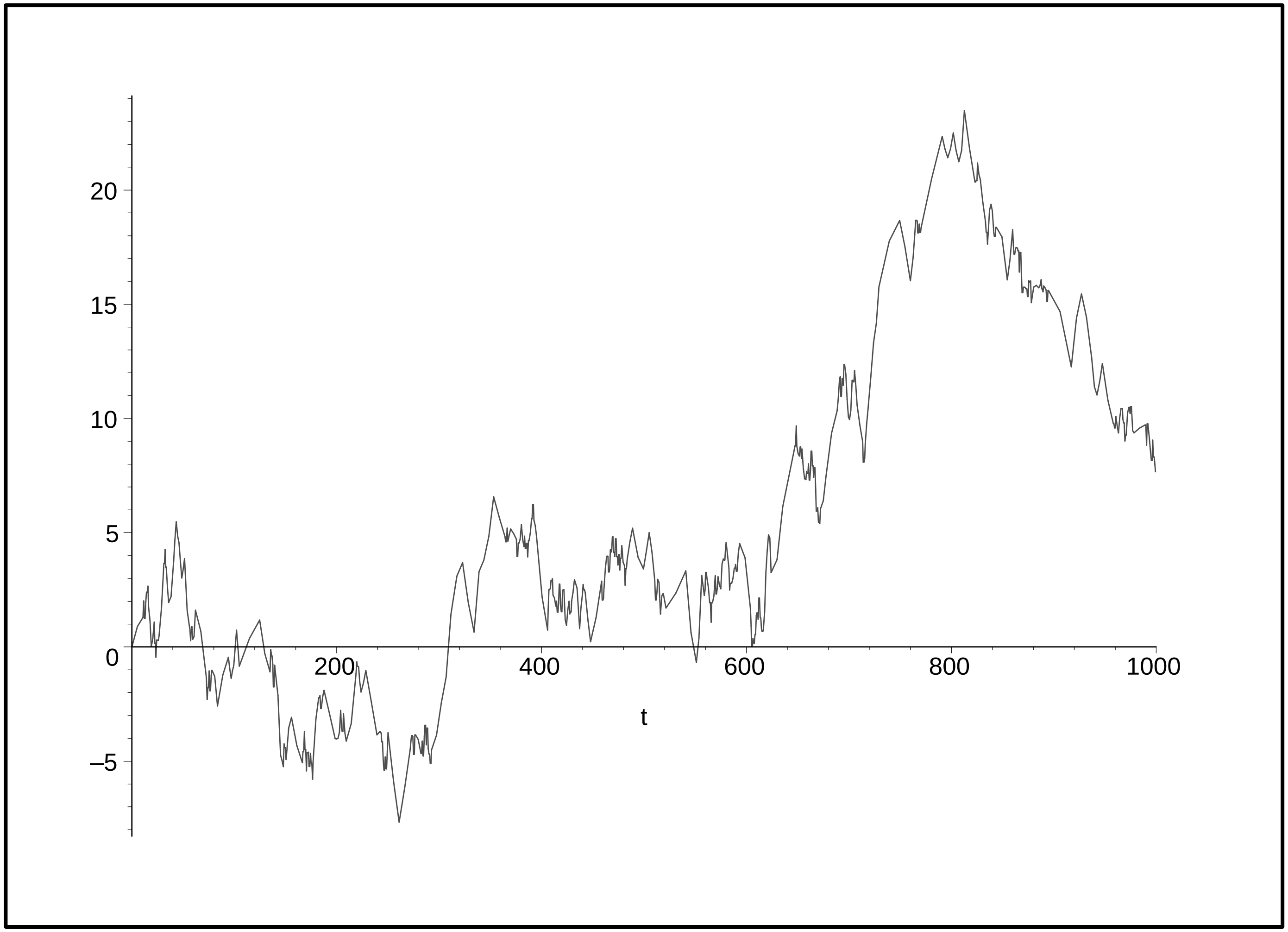
Simple random walk simulation



Longer time scale



Scaling limit: Brownian motion



The diffusion equation

Taking Fourier transforms in the classical diffusion equation

$$\frac{\partial c(x,t)}{\partial t} = \frac{\partial^2 c(x,t)}{\partial x^2}$$

yields

$$\frac{d\widehat{c}(k,t)}{dt} = (ik)^2 \widehat{c}(k,t) = -k^2 \widehat{c}(k,t)$$

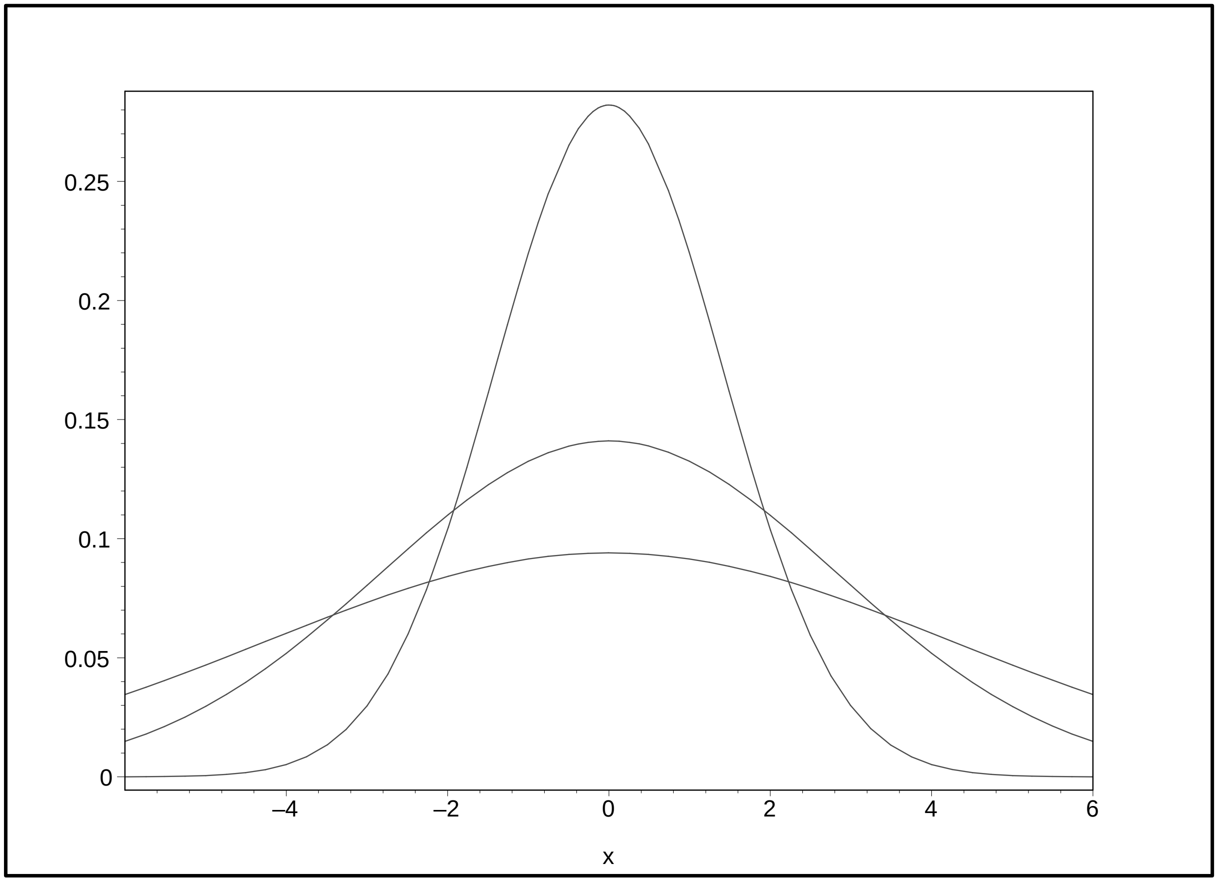
whose solution

$$\widehat{c}(k,t) = e^{-k^2 t}$$

inverts to the same limit density for the Brownian motion A_t.

For a cloud of diffusing particles $c(x,t)$ is the particle density.

Classical diffusion profile



Brownian motion Gaussian (Normal) density at time $t = 1, 4, 9$ showing square root spreading rate and fast tail decay.

