Variance Gamma and Normal Inverse Gaussian Risky Asset Models with Dependence through Fractal Activity Time

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Outline

- Geometric Brownian motion model and desired ‘stylized features’
- Fractal activity time models
- Distributional properties
- Construction of fractal activity time
- Data examples
- Pricing formulas

Geometric Brownian Motion Model

Geometric Brownian motion (GBM) or Black-Scholes model for risky asset:

$$P_t = P_0 e^{\{\mu t + \sigma B(t)\}}, \quad t > 0$$

where $\mu \in R$, $\sigma > 0$, and B is Brownian motion.
Log returns: $X_t = \log P_t - \log P_{t-1}$, and in GBM

$$X_t = \mu + \sigma(B(t) - B(t-1)), \quad t \geq 1.$$

According to this model, the log returns X_t, $t = 1, 2, 3, \ldots$ are i.i.d. Gaussian

‘Stylized features’

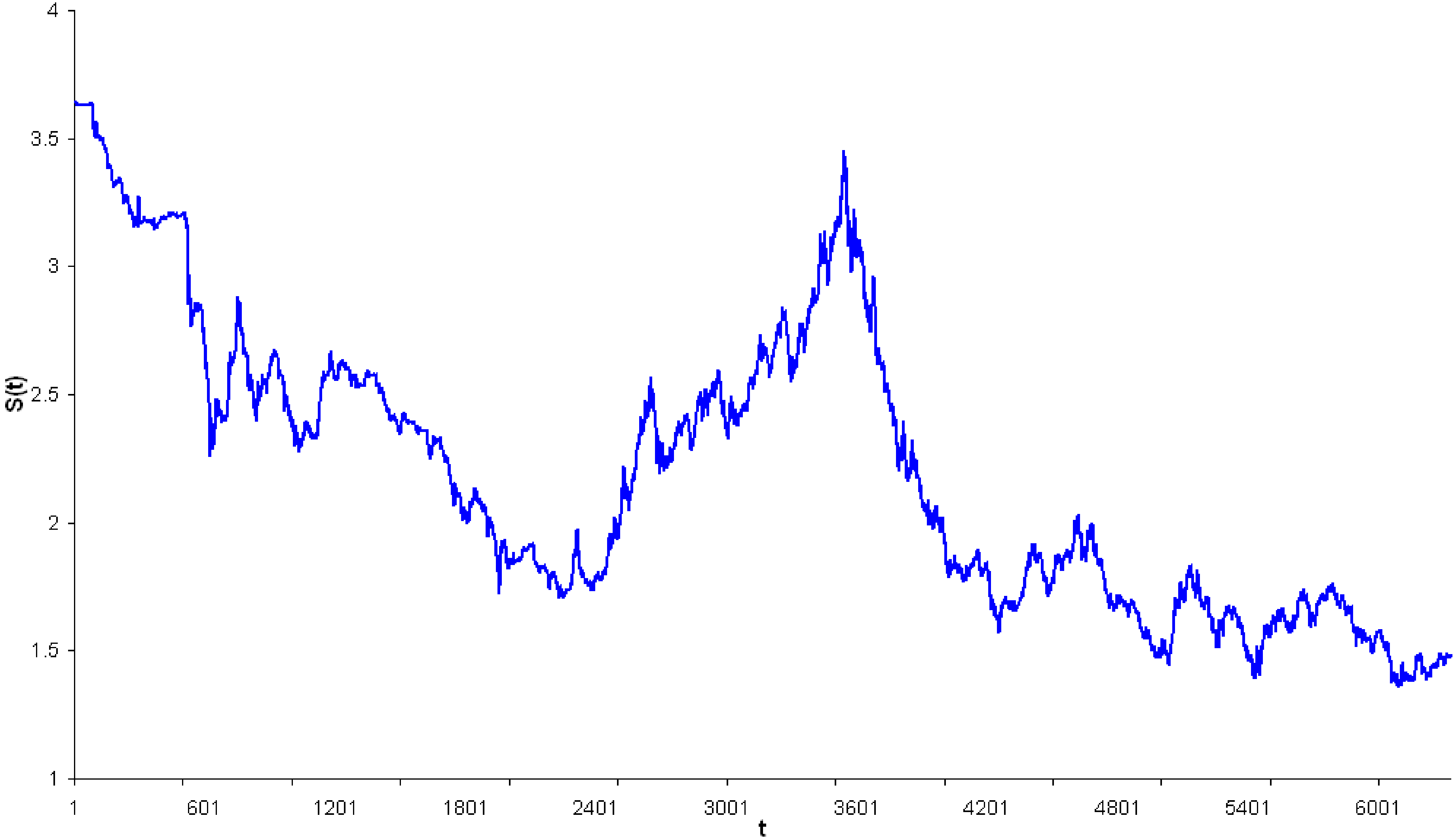
Features of log returns observed in practice (Granger 2005):

- Log returns are reasonably approximated by uncorrelated identically distributed random variables (independent in the Gaussian case)
- Squared and absolute log returns are dependent through time, with autocorrelation functions decreasing very slowly, remaining substantial after 50 to 100 lags
- Log returns have distributions that are heavier-tailed and higher-peaked than Gaussian distributions

Empirical evidence against GBM model

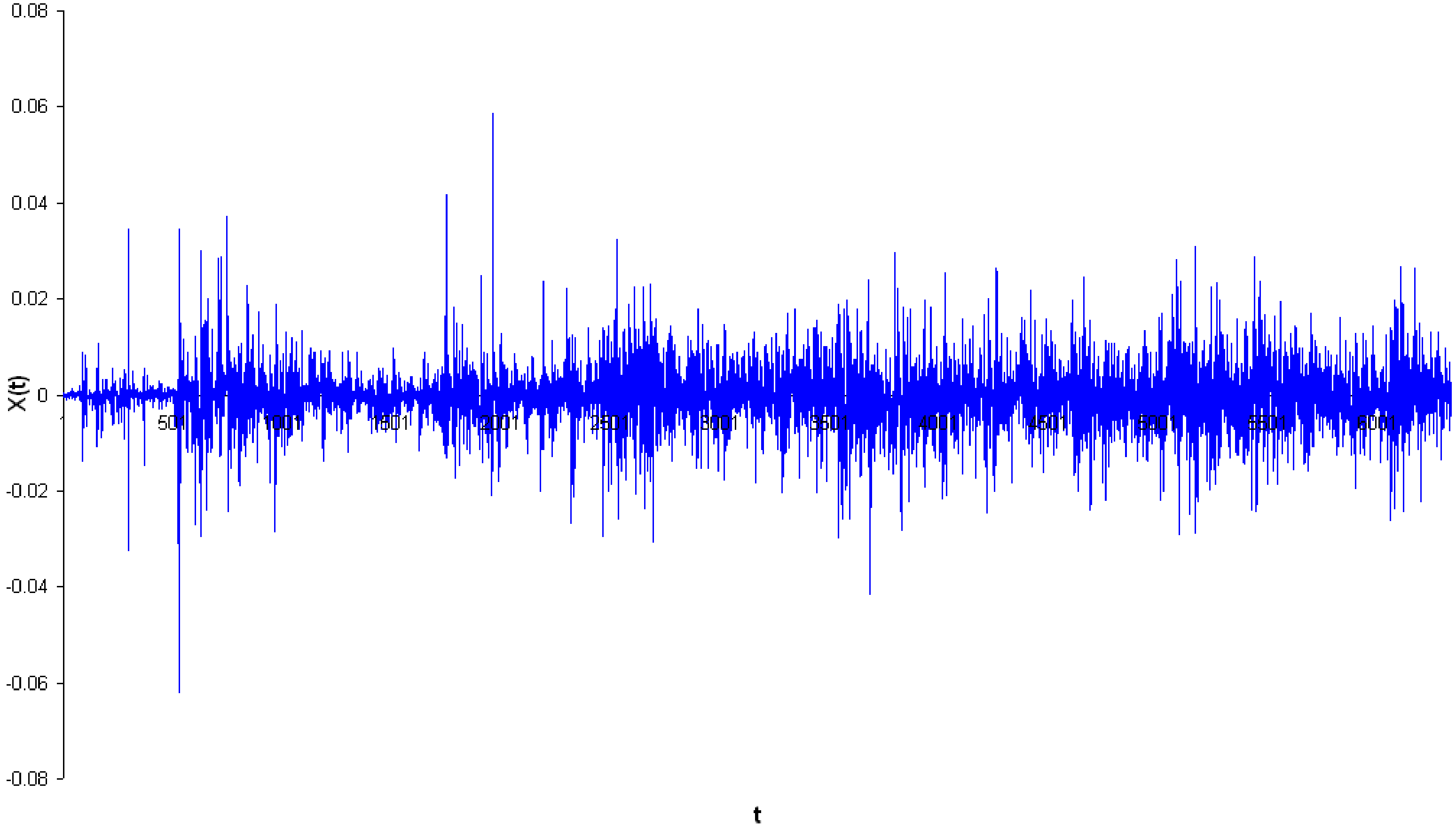
- Found in the literature (e.g. Heyde and Liu (2001), Seneta (2004))
- We present data of exchange rates between DM (N=6333), FF (N=6428), GBP (N=4510), JY (N=4510), CD (N=1700), NTD(N=1200), and the US dollar, for every working day over various periods of time 1971 and 2001

