Chapter 3. Discrete random variables (and related).

Exercises due in class Monday, September 15.

1. A player of Monopoly owns properties with respective rents \$90, \$150, \$200, \$150. Anyone landing on a given property has to pay the rent.

a. Give the probability mass function of the return from the \$90 property from one opponent on one pass of the board. Assume the probability of landing on the property in one pass of the board is 1/7.

x	0	90
p(x)	6/7	1/7

b. Calculate EX if X denotes the random return (a).

$$EX = \sum_x x\,p(x) = 0(6/7) + 90(1/7) = \frac{90}{7} = 12.9$$

c. Calculate E h(X) = E (1/X).

$$Eh(x) = \sum_x h(x)p(x) = \sum_h h\,p(h)$$

$$E\left(\tfrac{1}{x}\right) = \tfrac{1}{0}(6/7) + \tfrac{1}{90}(1/7)$$
$$= \text{undefined}$$

$E(X^2) = 0^2(6/7) + 90^2(1/7) = 90^2/7$
$E(X) = 90/7$

d. Calculate Var X.

$\mathrm{Var}X = E(X-EX)^2 = EX^2 - (EX)^2$
$= \frac{90^2}{7} - \left(\frac{90}{7}\right)^2 = 90^2(1/7)(6/7)$

e. Calculate sd X.

$SD = \sqrt{\mathrm{Var}X} = 90\sqrt{(1/7)(6/7)} = 31.5$

f. Determine P(X > 140). = 0

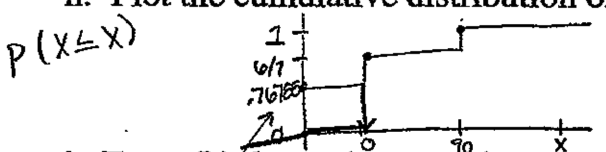
X	0	90
p(x)	6/7	1/7
	NO	NO

g. Determine P(|X-120| < 35). = 1/7

x	0	90
p(x)	6/7	1/7
	NO	YES

$|90-120| < 35$

h. Plot the cumulative distribution of X.



i. From (h) determine a random sample of X using a random number 0.76788.

get 0 from cumulative dist. above
$p(X=0) = P(U \text{ in } [0, 6/7]) = 6/7$

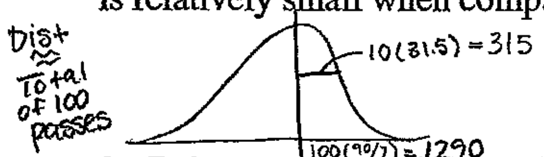
j. The **approximate** distribution of the **sum** of n = 100 **independent** samples of X is a bell curve having

passes

mean = n EX = 100(EX)

sd = $\sqrt{n}$ (sd X) = 10(sd X).

This would be the approximate distribution of the total rent these properties earn from 100 player passes of the board. Sketch this curve, identifying the above mean and sd as recognizable elements of your sketch. Notice that sd of this curve 10(sd X) is relatively small when compared with the mean 100(EX).



2. Poisson distribution. This distribution applies to the number X of occurrences of rare events (possibly 0, 1, 2, ... ad infinitum) when the expected number EX is given and specific conditions apply. Assuming the conditions hold:

a. pmf is $p(x) = e^{-EX}\frac{EX^x}{x!}$, x = 0, 1, 2, ... ad inf. For example, we expect four "aces of spades" in 4x52 = 208 draws of a single card (with-replacement, equal probability for all 52 cards). Getting an ace of spades on any single draw is relatively rare. What is the probability we actually obtain 3 aces of spades in the 208 tries? It is around p(3) for EX = 4. Calculate it.

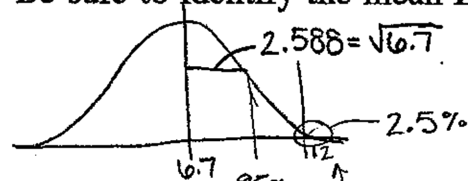
$$p(3) = e^{-4}\frac{4^3}{3!} = .195$$

b. We average around 6.7 lightning strikes in a harbor during the boating season. Assuming the distribution is Poisson, what is the probability of fewer than 4 strikes in a season? Hint: p(0)+p(1)+p(2)+p(3).