Heavy (power law) tails

If $P(X > x) \approx x^{-\alpha}$ then $f(x) \approx \alpha x^{-\alpha-1}$ and some moments

$$\mu_k = \int_{-\infty}^{\infty} x^k f(x) dx$$

do not exist. If $1 < \alpha < 2$ and $\mu_1 = 0$ then a typical X has FT

$$\widehat{f}(k) = 1 + (ik)^\alpha + \cdots$$

and $n^{-1/\alpha}(X_1 + \cdots + X_n)$ has FT

$$\begin{aligned} \widehat{f}(n^{-1/\alpha}k)^n &= \left(1 + (n^{-1/\alpha}\,ik)^\alpha + \cdots\right)^n \\ &= \left(1 + \frac{(ik)^\alpha}{n} + \cdots\right)^n \\ &\to e^{(ik)^\alpha} \equiv \widehat{g}(k) \quad \text{as } n \to \infty. \end{aligned}$$

The inverse Fourier transform $g(x)$ is called a stable density.

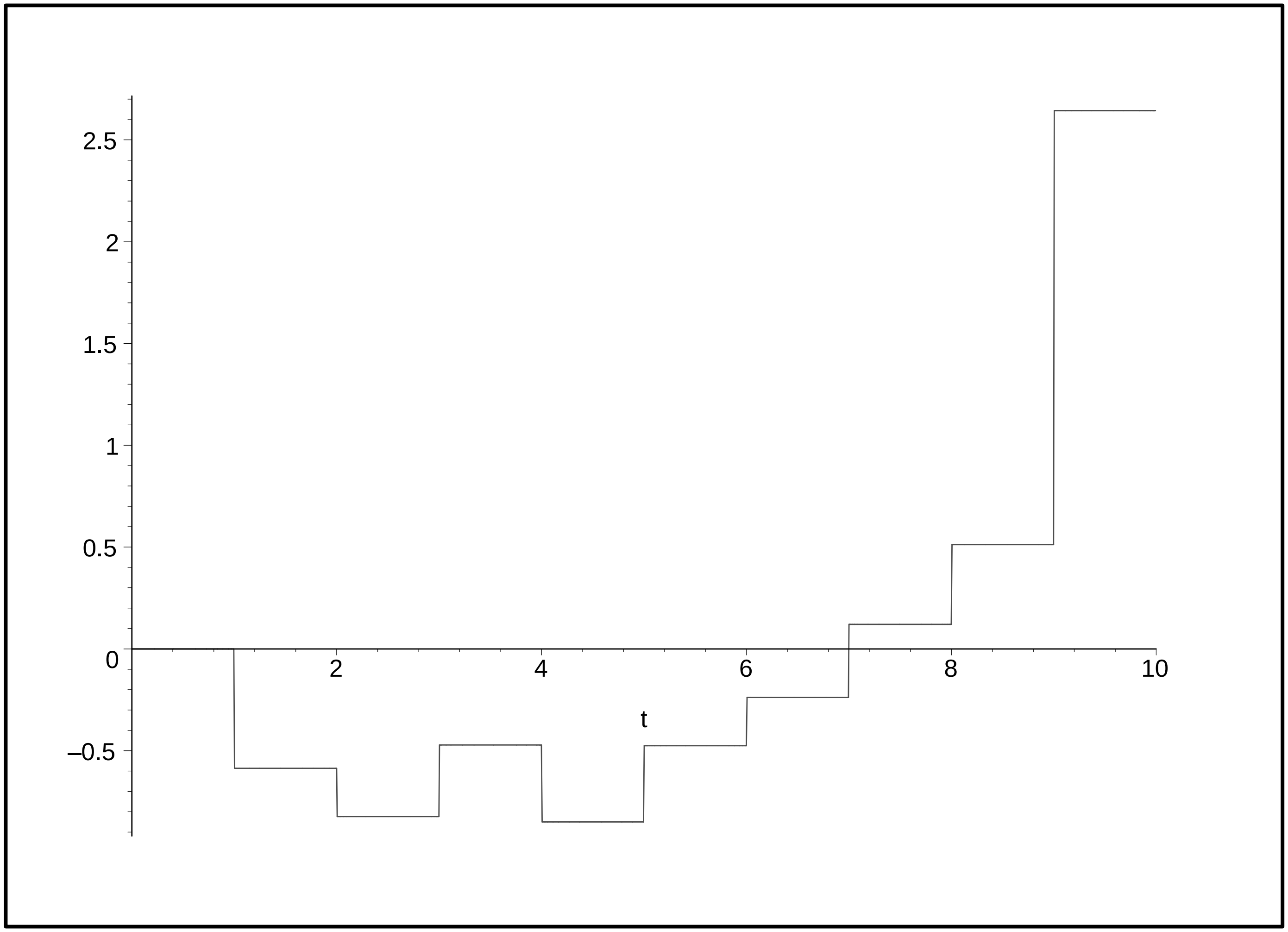
Lévy motion

If $S_n = X_1 + \cdots + X_n$ is particle location at time n then the scaling limit $c^{-1/\alpha} S_{[ct]} \Rightarrow A_t$ since

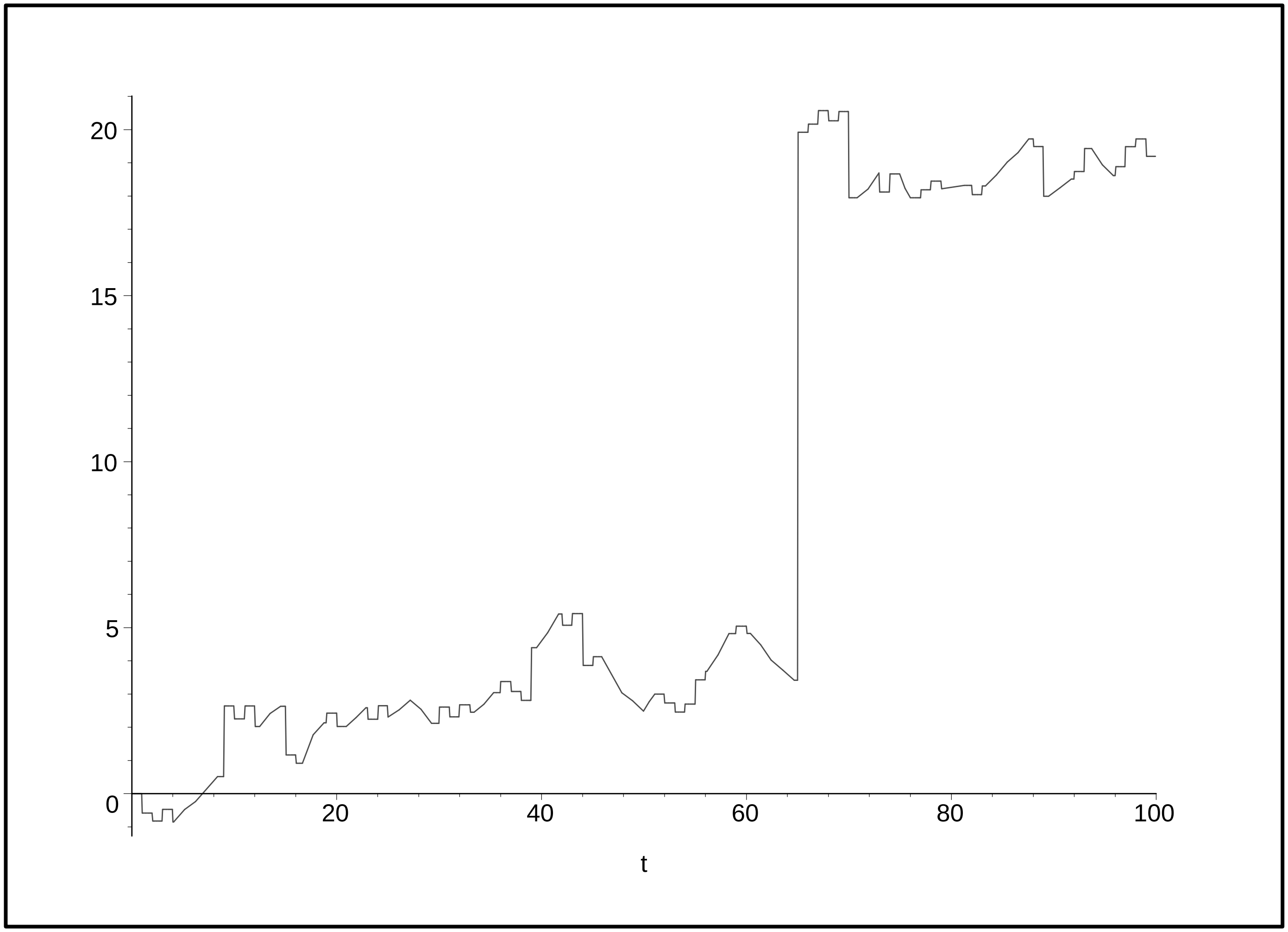
$$\begin{aligned} \widehat{f}(c^{-1/\alpha}k)^{[ct]} &= \left(1 + \frac{(ik)^\alpha}{c} + \cdots\right)^{[ct]} \\ &\to e^{t(ik)^\alpha} \equiv \widehat{c}(k,t). \end{aligned}$$