Price (exchange rate) for DM

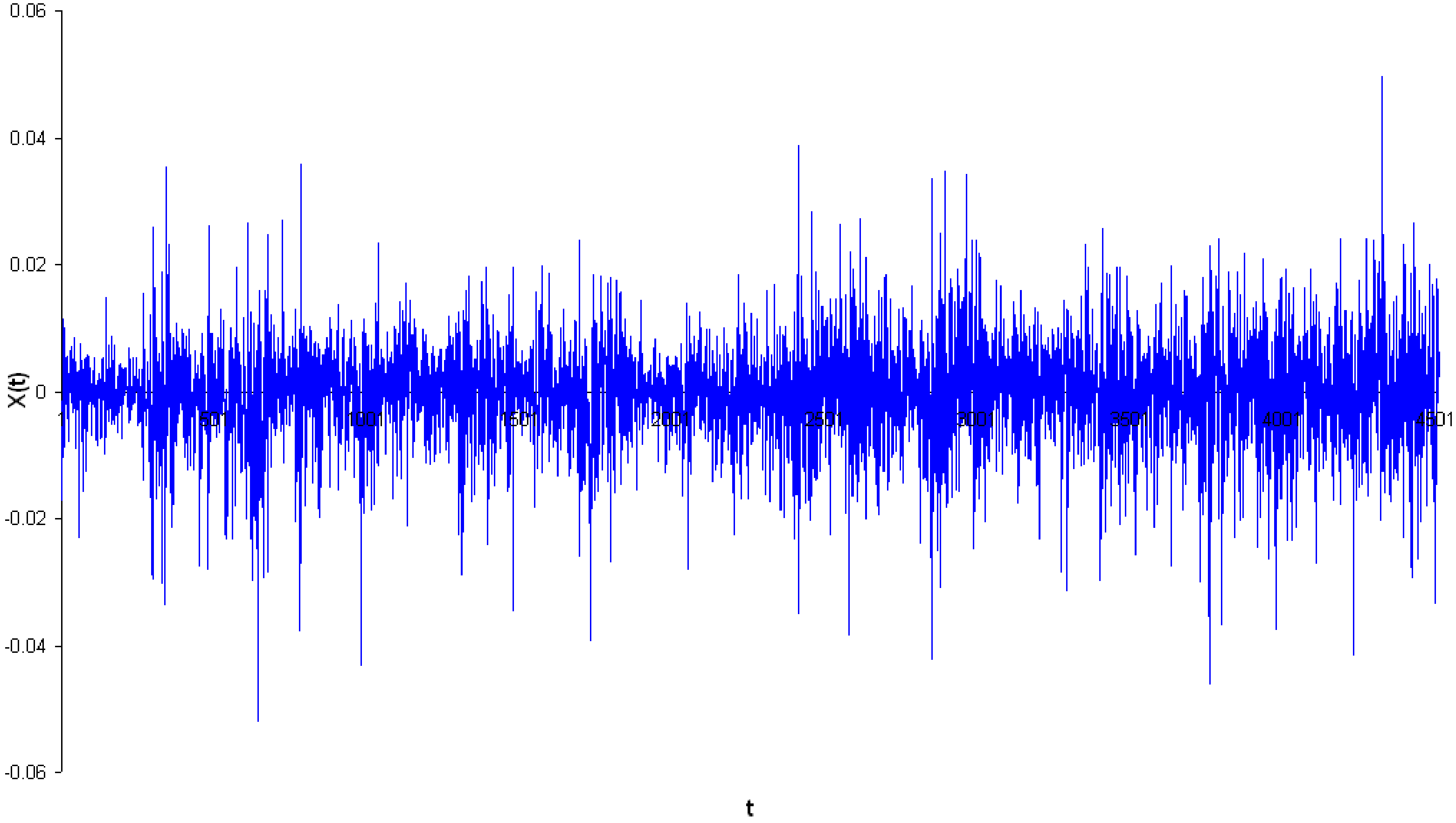


Price (exchange rate) for JY

Log returns for DM



Log returns for JY



Empirical autocorrelations

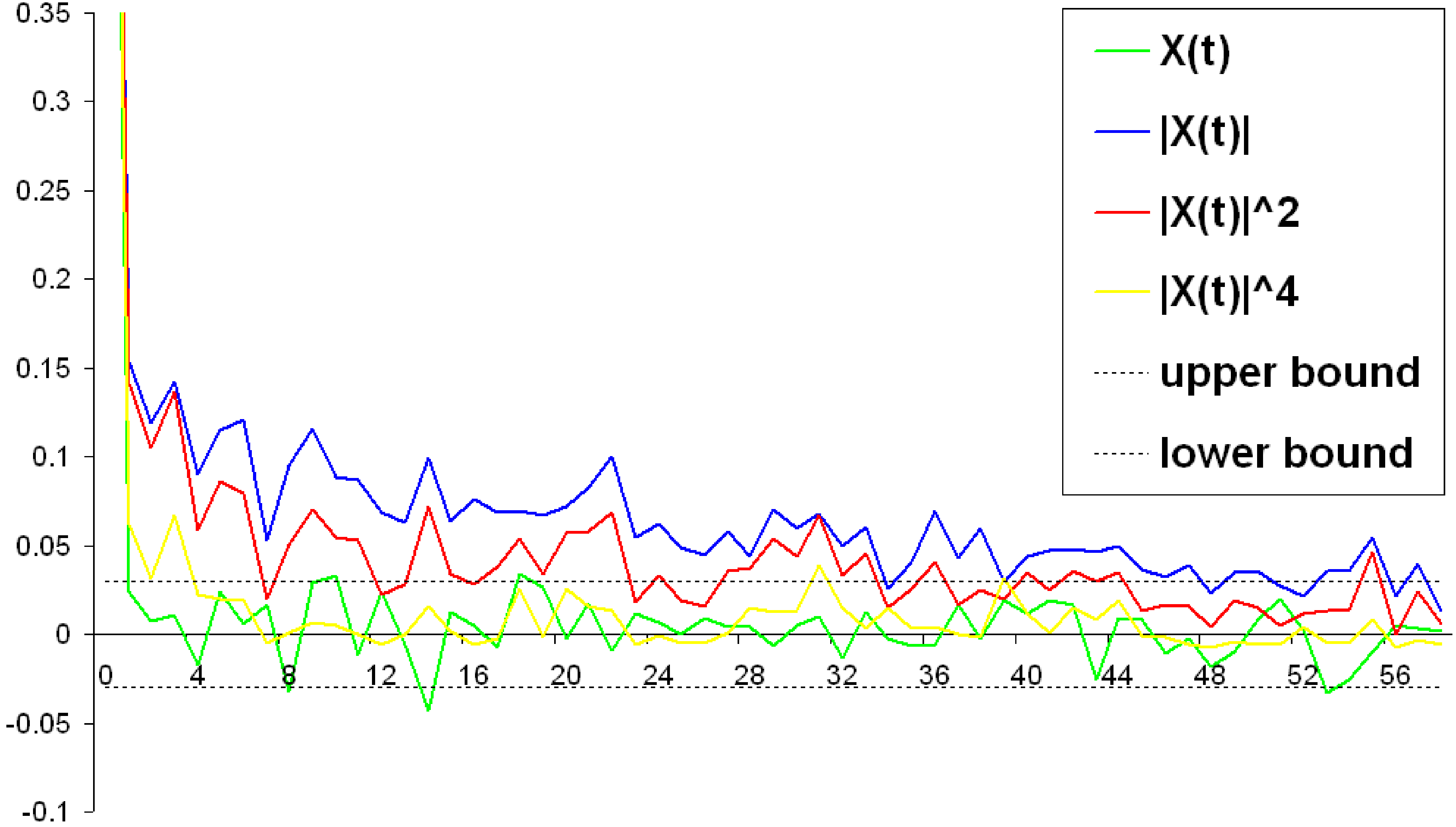
Based on the data set $X_t, t = 1, 2, \ldots, N-k$ using

$$\hat{r}_N(k) = \frac{1}{N} \sum_{t=1}^{N-k} (X_t - \bar{X}_N)(X_{t+k} - \bar{X}_N)$$

