STT351 Final Exam

Tests of hypotheses.

1. For the of score \( x = \) skin thickness, a 95% confidence interval \([0.822, 0.847]\) has been obtained for the population mean \( \mu \). It is desired to harness this CI to test

\[ H_0: \mu = 0.85 \text{ versus } H_a: \mu < 0.85. \]

1a. Which action, reject the null hypothesis or fail to reject the null hypothesis, is taken based on the test employing this CI?

\[ 95\% \ CI = [0.822, 0.847] \]

\[ H_0 : \mu = 0.85 \]

Reject \( H_0 \)

The 95\% CI falls entirely to the left of \( \mu = 0.85 \) therefore, reject \( H_0 \).

1b. Is this test one-sided or two-sided?

One sided because \( H_a : \mu < 0.85 \)

1c. What is the probability \( \alpha \), of type one error, for this test?

\[ \alpha = P(\text{type I err}) = \frac{1 - 0.95}{2} = 0.025 \]

1d. Ideally, what would be the desired probability of rejecting \( H_0 \) if \( \mu \) is 0.84?

\[ \text{desired probability} = \]

\[ 4 \]
2. Plots of $P(\text{reject } H_0 \mid p)$ are shown for two tests having the same $\alpha$, $H_0$, $H_a$.

2a. Determine $H_0$.  
$H_0 : p = 0.17$

2b. Determine $H_a$.  
$H_a : p > 0.17$

2c. Determine $\alpha$.  
$\alpha \approx 0.1$

2d. For the poorer test, determine $\beta(0.35)$.

$\beta(0.35) = 1 - 0.875 = 0.125$
(x, y) Data.

3. Given data

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>x²</th>
<th>y²</th>
<th>xy</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>6</td>
<td>0</td>
<td>36</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
<td>36</td>
<td>18</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>total</th>
<th>3</th>
<th>12</th>
<th>9</th>
<th>72</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>avg</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>24</td>
<td>6</td>
</tr>
</tbody>
</table>

To receive credit for #3 you must CALCULATE your answers from the GENERAL FORMULAS for obtaining slope and then intercept. Pretend that you cannot see the picture, only the averages above.

3a. Sample standard deviation $s_x$:

$$s_x = \sqrt{\frac{n}{n-1}} \sqrt{\frac{x^2 - \bar{x}^2}{n}}$$

$$s_x = \sqrt{\frac{3}{2-1}} \sqrt{3 - 2} = \sqrt{\frac{3}{2}} \sqrt{2}$$

3b. Estimated slope $\hat{\beta}_1$ of regression of $y$ on $x$.

$$\hat{\beta}_1 = r \frac{\bar{y}}{s_x} = \frac{\bar{xy} - \bar{x} \bar{y}}{\bar{x}^2 - \bar{x}^2} = \frac{6 - (1)(4)}{3 - 1^2} = \frac{6 - 4}{3 - 1} = \frac{2}{2} = 1$$

3c. Estimated y-intercept $\hat{\beta}_0$ of regression of $y$ on $x$.

$$\hat{y} = \bar{x} + \hat{\beta}_0 \rightarrow \hat{\beta}_0 = 6 - 3 = 3$$

3d. Fraction of $\sqrt{y^2 - \bar{y}^2}$ explained by regression of $y$ on $x$.

$$R^2 = \frac{\frac{\bar{x}^2 - \bar{x}^2}{\bar{x}}}{\sqrt{\frac{\bar{x}^2 - \bar{x}^2}{\bar{x}}}} = \frac{6 - (11)(4)}{\sqrt{3 - 2}} = \frac{6 - 4}{\sqrt{15}} = \frac{2}{\sqrt{15}} = \frac{1}{2}$$

3e. 95% t-based CI for $\mu_y$.

$$\sigma = 0.05$$

$$df = n - 2 = 2$$

$$t_{0.05} = 4.303$$

$$95\% \text{ CI: } \mu_y \pm 4.303 \frac{s}{\sqrt{3}} \rightarrow \mu_y \pm 4.303 \frac{\sqrt{2}}{\sqrt{3}}$$

3f. 95% regression-based t-based CI for $\mu_y$ if $\mu_x = 1.5$. 