The limit process A_t is called a Lévy motion.

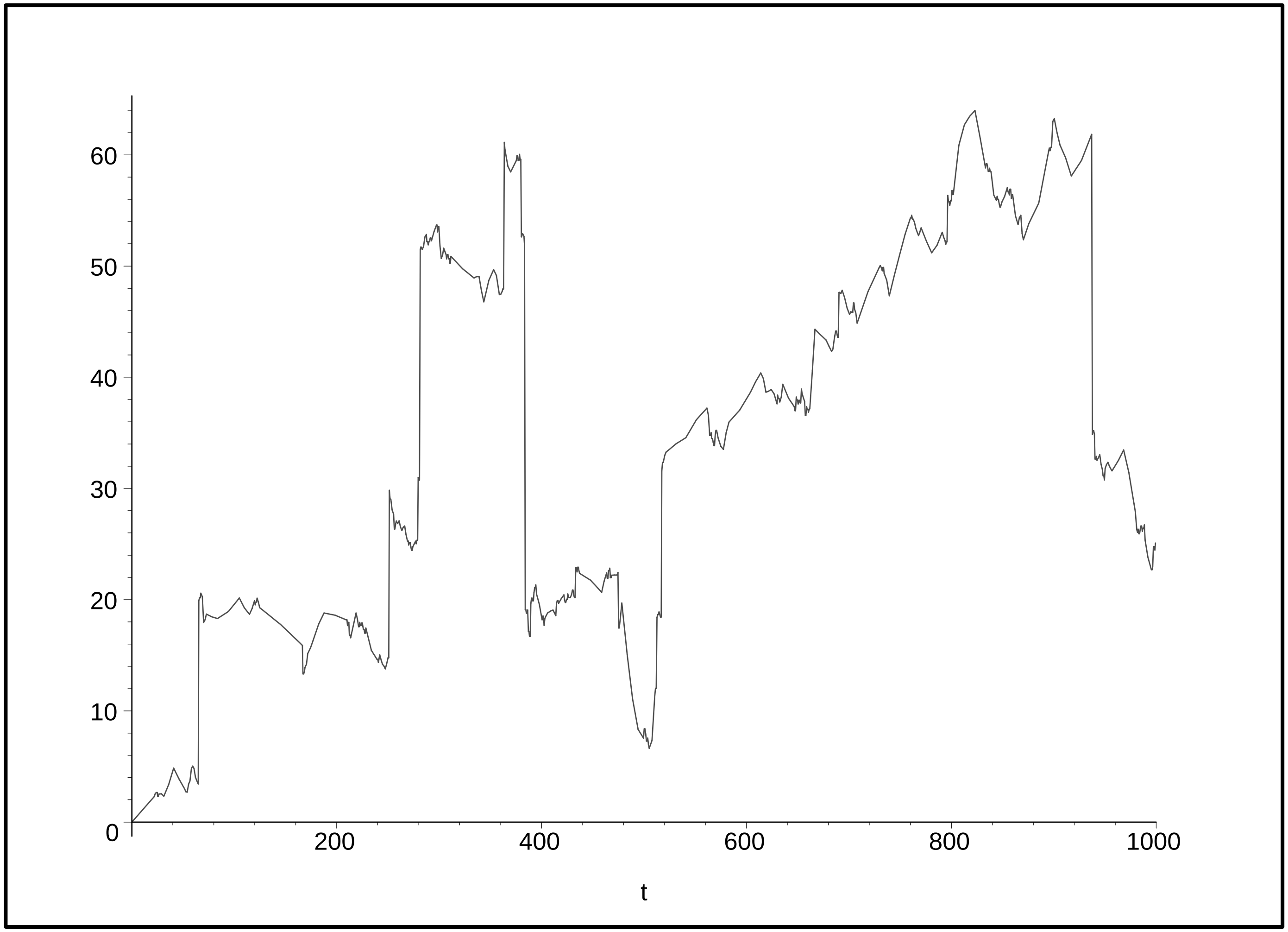
Heavy tail random walk simulation



Longer time scale



Scaling limit: Stable Lévy motion



Fractional diffusion equation

To solve the fractional diffusion equation

$$\frac{\partial c(x,t)}{\partial t} = \frac{\partial^\alpha c(x,t)}{\partial x^\alpha}$$

take Fourier transforms to get

$$\frac{d\widehat{c}(k,t)}{dt} = (ik)^\alpha \widehat{c}(k,t)$$

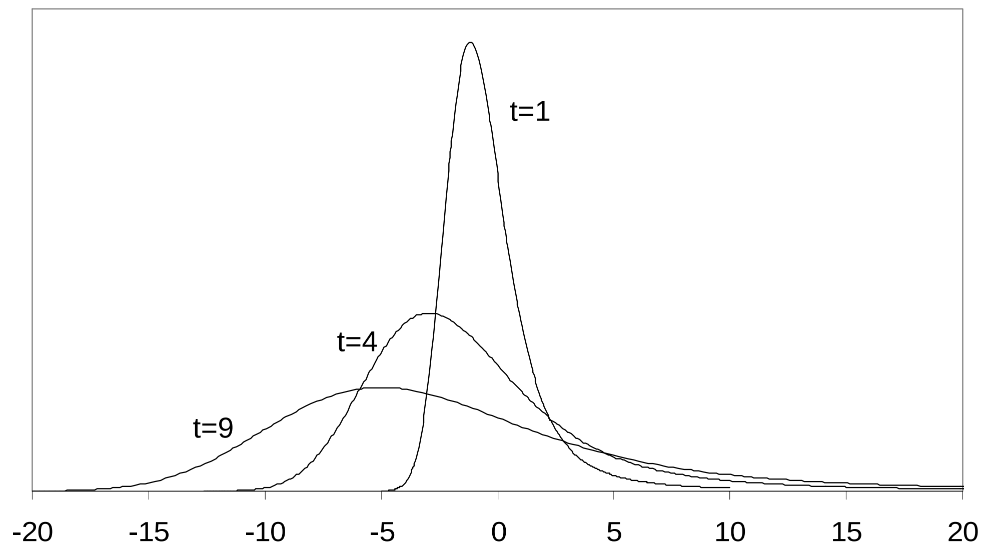
and solve to obtain

$$\widehat{c}(k,t) = e^{t(ik)^\alpha}$$

so the density of the Lévy motion solves this fractional PDE.