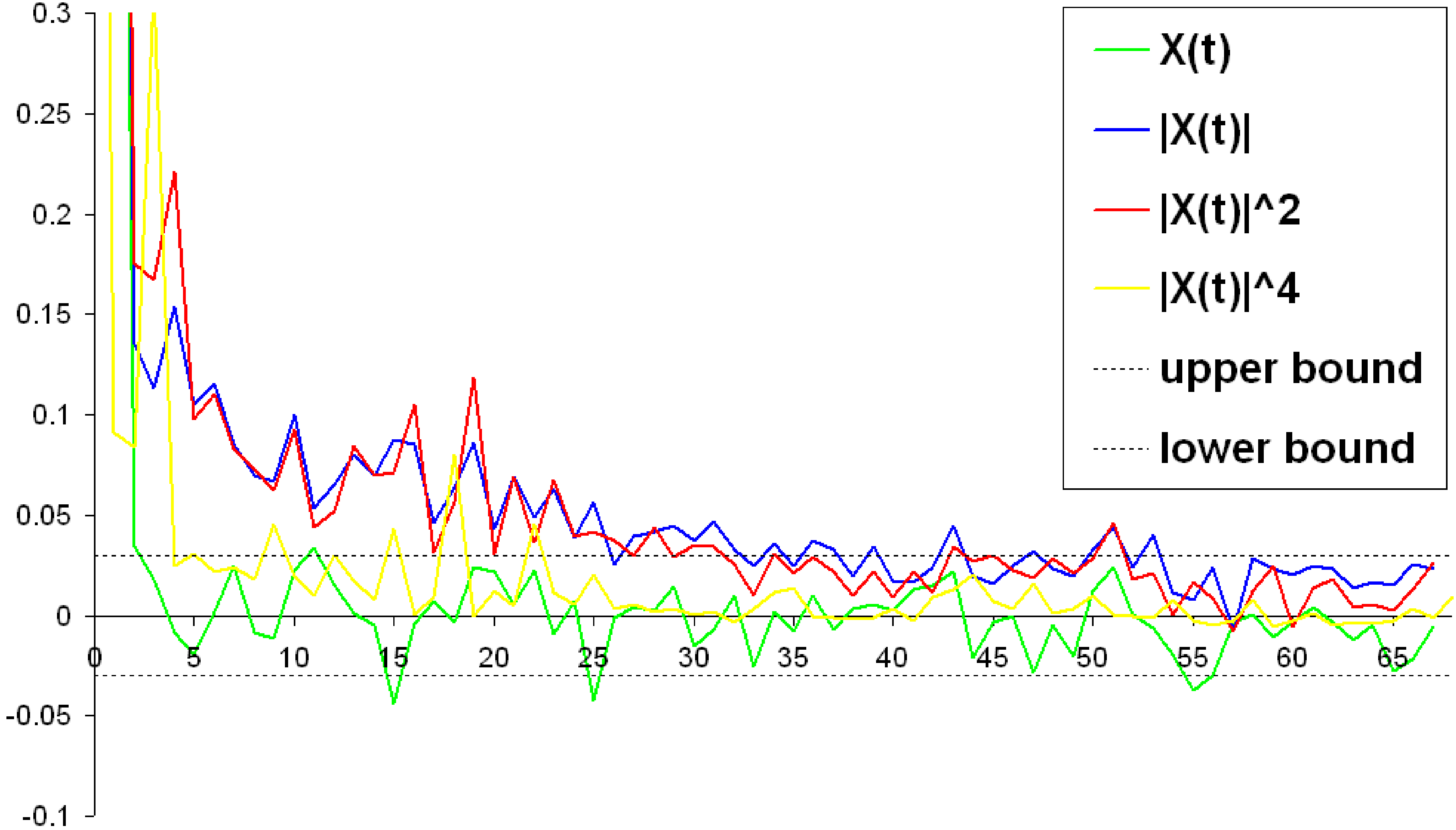
where k is the lag and $\bar{X}_N = \frac{1}{N} \sum_{t=1}^{N} X_t$, and the sample autocorrelations are appropriately normalized

$$\hat{\rho}_N(k) = \frac{\hat{r}_N(k)}{\hat{r}_N(0)}$$

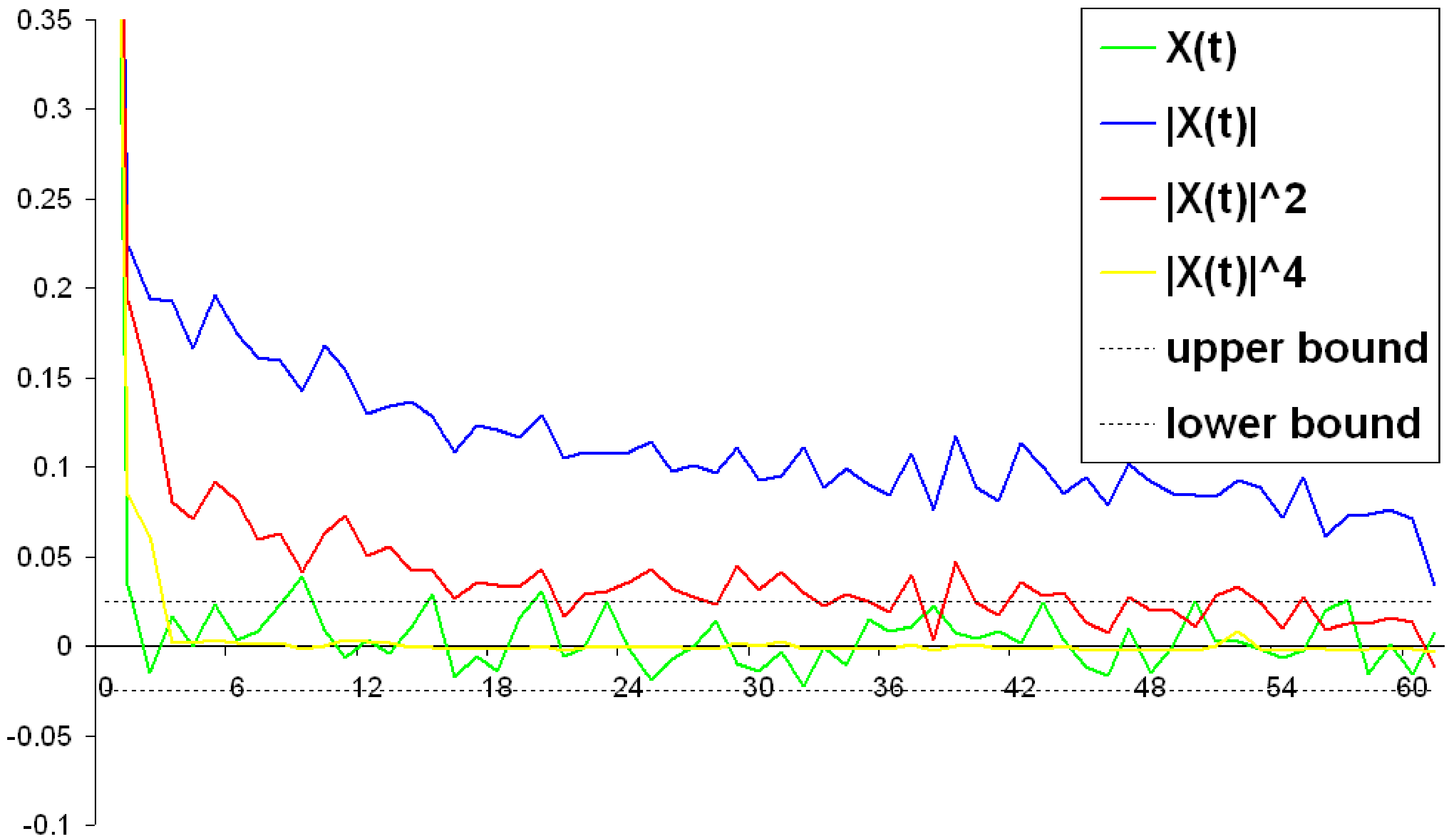
Autocorrelations for JY



Autocorrelations for GBP



Autocorrelations for DM



Some alternatives to GBM

- Use Lévy processes (independent increments, cadlag sample paths, continuous in probability, homogeneous if stationary increments) instead of Brownian motion in GBM model (Eberlein and Raible (1999))
- Mandelbrot (1997) proposed to model $X(t) = B_H(\theta(t))$, where B_H is fractional Brownian motion, that is zero mean Gaussian process with covariance $\frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}]$, and θ is a positive stochastic process independent of B_H