X

S
4a. Without calculation, plot the least squares line of $y$ on $x$ (per the usual vertical discrepancies). Label it so.

4b. Without calculation, plot the least squares line of $x$ on $y$ (per horizontal discrepancies). Label it so.

4c. Set up the design matrix for a fit of the polynomial model $y = \beta_0 + \beta_1 x + \beta_1 x^2$ to the above data.

$$
\text{design matrix } xx = \begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 3 & 9
\end{pmatrix}
$$

4d. Determine $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\beta}_2$ given

$$
\text{Pseudo-Inverse of } xx = \begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} & 0 \\
\frac{1}{60} & -\frac{1}{60} & \frac{1}{30} \\
\frac{1}{20} & -\frac{1}{20} & \frac{1}{10}
\end{pmatrix}
$$

$$
\begin{bmatrix}
\frac{1}{2} \\
-\frac{1}{2} \\
-\frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
\hat{\beta}_0 \\
\hat{\beta}_1 \\
\hat{\beta}_2
\end{bmatrix} = \begin{bmatrix}
1 \cdot (0) + 0 \cdot (1) + 0 \cdot (2) = 0 + 3 + 0 = 3 = \beta_0 \\
\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0 - \frac{1}{60} + \frac{2}{60} = \frac{1}{10} = 0.1 = \beta_1 \\
-\frac{1}{2} \cdot 0 - \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0 - \frac{5}{20} + \frac{6}{10} = \frac{1}{10} = 0.3 = \beta_2
\end{bmatrix}
$$
Probability.

5. Box I: \{4 R, 3 G, 7 Y\}, Box II: \{8 R, 2 B, 4 Y\}. Box I is chosen with probability 0.8, otherwise Box II. Then balls are selected from the chosen box with equal probability and without replacement. \[ P( \text{Z} ) = 0.8 \quad P( \text{T} | \text{Z} ) = 0.2 \]

5a. \( P( \text{R1} \mid \text{Y2} | \text{I} ) = \left( \frac{4}{14} \right) \left( \frac{7}{13} \right) \)

5b. \( P( \text{R1} ) = 0.8 \left( \frac{4}{14} \right) + 0.2 \left( \frac{8}{14} \right) \)

5c. \( P( \text{I} \mid \text{R1} ) = \frac{P( \text{I} \cap \text{R1} )}{P( \text{R1} )} = \frac{P( \text{I} ) P( \text{R1} | \text{I} )}{P( \text{R1} )} = \frac{(0.8) \left( \frac{4}{14} \right)}{0.8 \left( \frac{4}{14} \right) + 0.2 \left( \frac{8}{14} \right)} \)

5d. \( P( \text{R1} \cup \text{Y1} ) = P( \text{R1} ) + P( \text{Y1} ) - P( \text{R1} \cap \text{Y1} ) = P( \text{R1} ) + P( \text{Y1} ) - P( \text{R1} ) P( \text{Y1} | \text{R1} ) \)

\[ = 0.8 \left( \frac{4}{14} \right) + 0.2 \left( \frac{8}{14} \right) + 0.8 \left( \frac{7}{13} \right) + 0.2 \left( \frac{8}{14} \right) \]

6. \( P( \text{OIL} ) = 0.1, P(+ \mid \text{OIL}) = 0.8, P(+ \mid \text{OIL}^C) = 0.3. \)

6a. \( P( \text{OIL}^- ) = P( \text{OIL}^- ) P( -\text{OIL}^- ) = (0.1) (1 - 0.8) = (0.1) (0.2) \)

6b. \( P(+) = (0.1) (0.8) + (0.9) (0.3) \)

6c. \( P( \text{OIL} \mid + ) = \frac{P( \text{OIL} \cap + )}{P(+)} = \frac{(0.1) (0.8)}{(0.1) (0.8) + (0.9) (0.3)} \)

6d. Are events OIL, + independent? Why? \( \sqrt{0} \neq P( \text{OIL} ) P(+) \neq P( \text{OIL} ) P(+ \mid \text{OIL} ) \rightarrow (0.1) (0.8) + (0.9) (0.3) \neq 0.1 (0.8) \)

7. The number X of road service calls in one day is approximately Poisson distributed with mean 1.44. Sketch the normal approximation of the distribution of X. \( \mu = 1.44 \)

\[ \sigma = \sqrt{1.44} = 1.2 \]

\[ 10 \]
Expectation, Var, sd.