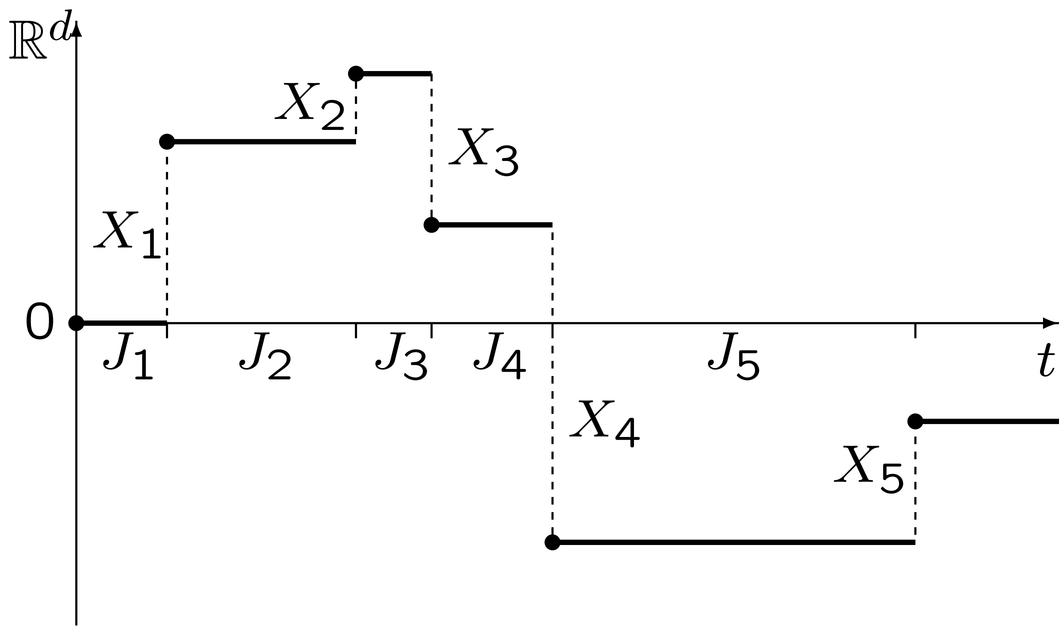
In this case $c(x,t)$ falls off like $x^{-\alpha-1}$ as $x \to \infty$.

Fractional diffusion profile



Stable $\alpha = 1.5$ Lévy motion density at time $t = 1, 4, 9$ showing super-diffusive spreading rate, skewness, and power law tail.

Continuous time random walks



The CTRW is a random walk with jumps X_n separated by random waiting times J_n. The random vectors (X_n, J_n) are i.i.d.

Heavy tail waiting times

If $P(J > t) \approx t^{-\beta}$ with $0 < \beta < 1$ then the mean waiting time is infinite. A typical J has LT

$$\tilde{f}(s) = 1 - s^{\beta} + \cdots$$

and $n^{-1/\beta}(J_1 + \cdots + J_n)$ has LT

$$\begin{aligned} \tilde{f}(n^{-1/\beta}s)^n &= \left(1 - (n^{-1/\beta}\,s)^{\beta} + \cdots\right)^n \\ &= \left(1 - \frac{s^{\beta}}{n} + \cdots\right)^n \\ &\to e^{-s^{\beta}} \equiv \tilde{g}(s) \quad \text{as } n \to \infty. \end{aligned}$$

The inverse Fourier transform $g(t)$ is a stable density with index $0 < \beta < 1$.

Waiting time process

If $T_n = J_1 + \cdots + J_n$ is the time of the nth particle jump, then the scaling limit $c^{-1/\beta} T_{[ct]} \Rightarrow D_t$ since

$$\tilde{f}(c^{-1/\beta}s)^{[ct]} = \left(1 - \frac{s^\beta}{c} + \cdots\right)^{[ct]}$$
$$\to e^{ts^\beta} \equiv \tilde{c}(k,t).$$

The limit process D_t is called a Lévy stable subordinator.

The number of jumps by time $t > 0$ is $N_t = \max\{n \geq 0 : T_n \leq t\}$ and note that $\{T_n \leq t\} = \{N_t \geq n\}$ (inverse processes)

CTRW scaling limits

Since $(c^{-1/\alpha}S_{[ct]}, c^{-1/\beta}T_{[ct]}) \Rightarrow (A_t, D_t)$ a continuous mapping argument yields

$$(c^{-1/\alpha}S_{[ct]}, c^{-\beta}N_{[ct]}) \Rightarrow (A_t, E_t)$$

where the inverse subordinator $E_t = \inf\{r > 0 : D_r > t\}$.

Another continuous mapping argument leads to

$$c^{-\beta/\alpha}S_{N_{[ct]}} = (c^{\beta})^{-1/\alpha}S(c^{\beta} \cdot c^{-\beta}N_{[ct]}) \Rightarrow A_{E_t}$$

Fractional governing equations

A simple conditioning argument shows that the limit process A_{E_t} has a density

$$m(x,t) = \int c(x,u)h(u,t)\,du$$

where $c(x,u)$ is the density of A_u and $h(u,t)$ is the density of E_t.

The density solves a space-time fractional diffusion equation

$$\frac{\partial^\beta m(x,t)}{\partial t^\beta} = \frac{\partial^\alpha m(x,t)}{\partial x^\alpha}$$

It is easy to check that this is equivalent to:

$$\frac{\partial c(x,u)}{\partial u} = \frac{\partial^\alpha c(x,t)}{\partial x^\alpha} \quad \text{and} \quad \frac{\partial h(u,t)}{\partial u} = -\frac{\partial^\beta h(u,t)}{\partial t^\beta}$$

Hitting time density

The time process is self-similar with $D_t \overset{d}{=} t^{1/\beta} D_1$. Then

$$\begin{aligned} P(E_t \leq u) &= P(D_u \geq t) \\ &= P(u^{1/\beta} D_1 \geq t) \\ &= P(D_1 \geq t u^{-1/\beta}) \end{aligned}$$

Take derivatives to see that

$$h(u,t) = \frac{t}{\beta} u^{-1-1/\beta} g(t u^{-1/\beta})$$

where $g(t)$ is the density of D_1.

Zolotarev duality

A negatively skewed stable $S_t = -A_t$ has density $P(x,t) = p(-x,t)$. The stable series representation for $1 < \alpha \leq 2$ gives

$$P(x,t) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\Gamma(1+k/\alpha)}{k!}t^{-k/\alpha}x^{k-1}\sin\left(\frac{\pi k}{\alpha}\right).$$

Substituting the hitting time density formula into the stable series representation for $0 < \beta = (1/\alpha) < 1$ leads to

$$h(u,t) = \alpha\,\frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\Gamma(1+k/\alpha)}{k!}t^{-k/\alpha}u^{k-1}\sin\left(\frac{\pi k}{\alpha}\right).$$

Hence $h(x,t) = \alpha P(x,t)$ for $x > 0$, or $E_t \overset{d}{=} S_t | S_t > 0$.

The process S_t has negative jumps.

Consequences of Zolotarev duality

Since $\frac{\partial h(u,t)}{\partial u} = -\frac{\partial^\beta h(u,t)}{\partial t^\beta}$ we also have

$$\frac{\partial P(u,t)}{\partial u} = -\frac{\partial^\beta P(u,t)}{\partial t^\beta}$$