Alternatives to GBM - Cont'd

- Fractal activity time GBM (FATGBM, Heyde (1999)):

$$\log P_t = \log P_0 + \mu t + \theta T_t + \sigma B(T_t),$$

  where $\mu \in R$, $\sigma > 0$, and $\theta \in R$.
  The process $\{T_t\}$ is positive, nondecreasing, and has stationary (but not independent) increments $\tau_t = T_t - T_{t-1}$, and $T_0 = 0$.
- Use Lévy processes to model the activity time T_t (Madan, Carr, and Chang (1998))
- Use $T_t = \int_0^t k(t,s)dL(s)$, where L is a strictly increasing Lévy process, and k is a deterministic Volterra type kernel ($k(t,s) = 0$ when $s > t \geq 0$) (Bender and Marquardt (2009)

Fractal activity time

- The process $\{T_t\}$ has an attractive interpretation of information flow or trading volume (Howison and Lamper (2001))
- The more information is released to the market, or the more 'frenzied' trading becomes, the faster the activity time flows
- If $T_t = t$, then FATGBM becomes classical Black-Scholes model, and $\log P_t$ is normal for any $t \geq 0$

Moments of log returns

$$X_t = \log P_t - \log P_{t-1} =^{\mathcal{D}} \mu + \theta\tau_t + \sigma\sqrt{\tau_t}B(1),$$

where $=^{\mathcal{D}}$ denotes equality in distribution. This gives

$$EX_t = \mu + \theta M_1, \; E(X_t - EX_t)^2 = \sigma^2 M_1 + \theta^2 M_2,$$

$$E(X_t - EX_t)^3 = 3\theta\sigma^2 M_2 + \theta^3 M_3,$$

$$E(X_t - EX_t)^4 = 3\sigma^4(M_2 + (E\tau_t)^2) + 6\sigma^2\theta^2(E\tau_t M_2 + M_3) + \theta^4 M_4,$$

where $M_1 = E\tau_t$, $M_i = E(\tau_t - E\tau_t)^i$, $i = 2, 3, 4$.

Skewness and excess kurtosis

Skewness:

$$\gamma_1 = \frac{3\theta\sigma^2 M_2 + \theta^3 M_3}{(\sigma^2 M_1 + \theta^2 M_2)^{3/2}}.$$

Excess kurtosis:

$$\gamma_2 = \frac{3\sigma^4 M_2 + 6\theta^2\sigma^2 M_3 + \theta^4(M_4 - M_2^2)}{(\sigma^2 M_1 + \theta^2 M_2)^2}.$$

The case of symmetric log returns corresponds to when $\theta = 0$, while when $\theta \neq 0$, the returns are skewed.

Covariances

Covariance of log returns:

$$cov(X_t, X_{t+k}) = \theta^2 cov(\tau_t, \tau_{t+k}),$$

Covariance of squared returns:

$$cov(X_t^2, X_{t+k}^2) = (\sigma^4 + 4\theta^2\mu^2 + 4\theta\mu\sigma^2) cov(\tau_t, \tau_{t+k}) + \theta^4 cov(\tau_t^2, \tau_{t+k}^2) +$$

$$(\theta^2\sigma^2 + 2\theta^3\mu)(cov(\tau_t^2, \tau_{t+k}) + cov(\tau_t, \tau_{t+k}^2)).$$

In the symmetric case,

$$cov(X_t, X_{t+k}) = 0,$$

$$cov(X_t^2, X_{t+k}^2) = \sigma^4 cov(\tau_t, \tau_{t+k})$$

.

Covariance of absolute returns

For $\mu = \theta = 0$ we also have

$$cov(|X_t|, |X_{t+k}|) = \frac{2}{\pi}\sigma^2 cov(\sqrt{\tau}_t, \sqrt{\tau_{t+k}}).$$

Conditional heteroscedasticity

The log return process $\{X_t\}$ has time dependent conditional variance. Define the σ-algebra of information available up to time t:

$$\mathcal{F}_t = \sigma(\{B(u), u \leq T_t\}, \{T_u, u \leq t\}).$$

Then

$$Var(X_t|\mathcal{F}_{t-1}) = E(X_t^2|\mathcal{F}_{t-1}) - E(X_t|\mathcal{F}_{t-1})^2 =$$

$$\theta^2 Var(\tau_t|\mathcal{F}_{t-1}) + \sigma^2 E(\tau_t|\mathcal{F}_{t-1}).$$

In the symmetric case, $Var(X_t|\mathcal{F}_{t-1}) = \sigma^2 E(\tau_t|\mathcal{F}_{t-1})$.
It is natural to interpret $\sigma\sqrt{\tau_t}$ as the volatility at time t, and $\{\sigma\sqrt{\tau_t}\}$ as stochastic volatility process.

Distribution theory

- Since $X_t =^{\mathcal{D}} \mu + \theta\tau_t + \sigma\sqrt{\tau_t}B(1)$, the conditional distribution of X_t given $\tau_t = V$ is normal with mean $\mu + \theta V$ and variance $\sigma^2 V$.
- The conditional distributions of X_t given $\tau_t = V$ are normal mixed or generalized hyperbolic distributions (Barndorff-Nielsen, Kent, and Sørensen (1982))

Gamma distribution of τ_t

If τ_t is distributed as $\Gamma(\alpha, \beta)$, where $\alpha, \beta > 0$, its density is

$$f_\Gamma(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \; x > 0.$$

The characteristic function of τ_t is

$$\phi_\Gamma(u) = \left(1 - \frac{iu}{\beta}\right)^{-\alpha}.$$

VG distribution of X_t

When τ_t has Gamma distribution, the distribution of X_t is Variance Gamma with density

$$f_{VG}(x) = \sqrt{\frac{2}{\pi}} \frac{\beta^\alpha e^{\frac{(x-\mu)\theta}{\sigma^2}}}{\sigma\Gamma(\alpha)} \Big(\frac{|x-\mu|}{\sqrt{\theta^2 + 2\beta\sigma^2}} \Big)^{\alpha - \frac{1}{2}} \times$$

$$K_{\alpha-\frac{1}{2}} \Big(\frac{|x-\mu|\sqrt{\theta^2 + 2\beta\sigma^2}}{\sigma^2} \Big),$$

where

$$K_\eta(\omega) = \frac{1}{2} \int_0^\infty z^{\eta-1} e^{-\omega/2(z+1/z)} dz, \ \omega > 0$$

is modified Bessel function of the third kind, or McDonalds function.

VG distribution of X_t - Cont'd

The characteristic function of X_t in the VG model is

$$\phi_{VG}(u) = e^{i\mu u}\left(1 - \frac{i\theta u}{\beta} + \frac{1}{2\beta}\sigma^2 u^2\right)^{-\alpha}.$$

We will use the notation $VG(\mu, \theta, \sigma^2, \alpha, \beta)$ for the VG model.

Inverse Gamma distribution of τ_t

Consider τ_t with inverse Gamma $R\Gamma(\delta, \epsilon)$, $\delta, \epsilon > 0$ marginal distribution (also called reciprocal Gamma).
The density is

$$f_{R\Gamma}(x) = \frac{\epsilon^{\delta}}{\Gamma(\delta)} x^{-\delta-1} e^{-\epsilon/x}, \; x > 0.$$

Moments of order k exist when $\delta > k$. For example, when $\delta \leq 2$, $Var(\tau_t) = \infty$.

Student's t distribution of X_t

When τ_t has inverse Gamma distribution, the distribution of X_t is Student's t with density

$$f_{St}(x) = \sqrt{\frac{2}{\pi}} \frac{(\delta-1)^{\delta} e^{\frac{(x-\mu)\theta}{\sigma^2}}}{\sigma\Gamma(\delta)} \left(\frac{\theta^2}{2\epsilon\sigma^2 + (x-\mu)^2}\right)^{\frac{\delta+1/2}{2}} \times$$

$$K_{\delta+1/2}\left(\frac{|\theta|\sqrt{2\epsilon\sigma^2 + (x-\mu)^2}}{\sigma^2}\right).$$

The above expressions of densities were given by Sørensen and Bibby (2003).
The characteristic function is

$$\phi_{St}(u) = \frac{2^{1-\delta/2} e^{i\mu u}}{\Gamma(\delta)} (\epsilon(\sigma^2 u^2 - 2i\theta u))^{\delta/2} K_{\delta}(\sqrt{2\epsilon(\sigma^2 u^2 - 2i\theta u)}).$$

Inverse Gaussian distribution of τ_t

Consider τ_t that has an inverse Gaussian distribution $IG(\delta, \gamma)$ with the density

$$f_{IG}(x) = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi x^3}} e^{-\frac{1}{2}(\delta^2/x+\gamma^2 x)},\ x > 0, \delta > 0, \gamma \geq 0.$$

The characteristic function of τ_t is

$$\phi_{IG}(x) = exp\Big\{\frac{\delta}{\gamma}\Big(1 - \sqrt{1 - \frac{2iu}{\gamma^2}}\Big)\Big\}.$$