$$P(\#\text{strikes}(X) < 4) = \underbrace{\left(e^{-6.7}\frac{6.7^0}{0!}\right)}_{.0012} + \underbrace{\left(e^{-6.7}\frac{6.7^1}{1!}\right)}_{.0083} + \underbrace{\left(e^{-6.7}\frac{6.7^2}{2!}\right)}_{.0276} + \underbrace{\left(e^{-6.7}\frac{6.7^3}{3!}\right)}_{.0617} = .0988 \approx 10\%$$

c. Refer to (b). In 10 seasons how many ~~seasons are we expecting~~ that have fewer than 4 strikes? (10)(.0988) = about 1 yr w/ fewer than 4 strikes

d. A Poisson distribution with EX ≥ 3 (a rule of thumb) is approximated by a bell curve having mean = EX and sd = $\sqrt{EX}$. Sketch the bell approximation of the Poisson having EX = 6.7. Be sure to identify the mean EX and sd $\sqrt{EX}$ as recognizable entities in your sketch.



e. Refer to (d). Would a season with 12 strikes be all that unusual? Place 12 on the axis of the curve in (d).

3. Binomial distribution. A process produces defective items at the rate of p = 0.3. A random sample of n = 10 such items (assumed independent) is selected for inspection. Let X denote the number of defective items among the 10.

we expect EX = np defectives in the sample

sd of X = $\sqrt{n\,p\,(1-p)}$

$$\text{pmf} = p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n$$

with

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}.$$

a. Calculate the probability p(5) of x = 5 defectives in the sample of 10.

$p(5) = \left(\frac{10!}{5!(10-5)!}\right) .3^5 (1-.3)^{10-5} = .103$

b. Calculate p(0) (note, 0! is defined to be one).

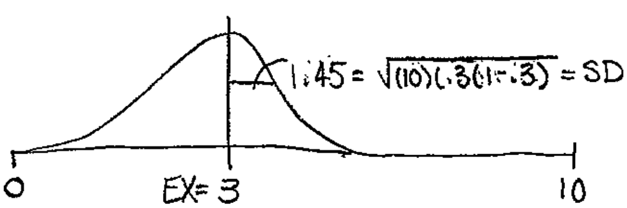
$p(0) = \left(\frac{10!}{0!(10-0)!}\right) .3^0 (1-.3)^{10-0} = .028$

c. For the binomial with n = 10 and p = 0.3 calculate EX and sd X.

$EX = np = (10)(.3) = 3$

$sd\,X = \sqrt{(10)(.3)(1-.3)} = 1.449$

d. For large enough np and n(1-p) we have a bell curve approximation of the binomial also. Sketch this approximation for the case n = 10 and p = 0.3. Be sure to label the mean and sd of the curve as recognizable entities. $EX = (10)(.3) = 3$



4. For random variables X, Y, and constants a, b, c, we have

$E(aX + bY + c) = a\,EX + b\,EY + c$ (linearity of expectation)

which holds irrespective of any dependence of X, Y. Also,

$\text{Var}(aX + b) = \text{Var}(aX) = a^2\,\text{Var}\,X$ (location b does not matter).

*Variance adds when indep.

If X, Y are **independent** random variables, Var=6.33

$\text{Var}(aX + bY + c) = a^2\,\text{Var}\,X + b^2\,\text{Var}\,Y$ (additivity with indep).

A business averages \$4.11 profit per sale. Profits are random and independent for different sales.

a. E(profit total from 100 sales)

$$= EX_1 + \ldots + EX_{100} = 100(4.11) = 411$$

b. Var(profit total from 100 sales)

$$Var = 100(VarX) = 100(6.33) = 633$$

Suppose NET profit = 0.9 profit - 1.2.

c. E =(net profit per sale)

$$E_{Net} = .9(\underbrace{4.11}_{\text{GROSS}}) - 1.2 = 2.5$$

d. Var(net profit per sale)

$$= Var(.9\underbrace{X}_{\text{GROSS PROFIT}} - \cancel{1.2})$$

$$= .9^2 \, Var X = .9^2(6.33) = 5.127$$

e. Total net profit from 100 independent sales is **approximately** bell curve distributed with mean and variance below. Sketch the curve with the numerical mean and sd of the curve on prominent display

E(total net profit from n sales) = n E(net from one sale) = $100(2.5) = 250$

Var(total net profit from n indep sales) = n Var(net from one sale). = $100(5.127) \doteq 51$