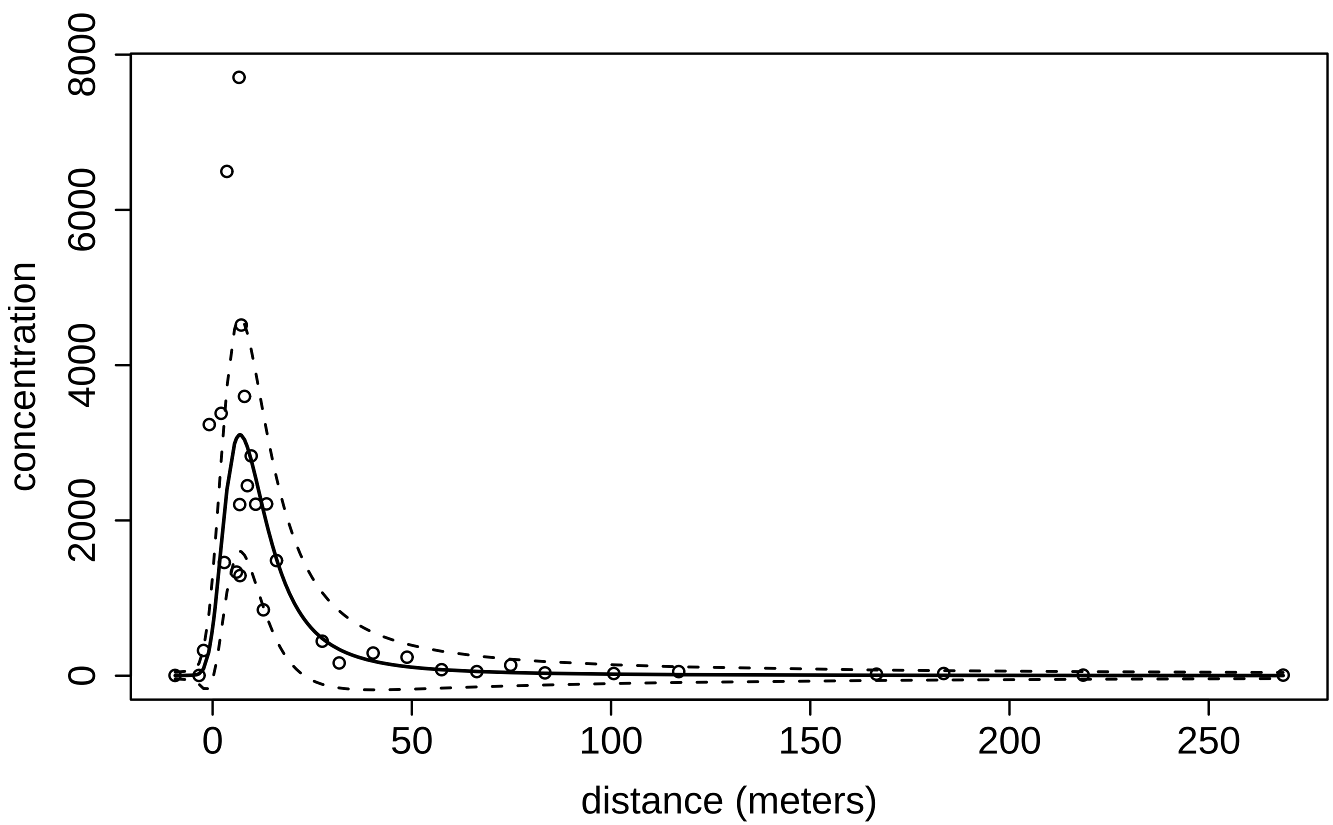
Since $\frac{\partial P(x,u)}{\partial u} = \frac{\partial^\alpha P(x,t)}{\partial(-x)^\alpha}$ we also have

$$\frac{\partial h(x,u)}{\partial u} = \frac{\partial^\alpha h(x,u)}{\partial(-x)^\alpha}$$

Here $d^{-\alpha} f(x)/d(-x)^\alpha$ has FT $(-ik)^\alpha \widehat{f}(k)$.

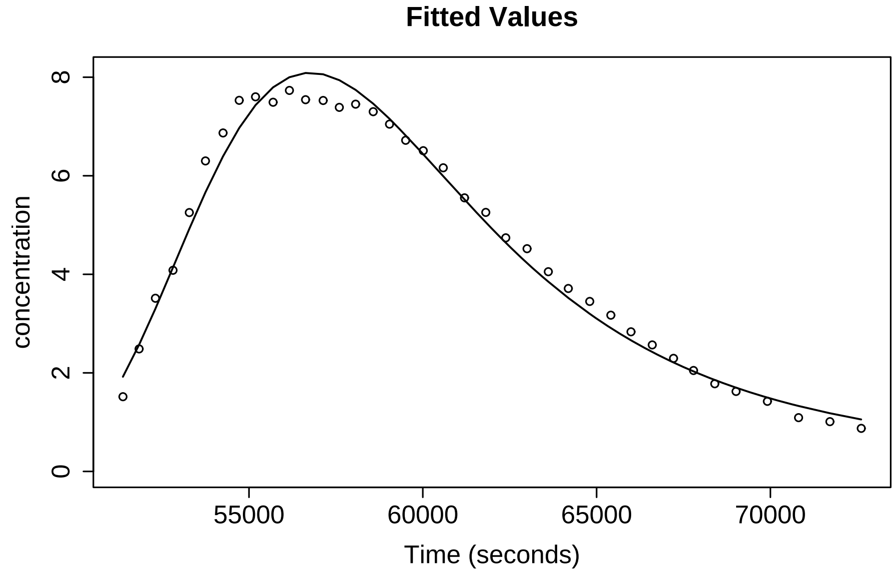
Tracer test in an underground aquifer

Positively skewed stable density $c(x,t)$ with $\alpha = 1.1$ gives a good fit. The positive skewness reflects downstream jumps attributed to high velocity channels.



Tracer test in a Michigan river

Negatively skewed stable fit $t \mapsto c(x,t)$ with $\alpha = 1.38$ for the Grand River.



A controversy in hydrology

Negative skewness comes from negative jumps in the CTRW.

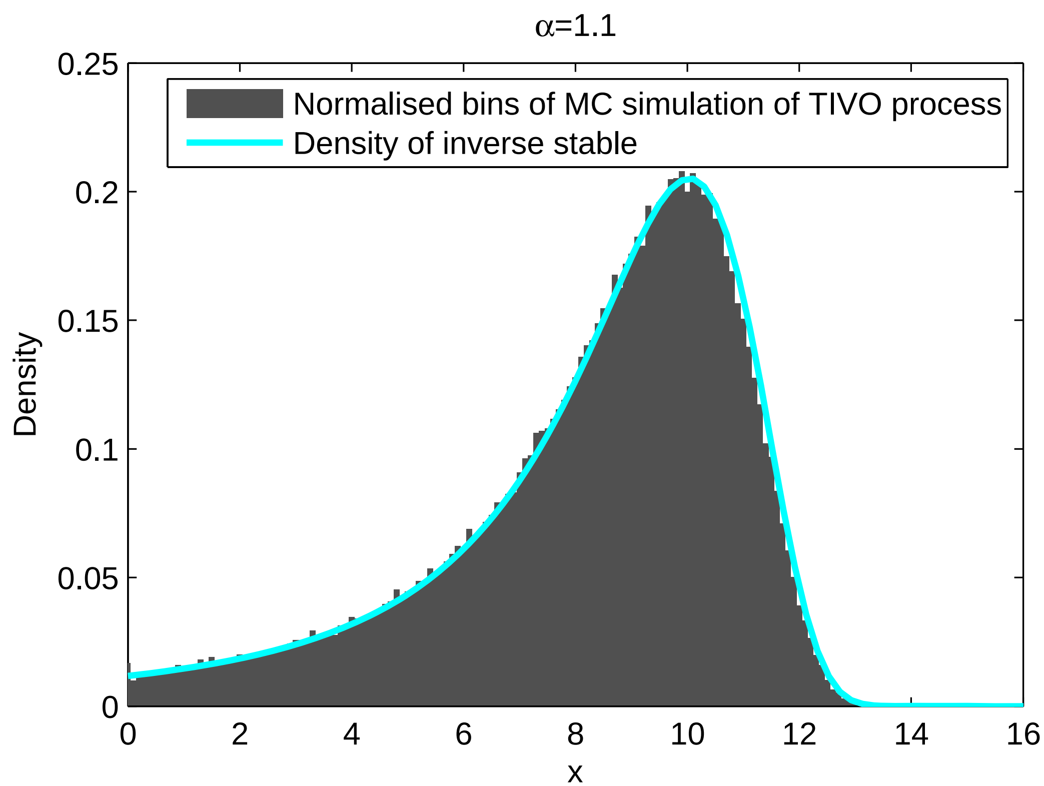
Particles cannot jump far upstream!

Duality equates negative skewness to power law waiting times.

Time-fractional model equivalent to negative space-fractional.