NIG distribution of X_t

When τ_t has $IG(\delta,\gamma)$ marginal distribution, then X_t has normal inverse Gaussian ($NIG(\alpha,\beta,\mu,\sigma,\delta)$) distribution, where $\beta=\theta/\sigma^2$, and $\alpha=\frac{\sqrt{\theta^2+\sigma^2\gamma^2}}{\sigma^2}$. The density of X_t is

$$f_{NIG}(x)=\frac{\sqrt{\theta^2+\gamma^2\sigma^2}}{\sigma^2\pi}\exp\{\delta\gamma+\frac{\theta^2}{\sigma}(x-\mu)\}\times$$

$$\frac{\sigma\delta}{\sqrt{\sigma^2\delta^2+(x-\mu)^2}}K_1\Big(\frac{\sqrt{(\theta^2+\gamma^2\sigma^2)(\sigma^2\delta^2+(x-\mu)^2)}}{\sigma^2}\Big).$$

Tail behavior

- If X_t has VG distribution, then as $x \to \infty$

$$P(|X_t| > x) \sim const(\alpha, \beta, \sigma) x^{\alpha-1} e^{-x\sqrt{2\beta/\sigma^2}}$$

- If X_t has NIG distribution, then as $x \to \infty$

$$P(|X_t| > x) \sim const(\alpha, \delta, \sigma) x^{-3/2} e^{-\alpha x}$$

- When X_t has Student distribution and $\mu = \theta = 0$ then

$$P(|X_t| > x) \sim const(\epsilon, \delta, \sigma) x^{-2\delta}$$

Here $f(x) \sim g(x)$ means that $lim_{x\to\infty} f(x)/g(x) = 1$.

GIG distribution of τ_t

The density of generalized inverse Gaussian (GIG) distribution $GIG(\alpha, \beta, \gamma)$ distribution is given by

$$f_{GIG}(x) = \frac{\left(\frac{\gamma}{\beta}\right)^{\alpha/2}}{2K_\alpha(\sqrt{\beta\gamma})} x^{\alpha-1} e^{-\frac{1}{2}\left(\frac{\beta}{x}+\gamma x\right)}, \; x > 0$$

The distributions considered for τ_t, Gamma, inverse Gamma, and inverse Gaussian, belong to GIG class (some as a limiting case when GIG parameter values are set to be 0)

GH distribution of X_t

When τ_t has GIG distribution, the distribution of X_t belongs to the class of generalized hyperbolic (GH) distributions. The density is

$$f_{GH}(x) = \Big(\frac{\gamma}{\beta}\Big)^{\alpha/2} \Big(\frac{\beta\alpha^2 + (x-\mu)^2}{\gamma\sigma^2 + \theta^2}\Big)^{\alpha/2-1/4} \times$$

$$K_{1-\alpha/2}\Big(\sqrt{\Big(\gamma + \frac{\theta^2}{\sigma^2}\Big)\Big(\beta + \frac{(x-\mu)^2}{\sigma^2}\Big)}\Big) \frac{e^{\frac{(x-\mu)\theta}{\sigma^2}}}{\sqrt{2\pi\sigma^2} K_\alpha(\sqrt{\gamma\beta})}$$

The characteristic function is

$$\phi_{GH}(u) = \frac{K_\alpha(\sqrt{\beta(\gamma - 2i\theta u + \sigma^2 u^2)})}{K_\alpha(\gamma\beta)} \Big(\frac{\gamma}{\gamma - 2i\theta u + \sigma^2 u^2}\Big)^{\alpha/2} e^{i\mu u}.$$

Other constructions of activity time

Heyde and Leonenko (2005) introduced the following construction:
Let $\eta_1(t), \ldots, \eta_\nu(t)$, $\nu \geq 1$ be independent copies of stationary Gaussian process $\eta(t)$ with $E\eta(t) = 0$, $E\eta^2(t) = 1$ and monotone correlation function $E\eta(t)\eta(t+s) = \rho_\eta(s)$, $t, s \geq 0$.
Consider the chi-square process

$$\chi_\nu^2(t) = \frac{1}{2}(\eta_1^2(t) + \ldots + \eta_\nu^2(t)).$$

Gamma process via chi-square

Take $\tau_t = \frac{2}{\nu}\chi^2_\nu(t)$ so the distribution of τ_t is $\Gamma(\alpha, \alpha)$ for $\alpha = \nu/2$. For $t = 1, 2, \ldots$ the activity time

$$T_t = \sum_{i=1}^{t} \tau_i = \frac{2}{\nu} \sum_{i=1}^{t} \chi^2_\nu(i).$$

This construction is considered by Finlay and Seneta (2006).

Drawback: α is an integer multiplier of $1/2$.

Advantage: flexible correlation structure $corr(\chi^2_\nu(t), \chi^2_\nu(t+s)) = \rho^2_\eta(s)$.

Inverse Gamma process via chi-square

Consider $\tau_t = [\frac{2}{\nu}\chi^2_\nu(t)]^{-1}$ with marginal distribution $R\Gamma(\nu/2, \nu/2)$ (Heyde and Leonenko (2005)). The covariance structure:

$$cov(\tau_t, \tau_{t+s}) = \sum_{k=1}^{\infty} C_k^2(\nu)\rho_\eta^{2k}(s), \;\; \nu > 4,$$

where C_k are coefficients from the expansion of $G(x) = \frac{\nu}{2x}$ using Laguerre polynomials.

Expansion using Laguerre polynomials

The density of χ^2_ν is $f_\Gamma(\nu/2, 1)$. Consider
$L_2((0, \infty), f_\Gamma(\nu/2, 1))$.
Complete orthogonal system of functions is

$$e_k(u) = L_k^{\nu/2-1}(u)\left\{k! \frac{\Gamma(\nu/2)}{\Gamma(\nu/2 + k)}\right\}^{1/2},$$

where

$$L_k^\beta(u) = \frac{1}{k!} u^{-\beta} e^u \frac{d^k}{du^k}\{u^{\beta+k} e^{-u}\}$$

are generalized Laguerre polynomials of index β, $k \geq 0$.

Expansion - Cont'd

Note that $\tau_t = G(\chi^2_\nu(t))$ with $G(x) = \frac{\nu}{2x} \in L_2((0,\infty), f_\Gamma(\nu/2,1))$. This function can be expanded

$$G(x) = \sum_{k=1}^{\infty} C_k(\nu) e_k(x),$$

where

$$C_k(\nu) = \frac{\nu}{2} \int_0^\infty \frac{f_\Gamma(\nu/2,1)(x) e_k(x) dx}{x}.$$

Chi-square construction for $R\Gamma$

- Flexible correlation structure: long- or short- range dependence possible with different choices of ρ_η
- The distribution of τ_t is $R\Gamma(\nu/2, \nu/2)$, where ν is an integer
- Correlation structure is defined when $\nu > 4$

Key ingredients for the construction

- We consider the construction of τ_t with Gamma or IG marginals using Ornstein-Uhlenbeck (OU) processes
- Gamma and IG distributions are self-decomposable: for any $c \in (0,1)$ there exits r.v. X_c independent of X such that $X =^{\mathcal{D}} cX + X_c$
- Gamma and IG distributions have additivity property in one of the parameters
- The variances of Gamma and IG distributions are proportional to the parameter in which the additivity property holds

Why not other distributions

- Inverse Gamma distribution (leading to Student's t distribution of the returns), does not have these properties
- For inverse Gamma distribution of τ_t, construction via chi-square processes is available
- Construction via chi-square processes also works for Gamma distribution of τ_t
- In construction using OU processes, we do not need any of the parameters to be integers

Construction using OU processes

- Idea is due to Barndorff-Nielsen (1998), further developed in Barndorff-Nielsen and Shephard (2001) for continuous time stochastic volatility models
- Superpositions investigated by Barndorff-Nielsen (2001), Barndorff-Nielsen and Leonenko (2005), Leonenko and Tauffer (2005)
- OU process is stationary solution of the stochastic differential equation

$$dy(t) = -\lambda y(t) + dZ(\lambda t), \quad t \geq 0, \tag{1}$$

where $Z(t), t \geq 0$ is a non-decreasing Lévy process, and $\lambda > 0$

OU processes

Theorem 1. There exists a stationary process $y(t), t \geq 0$, which has marginal $\Gamma(\alpha, \beta)$ or $IG(\delta, \gamma)$ distribution and satisfies equation (1). The process y has all moments, and the correlation function of y is given by $r_y(h) = corr(y(t), y(t+h)) = e^{-\lambda h}$, $h \geq 0$.