8. Random variables $X$, $Y$ have
   
   $E X = 5$ \hspace{1cm} Var $X = 3$

   $E Y = 9$ \hspace{1cm} Var $Y = 4$

8a. $E(6X - 7Y + 11 - X) = 6E X - 7E Y + 11 - E X = 6(5) - 7(9) + 11 - 5$
   \hspace{1cm} = 30 - 63 + 11 - 5 = -33 + 11 - 5 = -22 - 5 = -27$

8b. If $X$, $Y$ are independent $E(XY) = (E X)(E Y) = (5)(9) = 45$

8c. If $X$, $Y$ are independent $Var(6X - 7Y + 11 - X) =$
   \hspace{1cm} $Var(5X - 7Y + 11) = 5^2 Var X - 7^2 Var Y = 25(3) - 49(4)$
   \hspace{1cm} = 75 - 196 = -121$

9. The distribution of r.v. $W$ is:
   
   \begin{tabular}{c|c}
   $w$ & $p(w)$ \\
   \hline
   0   & 0.9 \\
   20  & 0.1 \\
   \end{tabular}

9a. $E W = 0(0.9) + 20(0.1)$

9b. $Var W = \left[0^2(0.9) + 20^2(0.1)\right] - \left[0(0.9) + 20(0.1)\right]^2$

9c. Let $T$ denote the total of 100 independent plays of the lottery whose returns are distributed as $W$. Sketch the bell-approximation of the distribution of $T$.

9d. $E\left(\frac{1}{1+W}\right) = \frac{1}{1 + 0}(0.9) + \frac{1}{1 + 20}(0.1)$
   \hspace{1cm} = 0.9 + \frac{1}{21}(0.1)$
Continuous models.

10. Lifetime $T$ of an electronic component is exponentially distributed with mean $\mu = E[T] = 8$ years.

10a. Determine $P(T > 8)$. 

\[ P(T > 8) = 1 - P(T \leq 8) = 1 - e^{-\frac{8}{8}} = 1 - e^{-1} \]

10b. Determine $P(T > 20 \mid T > 12)$. 

\[ P(T > 20 \mid T > 12) = e^{-1} \]

11. Time $T$ and wear $W$ are jointly distributed with density:

\[ f(t, w) = \frac{1}{6} (t + w), \quad 0 < t < 1, \quad 0 < w < 3 \]

\[ = 0 \text{ elsewhere.} \]

11a. Verify that $f$ is a joint probability density.

\[
\int_0^3 \int_0^{3-t-w} dw \, dt = \frac{1}{6} \int_0^3 \left[ (3 - t - w)^2 \right]_{w=0}^{w=3-t} dt = \frac{1}{6} \left[ \frac{3}{2} t^2 + \frac{3}{2} t \right]_{0}^{1} = \frac{1}{6} \left[ \frac{15}{4} \right] = 1
\]

11b. Determine the marginal density $f_T(t)$.

\[
f_T(t) = \frac{1}{6} \int_0^{3-t} (t + w) \, dw = \frac{1}{6} \int_0^{3-t} \frac{1}{2} (tw + \frac{w^2}{2}) \, dw = \frac{1}{6} \left[ \frac{3t^2}{2} - \frac{3t}{2} \right] = \frac{1}{2} t + \frac{9}{12}
\]

11c. Determine the conditional density $f_{W \mid T}(w \mid t)$.

\[ f_{W \mid T}(w \mid t) = \frac{f(t, w)}{f_T(t)} = \frac{\frac{1}{6} (t + w)}{\frac{1}{2} t + \frac{9}{12}} \]