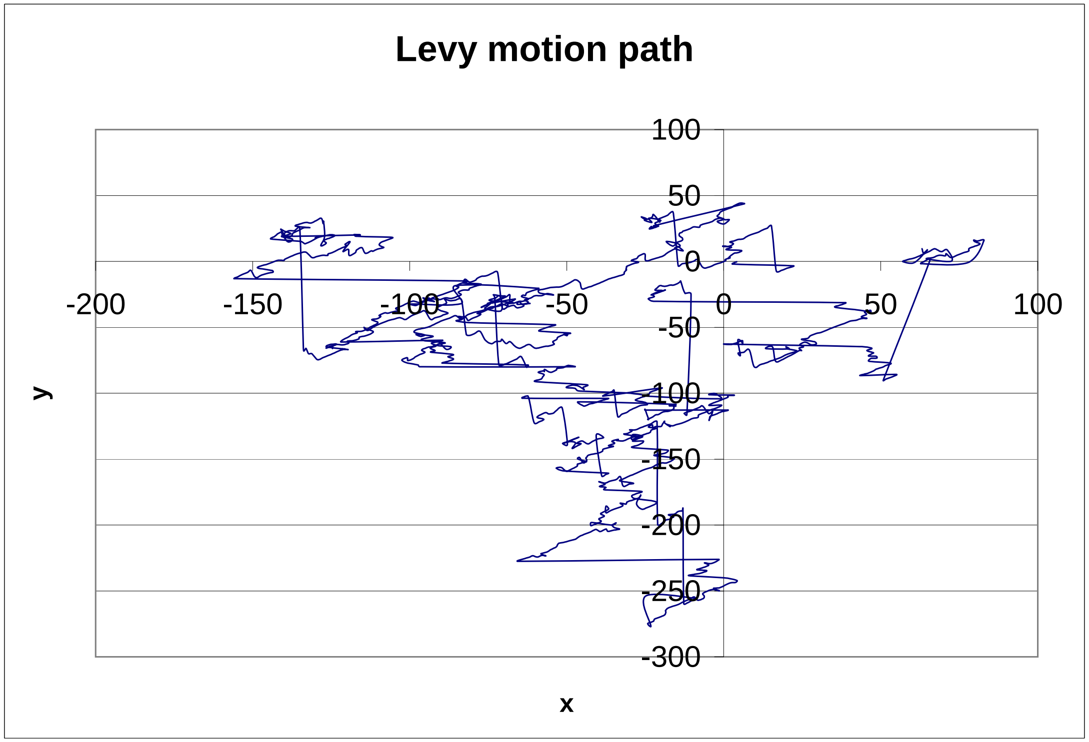
Open problem: Markovian subordinator

Is there a stable-like Markov process Z_t with the same probability density functions as E_t?



Open problem: Fractal properties

Lévy particle traces are random fractals with dimension α. Fractal properties of CTRW limits are unexplored.

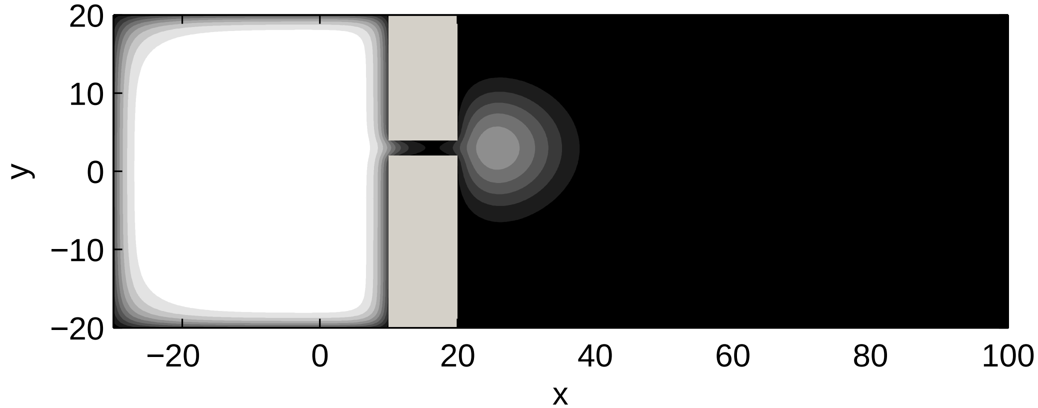


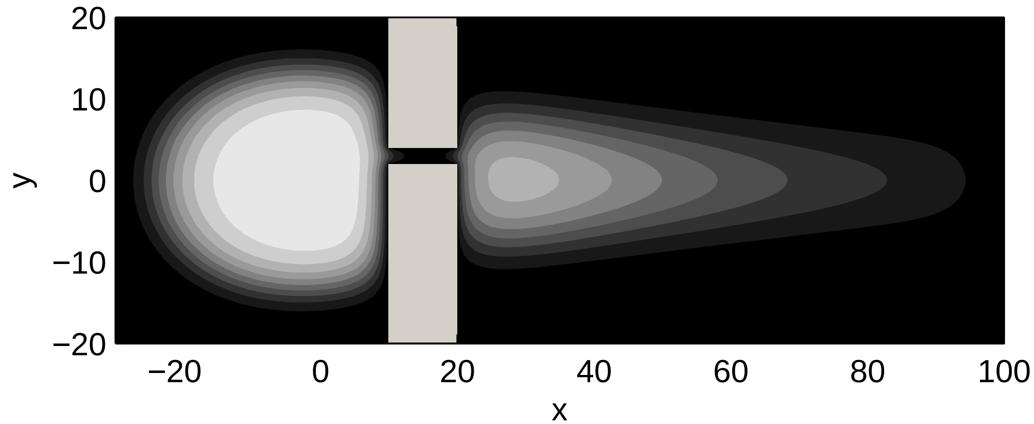
Biological species growth and dispersion

Fractional derivatives model fast spreading via long movements.

$$\frac{\partial P}{\partial t} = C\frac{\partial^\alpha P}{\partial x^\alpha} + D\frac{\partial^2 P}{\partial y^2} + rP\left(1 - \frac{P}{K}\right)$$

Compare $\alpha = 2$ (top) to $\alpha = 1.7$ (bottom).