This theorem is a special case of a more general result (Sato (1999)). The unique strong stationary solution of equation (1) exists if $\int_2^\infty \log x \rho(dx) < \infty$, where $\rho(\cdot)$ is Lévy measure of $Z(1)$.
The solution is given by

$$y(t) = e^{-\lambda t} y(0) + \int_0^t e^{-\lambda(t-s)} dZ(\lambda s).$$

OU processes Cont'd

- The law of Z is determined uniquely by that of y
- Lévy-Khinchin representation:

$$\kappa_y(u) = \log E e^{iuy} = iua - \int_0^\infty (e^{iux} - 1) Q(dx), \;\; u \in R,$$

where $\int_0^\infty (1 \wedge x) Q(dx) < \infty$, and $Q(-\infty, 0) = 0$

- When y is self-decomposable $Q(dx) = \frac{q(x)}{x} dx$, with canonical function q decreasing on $(0, \infty)$
- The cumulant function of $Z(1)$ is related to that of y:

$$\kappa_{Z(1)}(u) = \log E e^{iuZ(1)} = u \frac{\partial}{\partial u} \kappa_{y(t)}(u).$$

Gamma OU process

When y has $\Gamma(\alpha, \beta)$ marginal distribution,

$$q_\Gamma(x) = \alpha e^{-\beta x} 1_{\{x>0\}},$$

and Lévy process $Z(t)$ is a compound Poisson process

$$Z(t) = \sum_{n=1}^{N(t)} Z_n,$$

where $N(t)$ is a Poisson process with intensity α, and Z_n are independent identically distributed $\Gamma(1, \beta)$ random variables.

IG OU process

In the IG case, the canonical function is

$$q_{IG}(x) = \frac{\delta x^{-1/2}}{\sqrt{2\pi}} e^{-\gamma^2 x/2} 1_{\{x>0\}}.$$

$Z(t) = Z_1(t) + Z_2(t)$, where Z_1 and Z_2 are independent.
Z_1 is a Lévy process with inverse Gaussian marginals,
Z_2 is a compound Poisson process

$$Z_2(t) = \frac{1}{\gamma^2} \sum_{k=1}^{N(t)} W_n^2,$$

where $N(t)$ is Poisson process with intensity $\delta\gamma/2$, and $W_1, W_2, \ldots$ are independent N(0,1).

Distributions of OU processes

- It is important to specify the distribution of $T_t = \sum_{i=1}^{t} \tau_i$, when τ is OU type process
- Distribution of T_t can be obtained from distribution of τ_1 and transition probability $P(t, B; x)$ from x to B in time t:

$$P\Big(\sum_{i=1}^{t} \tau_i \leq x\Big) = \int_{x_1+x_2+\ldots+x_t \leq x} f(x_1)dx_1 P(1, dx_2; x_1)$$

$$P(1, dx_3; x_2) \ldots P(1, dx_t; x_{t-1}),$$

where $f(\cdot)$ is either $\Gamma(\alpha, \beta)$ or IG (δ, γ) density for VG and NIG models respectively

Transition probability for Gamma process

It was shown in Zhang, Zhang and Sun (2006) that temporally homogeneous transition function $P(t, y; x, \lambda, \alpha, \beta)$ from x to $y(\cdot) \leq y$ after time interval t is

$P(t, y; x, \lambda, \alpha, \beta) = 0$, if $y < e^{-\lambda t} x$,

$P(t, y; x, \lambda, \alpha, \beta) = e^{-\lambda \alpha t}$, if $y = e^{-\lambda t} x$,

$$P(t, y; x, \lambda, \alpha, \beta) = e^{-\lambda \alpha t} + \sum_{n=1}^{\infty} \frac{(\lambda \alpha t)^n e^{-\lambda \alpha t}}{n!} \int_0^{y - e^{-\lambda t} x} f_n(u) du,$$

if $y > e^{-\lambda t} x$.

Transition probability Cont'd

The sequence of functions in the transition probability formula is defined by

$$f(w) = \frac{e^{-\beta w} - e^{-\beta w e^{\lambda t}}}{\lambda t w}, w > 0,$$

and $f(w) = 0, w \leq 0.$

$$f_1(x) = f(x)$$

$$f_n(x) = \int_0^\infty f(y) f_{n-1}(x - y) dy, \; n \geq 2.$$

Transition probability for IG process

Using representation of Z and results from Zhang and Zhang (2008), the transition probability of inverse Gaussian OU process can be expressed as follows:

$P(t, y; x, \lambda, \gamma, \delta) =$

$$\sum_{n=1}^{\infty} \frac{exp\{-\delta\gamma t(1 - e^{-1/2\lambda t})\}(\delta\gamma t(1 - e^{-1/2\lambda t}))^n}{n!} \int_0^{y - e^{-\lambda t} x} f_n(u) du,$$

for $y > e^{-\lambda t} x$,

$P(t, x; y, \lambda, \gamma, \delta) = 0$, if $y \leq e^{-\lambda t} x$.

Transition probability Cont'd

Function f_1 is the inverse Gaussian density with parameters $(\delta(1-e^{-1/2\lambda t}), \gamma)$, and

$$f_n(u) = \int_0^\infty f_{n-1}(u-x) f(x) dx, n \geq 2,$$

where

$$f(u) = \frac{e^{-1/2\gamma^2 u} - e^{-1/2\gamma^2 u e^{\lambda t}}}{\sqrt{2\pi u^3}\gamma(e^{1/2\lambda t} - 1)}, \quad u > 0.$$

Sup-OU processes

- We use discrete version of superposition introduced by Barndorff-Nielsen (1998)
- Let $\tau^{(k)}(t),\ k \geq 1$ be the sequence of independent processes such that each $\tau^{(k)}(t)$ is solution of the equation

$$d\tau^{(k)}(t) = -\lambda^{(k)}\tau^{(k)}(t) + dZ^{(k)}(\lambda^{(k)}t), \quad t \geq 0,$$

  in which Lévy processes $Z^{(k)}$ are independent and are such that the distribution of $\tau^{(k)}$ is either $\Gamma(\alpha_k, \beta)$ or $IG(\delta_k, \gamma)$

- Finite superposition: $\tau_t^m = \sum_{k=1}^{m} \tau^{(k)}(t)$

Infinite superpositions

- Infinite superposition: $\tau_t^\infty = \sum_{k=1}^{\infty} \tau^{(k)}(t)$
- Well-defined in the sense of mean-square or almost-sure convergence provided that $\sum_{k=1}^{\infty} \alpha_k < \infty$ in case of the VG model, and $\sum_{k=1}^{\infty} \delta_k < \infty$ in case of NIG model
- For VG model, the marginal distribution of τ_t^∞ is $\Gamma(\sum_{k=1}^{\infty} \alpha_k, \beta)$ and for NIG model, the marginal distribution of τ_t^∞ is $IG(\sum_{k=1}^{\infty} \delta_k, \gamma)$
- For finite superpositions, sums go to m instead of ∞

Covariance functions

- Finite superposition:

$$R_{\tau^m}(t) = cov(\tau_s^m, \tau_{t+s}^m) = \sum_{k=1}^{m} Var(\tau^{(k)}(t))e^{-\lambda^{(k)}t}$$

- For the VG model, $Var(\tau^{(k)}) = \alpha_k/\beta^2$, and for NIG model $Var(\tau^{(k)}) = \delta_k/\gamma^3$
- Infinite superposition: summation to ∞ instead of m
- Infinite superposition: let $0 < H < 1$, choose $\alpha_k = k^{-(1+2(1-H))}$ in case of VG model, and choose $\delta_k = k^{-(1+2(1-H))}$ in case of NIG model
- Choose $\lambda^{(k)} = 1/k$

Covariances for infinite superposition

With chosen parameters

$$R_{\tau^\infty}(t) = c \sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}} e^{-t/k}.$$

The constant c equals $\frac{1}{\beta^2}$ in VG model, and $\frac{1}{\gamma^3}$ in NIG model.

Lemma. For infinite superposition, the covariance function of τ^∞ can be written as $R_{\tau^\infty}(t) = \frac{L(t)}{t^{2(1-H)}}$,
where L is a slowly varying at infinity function, bounded on every bounded interval.

Remark. If $1/2 < H < 1$, the process τ_t^∞ has long range dependence.

Asymptotic self-similarity

- Finite superposition notation $T_t^m = \sum_{i=1}^t \tau_i^m$
- Infinite superposition notation $T_t^\infty = \sum_{i=1}^t \tau_i^\infty$
- Empirical evidence in support of approximate self-similarity (Heyde (1999), Heyde and Liu (2001))
- Exact self-similarity for increasing T is not possible (Heyde and Leonenko (2005))

Self-similarity

Exact self-similarity: $T_{ct} - ET_{ct} =^{\mathcal{D}} c^H(T_t - ET_t)$, $0 < H < 1$.
Note that $ET_t = tE\tau_1$.

If this were true, then for all $t > 0$, $c > 0$, and $\Delta > 0$
$T_{t+\Delta} - T_t - \Delta E\tau_1 =^{\mathcal{D}} T_\Delta - \Delta E\tau_1 =^{\mathcal{D}} \Delta^H(T_1 - E\tau_1)$.

And therefore
$P(T_{t+\Delta} - T_t < 0) = P(T_1 < E\tau_1 - \Delta^{1-H}) > 0$ if $\Delta < (E\tau_1)^{H-1}$.

Asymptotic self-similarity of T_t

Let $D[0,1]$ be Skorokhod space, and for $t \in [0,1]$ consider random functions $T^m_{[Nt]}$ and $T^\infty_{[Nt]}$.

Theorem 2. For a fixed $m < \infty$ (finite superposition)

$$\frac{1}{c_m N^{1/2}}\left(T^m_{[Nt]} - ET^m_{[Nt]}\right) \to B(t), \quad t \in [0,1],$$

as $N \to \infty$ in the sense of weak convergence in $D[0,1]$. The process $B(t)$ is Brownian motion, and the norming constant c_m is given by

$$c_m = \left(\sum_{k=1}^{m} Var(\tau^{(k)}) \frac{1 - e^{-\lambda^{(k)}}}{1 + e^{-\lambda^{(k)}}}\right)^{1/2},$$ where $Var(\tau^{(k)}) = \alpha_k/\beta^2$

for the VG model, and $Var(\tau^{(k)}) = \delta_k/\gamma^3$ for the NIG model.

Ingredients of proof of Theorem 2

- Each OU process in the finite superposition is β-mixing (absolutely regular) under the condition of existence of unique strong stationary solution of (1) $\int_2^\infty \log x \rho(dx) < \infty$ (Jongbloed et al. (2005))
- Masuda (2004) showed β-mixing with exponential rate under a stronger condition of existence of the absolute moment of order $p > 0$ of the marginal distribution: there exists $a > 0$ such that the mixing coefficient $\beta_y(t) = O(e^{-at})$
- Finite sum of β-mixing processes is also β-mixing
- β-mixing ensures that conditions of Theorem 20.1 Billingsley (1968) are satisfied

β-mixing

β-mixing (absolute regularity) is present when

$$\beta(n) = \sup_{j \geq 0} \beta(\mathcal{F}_0^j, \mathcal{F}_{j+n}^{\infty}) \to 0, \quad n \to \infty,$$

where σ-algebra $\mathcal{F}_i^j$ is generated by $\{y(t), i \leq t \leq j\}$ for $j \geq 0$, $j \geq 0$, and for two σ-algebras $\mathcal{A}$ and $\mathcal{B}$

$$\beta(\mathcal{A}, \mathcal{B}) = \sup \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} |P(A_i \cap B_j) - P(A_i)P(B_j)|$$

where the supremum is taken over all pairs of finite partitions $\{A_1, \ldots A_I\}$ and $\{B_1, \ldots B_J\}$ of Ω such that $A_i \in \mathcal{A}$,and $B_j \in \mathcal{B}$, $i = 1, \ldots, I$, $j = 1, \ldots J$ (Bradley (2005)).

β-mixing - Cont'd

Since OU process y is stationary Markov, it was shown in Davydov (1973) that β-mixing condition becomes

$$\beta_y(t) = \int_0^\infty \pi(dx)||P_t(x,\cdot) - \pi(\cdot)||_{TV} \to 0, \ t \to \infty,$$

where $\pi(\cdot)$ is the initial distribution, and $||\cdot||_{TV}$ is total variation norm.

Asymptotic self-similarity

Theorem 3. For infinite superposition and $1/2 < H < 1$

$$\frac{1}{c_\infty N^H L(N)^{1/2}} \left(T^\infty_{[Nt]} - ET^\infty_{[Nt]} \right) \to B_H(t), \quad t \in [0,1],$$

as $N \to \infty$ in the sense of weak convergence in $D[0,1]$. The process B_H is fractional Brownian motion.

The constant $c_\infty = \frac{\alpha(H)}{H(2H-1)\beta^2}$ for the VG model, and

$c_\infty = \frac{\alpha(H)}{H(2H-1)\gamma^3}$ for the NIG model, where

$\alpha(H) = \sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}}$ is Riemann zeta-function.

Proof ingredients

- Follows from a more general results in Barndorff-Nielsen and Leonenko (2005) and Leonenko and Tauffer (2005)
- Proof is based on a linear process type representation of sup-OU process $\tau_t^{\infty} = \sum_{j=0}^{\infty} a_j \epsilon_{n-j}$, where ϵ_j are independent with the same variance but not identically distributed
- Proof follows from Davydov (1970)

Empirical evidence - Skewness

	n	$\hat{\gamma}_1$	$\sqrt{\frac{n}{6}}\lvert\hat{\gamma}_1\rvert$		$\gamma_1 = 0$
DM	6333	-0.035213296	1.144025741	< 1.96	Retain
FF	6428	0.320116571	10.477808702	> 1.96	Reject
GBP	4510	-0.001525777	0.041831526	< 1.96	Retain
JY	4510	-0.414678054	11.369037463	> 1.96	Reject
CD	1700	-0.093129341	1.567600399	< 1.96	Retain
NTD	1200	-0.265079853	3.748795232	> 1.96	Reject

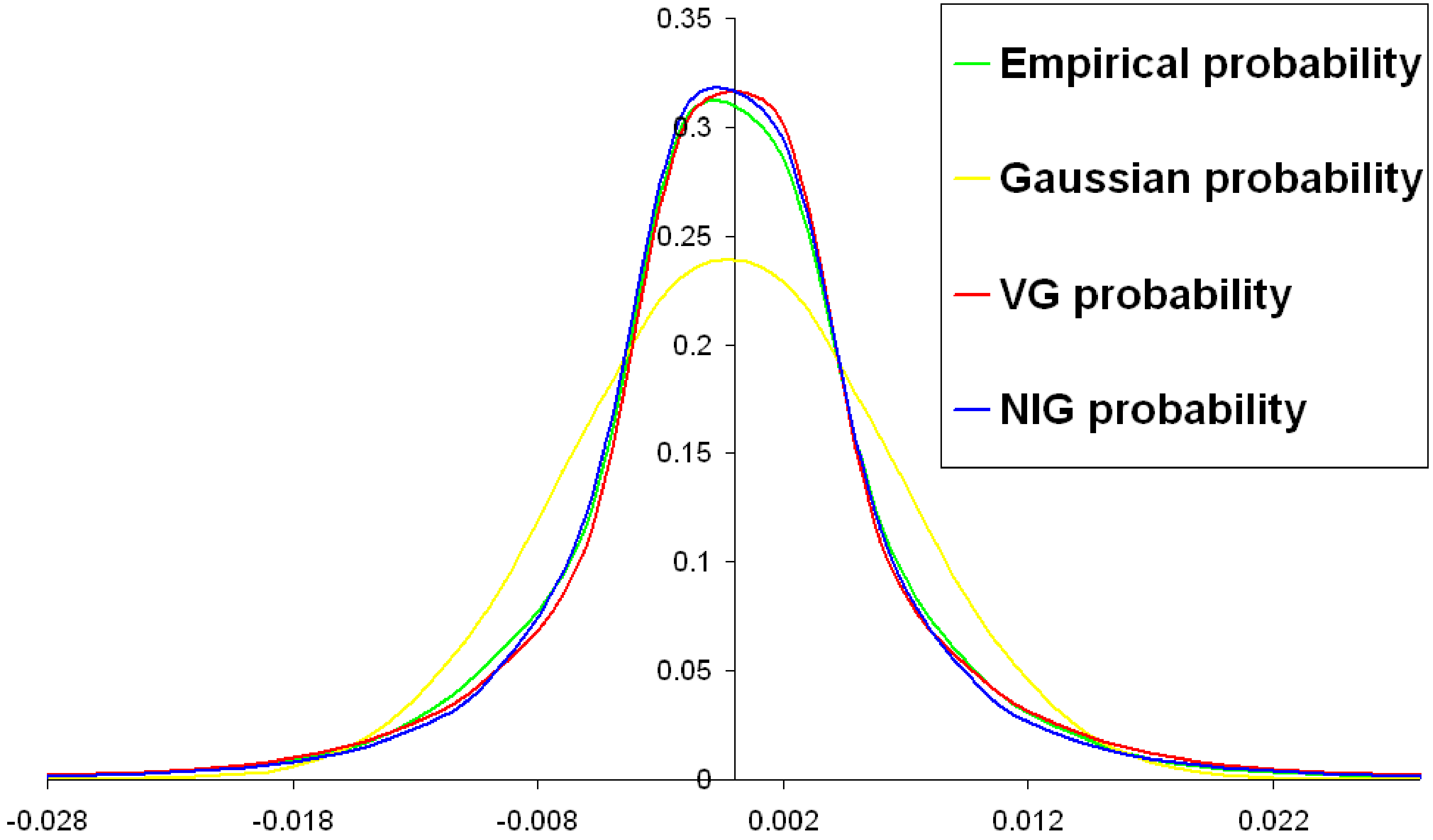
Table 1: Testing the hypothesis $\gamma_1 = 0$ for DM, FF, GBP, JY, CD, and NTD