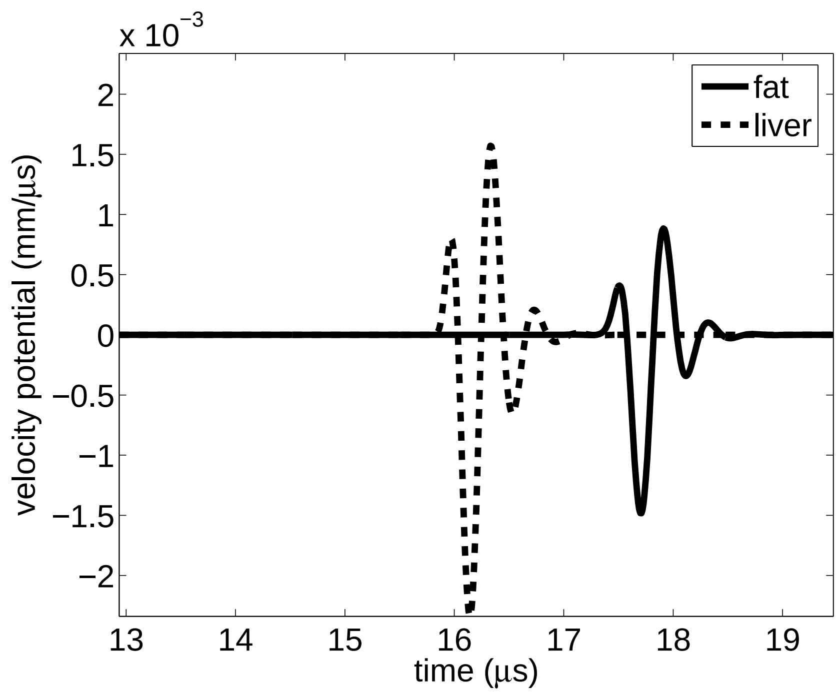




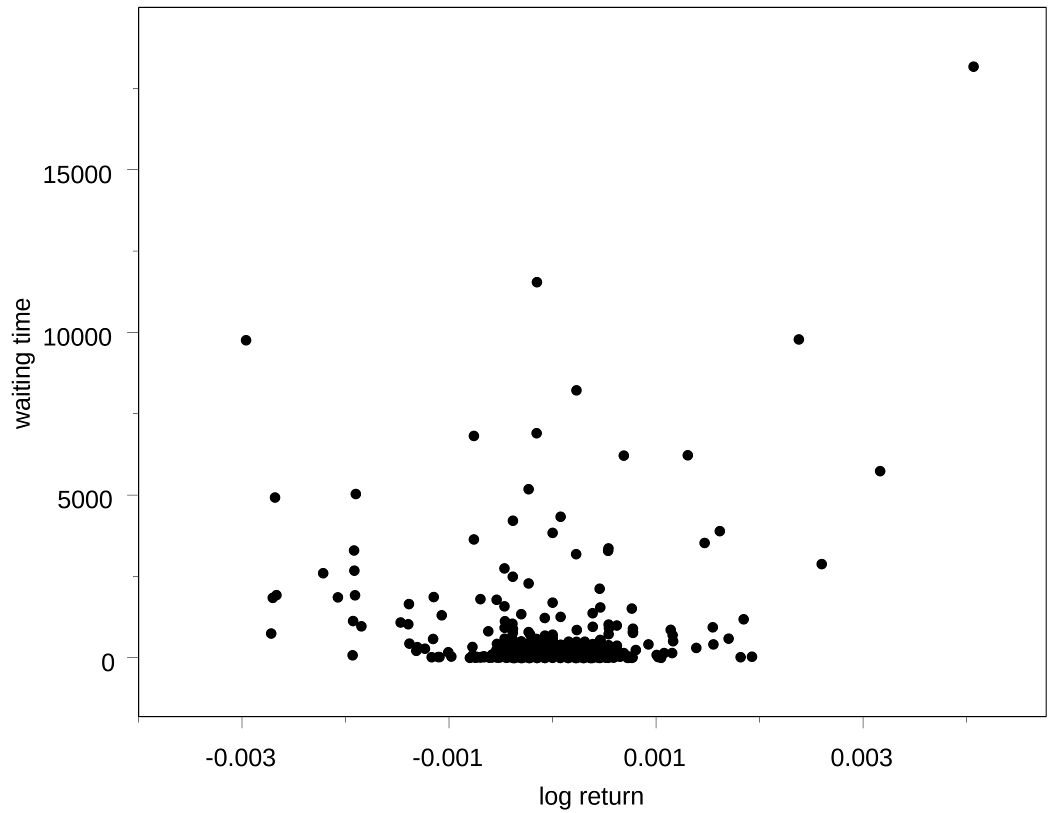
Sound wave propogation

We use $\beta = 2.5$ for human fat tissue and $\beta = 2.1$ for liver tissue.

$$\frac{\partial^2}{\partial t^2} c(t,x) + C \frac{\partial^\beta}{\partial t^\beta} c(t,x) = D \frac{\partial^2}{\partial x^2} c(t,x)$$

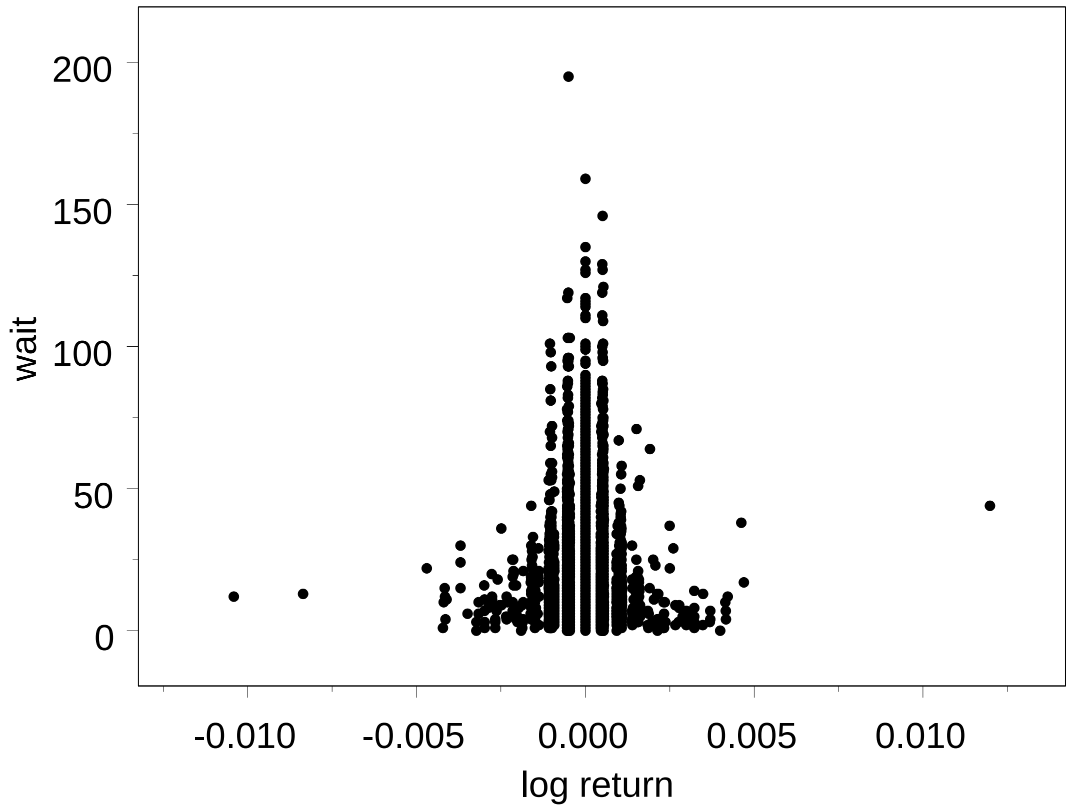


LIFFE BTP bond futures Sept 1997 delivery



Log returns and waiting times (sec) are dependent random variables. Long waiting times are associated with large returns.

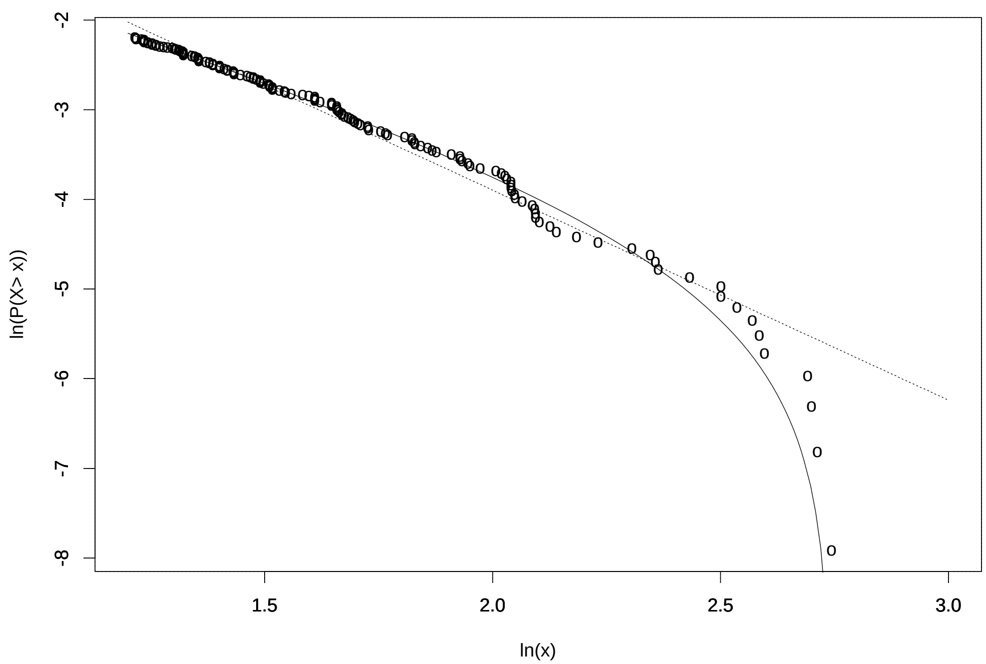
General Electric stock October 1999



Long waiting times and large returns appear asymptotically independent.

Tempered stables in finance

AMZN stock price changes fit a tempered power law model $P(X > x) \approx x^{-0.6} e^{-0.3x}$ for x large



References

1. B. Baeumer and M.M. Meerschaert (2001) Stochastic solutions for fractional Cauchy problems. *Fractional Calculus and Applied Analysis* **4**, 481–500.

2. B. Baeumer, M. Kovács, and M.M. Meerschaert (2007) Fractional reproduction-dispersal equations and heavy tail dispersal kernels. *Bull. Math. Biology* **69**, 2281–2297.

3. B. Baeumer, M. Kovács, and M.M. Meerschaert (2008) Numerical solutions for fractional reaction-diffusion equations. *Comput. Math. Appl.* **55**, 2212-2226.

4. **B. Baeumer, M.M. Meerschaert and E. Nane (2009) Space-time duality for fractional diffusion.** *J. Applied Probab.*, **to appear.** `www.stt.msu.edu/~mcubed/duality.pdf`

5. P. Becker-Kern, M.M. Meerschaert and H.P. Scheffler (2003) Hausdorff dimension of operator stable sample paths. *Monatshefte fur Mathematik*, **140**, No. 2, 90–101.

6. P. Becker-Kern, M.M. Meerschaert and H.P. Scheffler (2004) Limit theorems for coupled continuous time random walks. *The Annals of Probability* **32**, No. 1B, 730–756.

7. P. Chakraborty, M.M. Meerschaert and C.Y. Lim (2009) Parameter Estimation for Fractional Transport: A particle tracking approach. *Water Resources Research*, to appear. `www.stt.msu.edu/~mcubed/fADEfit.pdf`

8. J.F. Kelly and R.J. McGough, M.M. Meerschaert (2008) Time-Domain 3D Greens Functions for Power Law Media. *J. Acoustical Soc. Amer.* **124**, 2861-2872.

9. M. Kleinz and T.J. Osler (2000) A child's garden of fractional derivatives. *College Math. J.* **31**, No. 2, 82–87.

10. M.M. Meerschaert and H.P. Scheffler (2001) *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*. *Wiley Interscience*, New York.

11. M.M. Meerschaert, D.A. Benson, H.P. Scheffler and P. Becker-Kern (2002) Governing equations and solutions of anomalous random walk limits. *Phys. Rev. E* **66**, 102–105.

12. M.M. Meerschaert and H.P. Scheffler (2004) Limit theorems for continuous-time random walks with infinite mean waiting times. *J. Appl. Probab.* **41**(3), 623–638.

13. M.M. Meerschaert and Y. Xiao (2005) Dimension results for sample paths of operator stable Lévy processes. *Stochastic Processes and Their Applications* **115**, 55–75.

14. M.M. Meerschaert and E. Scalas (2006) Coupled continuous time random walks in finance. *Physica A*, **370**, 114–118.

15. M.M. Meerschaert and H.P. Scheffler (2008) Triangular array limits for continuous time random walks. *Stochastic Processes and their Applications* **118**, No. 9, 1606-1633.

16. M.M. Meerschaert, P. Roy and Q, Shao (2009) Parameter estimation for tempered power law distributions. Preprint at `www.stt.msu.edu/~mcubed/TempPareto.pdf`

17. K. Miller and B. Ross (1993) *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley and Sons, New York.

18. K.B. Oldham and J. Spanier (1974) *The Fractional Calculus*. Academic Press, New York.

19. S. Samko, A. Kilbas and O. Marichev (1993) *Fractional Integrals and derivatives: Theory and Applications*. Gordon and Breach, London.

20. G. Samorodnitsky and M. Taqqu (1994) *Stable non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapman and Hall, London.

21. I.M. Sokolov and J. Klafter (2005) From diffusion to anomalous diffusion: a century after Einstein's Brownian motion. *Chaos* **15**, No. 2, 26–103.

Derivatives of power laws

If both p and α are integers then

$$\mathbb{D}_1\left[x^p\right] = p x^{p-1}$$
$$\mathbb{D}_2\left[x^p\right] = p(p-1) x^{p-2}$$
$$\vdots$$
$$\mathbb{D}_\alpha\left[x^p\right] = \frac{p!}{(p-\alpha)!} x^{p-\alpha}$$

For $p > 0$ the Gamma function extends $p! = \Gamma(p+1)$ via

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx.$$

Use the property $\Gamma(p+1) = p\Gamma(p)$ to get

$$\mathbb{D}_\alpha\left[x^p\right] = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha}.$$

Fractional derivatives of power laws

If $p > 0$ then the Laplace transform

$$\text{LT}\{x^p\} = \int_0^\infty e^{-sx} x^p dx \qquad \boxed{\text{substitute } y = sx}$$
$$= \int_0^\infty e^{-y} (y/s)^p dy/s = s^{-p-1}\Gamma(p+1).$$

Then

$$\begin{aligned}\text{LT}\{\mathbb{D}_\alpha x^p\} &= s^\alpha s^{-p-1}\Gamma(p+1) \\ &= s^{-(p-\alpha)-1}\Gamma(p-\alpha+1)\cdot\frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} \\ &= \text{LT}\left\{\frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}x^{p-\alpha}\right\}\end{aligned}$$

and the uniqueness of the LT yields

$$\mathbb{D}_\alpha[x^p] = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}x^{p-\alpha}.$$

Difference quotients

The derivative $\mathbb{D}_1 f(x) = \lim_{h\to 0} h^{-1}\Delta f(x)$ where

$$\Delta f(x) = f(x) - f(x-h).$$

For positive integers α, $\mathbb{D}_\alpha f(x) = \lim_{h\to 0} h^{-\alpha}\Delta^\alpha f(x)$ where

$$\begin{aligned}
\Delta^2 f(x) &= (f(x) - f(x-h)) - (f(x-h) - f(x-2h)) \\
&= f(x) - 2f(x-h) + f(x-2h), \\
\Delta^3 f(x) &= f(x) - 3f(x-h) + 3f(x-2h) - f(x-3h) \\
&\vdots \\
\Delta^\alpha f(x) &= \sum_{m=0}^{\alpha} \binom{\alpha}{m} (-1)^m f(x-mh). \quad \text{Here } \binom{\alpha}{m} = \frac{\alpha!}{m!(\alpha-m)!}
\end{aligned}$$

Fractional difference quotients

For $\alpha > 0$ define $\mathbb{D}_\alpha f(x) = \lim_{h\to 0} h^{-\alpha}\Delta^\alpha f(x)$ where

$$\Delta^\alpha f(x) = \sum_{m=0}^{\infty} \binom{\alpha}{m} (-1)^m f(x-mh), \qquad \binom{\alpha}{m} = \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)}$$

Since $f(x-h)$ has FT $e^{-ikh}\widehat{f}(k)$, and using the Binomial formula

$$(1+z)^\alpha = \sum_{m=0}^{\infty} \binom{\alpha}{m} z^m \quad \text{for any complex } |z| \le 1$$

we see that $\Delta^\alpha f(x)$ has FT

$$\sum_{m=0}^{\infty} \binom{\alpha}{m} (-1)^m e^{-ikmh}\widehat{f}(k) = (1-e^{-ikh})^\alpha \widehat{f}(k)$$

and then the FT of $h^{-\alpha}\Delta^\alpha f(x)$ is

$$h^{-\alpha}(ikh)^\alpha \left(\frac{1-e^{-ikh}}{ikh}\right)^\alpha \widehat{f}(k) \to (ik)^\alpha \widehat{f}(k) \quad \text{as } h \to 0.$$

Random walk simulation code (Maple)

```
> N:=1000:
> J:=random[uniform[-1,1]](N): # jump distribution
> n:='n':T:=0:
> for n from 1 to N do
>   T:=T+1;
>   S[n]:=T;
> od:n:='n':
> plot(sum(J[n]*Heaviside(t-S[n]),n=1..1000),t=0..10);
```

See http://www.maplesoft.on.ca/

Heavy tail random walk simulation code (Maple)

```
> lambda:=1:N:=1000:alpha:=1.5:C:=.1:
> P:=random[uniform[0,1]](N):
> J:=random[uniform[0,1]](N):
> n:='n':T:=0:
> for n from 1 to N do
>   T:=T+1;
>   S[n]:=T;
> od:n:='n':
> plot(sum((2*floor(2*P[n])-1)*(C/J[n])^(1/alpha)
   *Heaviside(t-S[n]),n=1..1000),t=0..1000);
```

See `http://www.maplesoft.on.ca/`

Heavy tailed jumps $U^{-1/\alpha}$ where $U \sim \text{Uniform}[0,1]$.