Empirical evidence -kurtosis

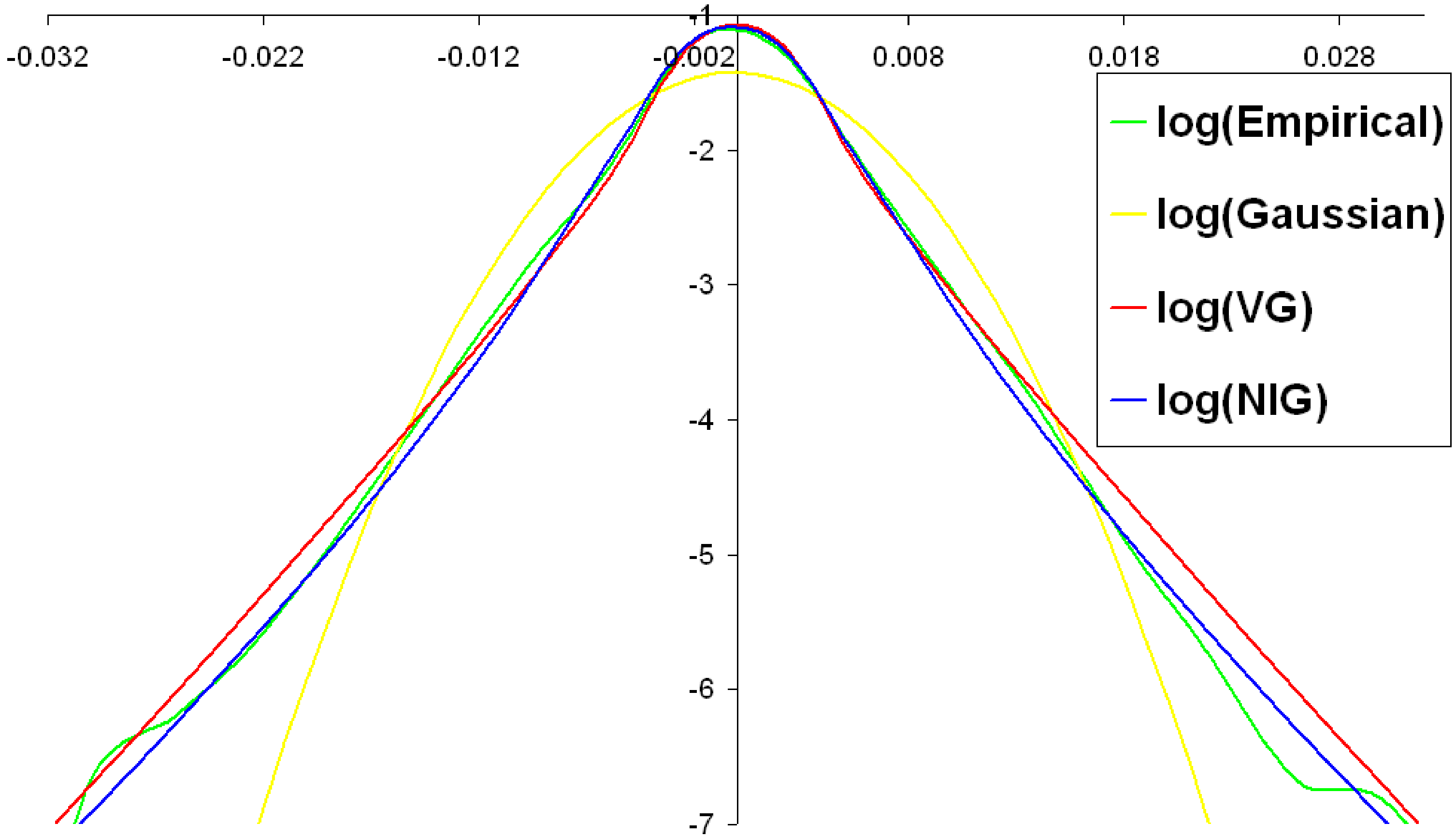
	n	$\hat{\gamma_2}$	$\sqrt{\frac{n}{24}}\lvert\hat{\gamma_2}\rvert$		$\gamma_2 = 0$
DM	6333	5.289831117	85.929231975	> 1.96	Reject
FF	6428	11.08034788	181.336700363	> 1.96	Reject
GBP	4510	2.720264185	37.290115946	> 1.96	Reject
JY	4510	2.611152007	35.794376749	> 1.96	Reject
CD	1700	2.651642996	22.316901277	> 1.96	Reject
NTD	1200	2.106045092	14.891987660	> 1.96	Reject

Table 2: Testing the hypothesis $\gamma_2 = 0$ for DM, FF, GBP, JY, CD, and NTD

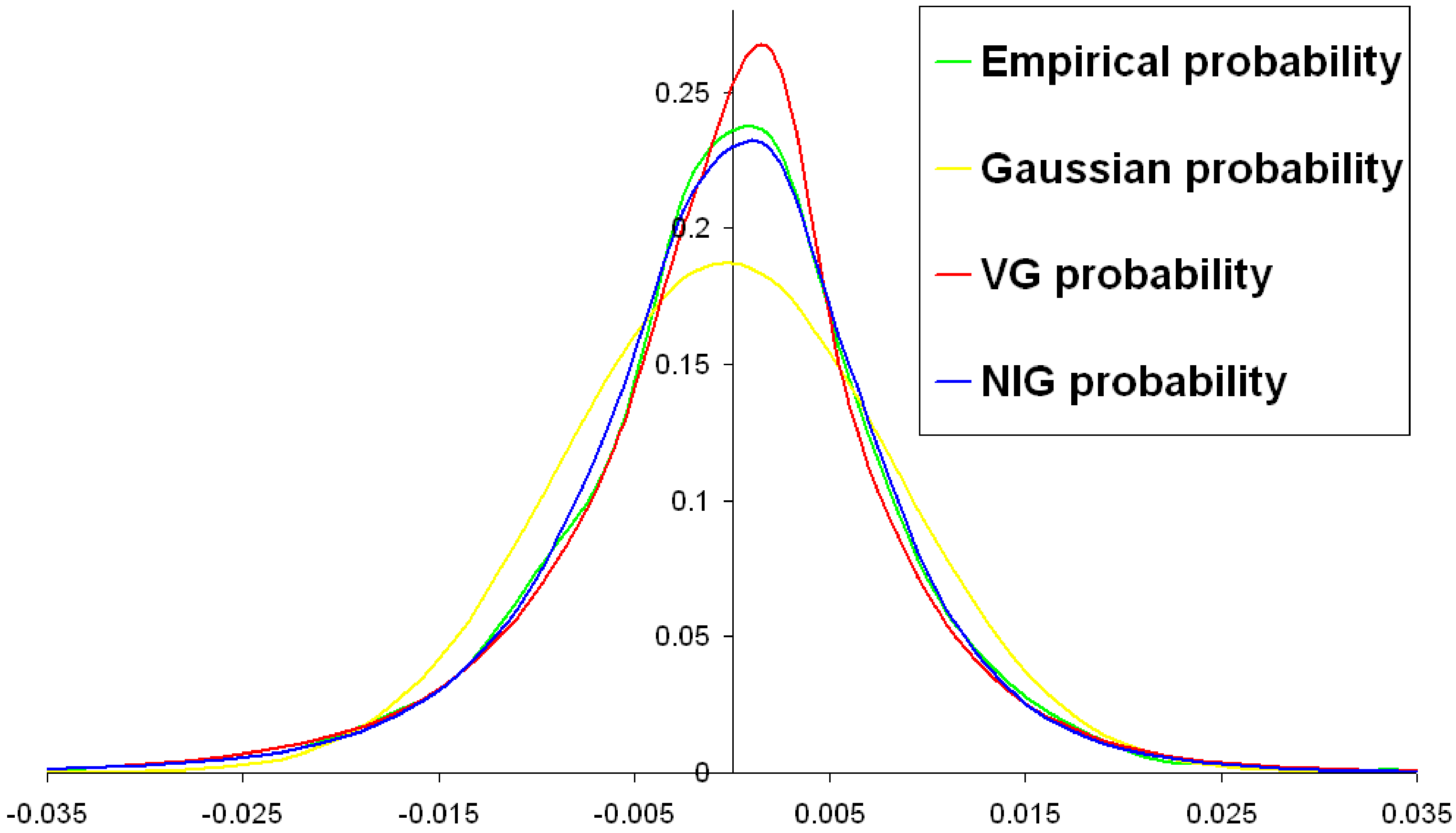
Densities for DM data



Log densities for DM data



Densities for JY data



Log densities for JY data