18-02/8

STT 351 HW5
Due at the close of class 10 - 8 - 08.

Define function $x/y^2 + y$ for $0 < x < 1$ and $1 < y < 2$.

a. Integrate the above function over the indicated doomain of x, y values.

$$\int_1^2\int_0^1 \frac{x}{y^2}+y\,dx\,dy = \int_1^2\left[\frac{x^2}{2y^2}+xy\right]_0^1 dy = \int_1^2 \frac{1}{2y^2}+y\,dy = \left.\frac{1}{-2y}+\frac{y^2}{2}\right]_1^2 = \left(-\frac{1}{4}+\frac{4}{2}\right)-\left(-\frac{1}{2}+\frac{1}{2}\right) = \frac{7}{4}$$

b. From the above, determine the probability density f(x, y) (or $f_{X,Y}(x, y)$) that is a constant multiple of the function given there.

$$\int_1^2\int_0^1 \frac{1}{4}\left(\frac{x}{y^2}+y\right)dx\,dy = 1$$

$$c\frac{7}{4}=1 \Rightarrow c=\frac{4}{7}$$

$$f(x,y) = \frac{4}{7}\left(\frac{x}{y^2}+y\right)$$ non-negative, integrates to one

Determine

c. $EX = \int\int x\,f(x, y)\,dx\,dy$

$$\int_1^2\int_0^1 x\,\frac{4}{7}\left(\frac{x}{y^2}+y\right)dx\,dy = \int_1^2\int_0^1 \frac{4}{7}\frac{x^2}{y^2}+\frac{4}{7}xy\,dx\,dy = \int_1\left[\frac{4}{21}\frac{x^3}{y^2}+\frac{4x^2y}{7\cdot 2}\right]_0^1 dy = \int_1^2 \frac{4}{21y^2}+\frac{1}{2}y\,dy$$

$$\left.-\frac{4}{21y}+\frac{1}{4}y^2\right]_1^2 = \left(-\frac{4}{42}+\frac{4}{4}\right)-\left(-\frac{4}{21}+\frac{1}{4}\right) = \frac{11}{21}$$

d. E Y

$$\int_1^2\int_0^1 y\,\frac{4}{7}\left(\frac{x}{y^2}+y\right)dx\,dy = \int_1^2\int_0^1 \frac{4}{7}\frac{x}{y}+\frac{4}{7}y^2\,dx\,dy = \int_1^2\left[\frac{4}{14}\frac{x^2}{y}+\frac{4}{7}xy^2\right]_0^1 dy = \int_1^2 \frac{4}{14y}+\frac{4}{7}y^2\,dy = \left[\frac{4}{14}\ln y+\frac{4y^3}{7\cdot 3}\right]_1^2 = \frac{2\ln(2)}{7}+\frac{4}{3}$$

e. $EX^2 = \int\int x^2 f(x, y)\,dx\,dy$

$$\int_1^2\int_0^1 x^2\frac{4}{7}\left(\frac{x}{y^2}+y\right)dx\,dy = \int_1^2\int_0^1 \frac{4}{7}\frac{x^3}{y^2}+\frac{4}{7}x^2y = \int_1\left[\frac{4}{28}\frac{x^4}{y^2}+\frac{4}{21}x^3y\right]_0^1 dy = \int_1^2 \frac{4}{28y^2}+\frac{4}{21}y\,dy = \left[-\frac{4}{28y}+\frac{4}{42}y^2\right]_1^2 = \frac{7}{4}$$

f. EY^2

$$\int_1^2\int_0^1 y^2\frac{4}{7}\left(\frac{x}{y^2}+y\right)dx\,dy = \int_1^2\int_0^1 \frac{4}{7}x+\frac{4}{7}y^3\,dx\,dy = \int_1^2\left[\frac{4}{14}x^2+\frac{4}{7}y^3x\right]_0^1 dy = \int_1^2 \frac{4}{14}+\frac{4}{7}y^3\,dy = \left[\frac{4}{14}y+\frac{4}{28}y^4\right]_1^2 = \left(\frac{8}{14}+\frac{16}{28}\right)-\left(\frac{4}{14}+\frac{4}{28}\right) = \frac{17}{7}$$

g. $E(XY) = \int\int xy\,f(x, y)\,dx\,dy$

$$\int_1^2\int_0^1 xy\,\frac{4}{7}\left(\frac{x}{y^2}+y\right)dx\,dy = \int_1^2\int_0^1 \frac{4}{7}\frac{x^2}{y}+\frac{4}{7}xy^2\,dx\,dy = \int_1^2\left[\frac{4}{21}\frac{x^3}{y}+\frac{4}{14}x^2y^2\right]_0^1 dy = \int_1^2 \frac{4}{21y}+\frac{4}{14}y^2\,dy$$

$$= \left[\frac{4}{21}\ln|y|+\frac{4}{3\cdot 14}y^3\right]_1^2 = \left(\frac{4}{21}\ln|2|+\frac{32}{3\cdot 14}\right)-\left(0+\frac{4}{3\cdot 14}\right) = \frac{2}{21}(7+\ln 4)$$

h. Var X

$$= EX-(EX)^2 = \frac{7}{4}-\left(\frac{11}{21}\right)^2$$

i. $\sigma_X = \text{sd X}$

$= \sqrt{\text{var} X} = \sqrt{\frac{7}{4} - \left(\frac{11}{21}\right)^2}$

j. Var Y

$= EY - (EY)^2 = \frac{17}{7} - \left(\frac{4}{3} + \frac{2\ln 2}{7}\right)^2$

k. $\sigma_Y = \text{sd Y}$

$= \sqrt{\text{Var} Y} = \sqrt{\frac{17}{7} - \left(\frac{4}{3} + \frac{2\ln 2}{7}\right)^2}$

l. Covariance of X with Y defined by E(XY) - (E X)(E Y)

$= \left(\frac{2}{21}\right)(7 + \ln 4) - \left(\frac{11}{21}\right)\left(\frac{4}{3} + \frac{2\ln 2}{7}\right)$

m. Covariance of X with X

$= E(XY) - (EX)(EX) = \left(\frac{7}{4}\right) - \left(\frac{11}{21}\right)\left(\frac{11}{21}\right)$

n. Correlation between X, Y defined by $\frac{E\,XY - (E\,X)(E\,Y)}{\sigma_X\ \sigma_Y}$

$$= \frac{\frac{2}{21}(7 + \ln 4) - \left(\frac{11}{21}\right)\left(\frac{4}{3} + \frac{2\ln 2}{7}\right)}{\sqrt{\left(\frac{7}{4}\right) - \left(\frac{11}{21}\right)^2} \cdot \sqrt{\left(\frac{17}{7}\right) - \left(\frac{4}{3} + \frac{2\ln 2}{7}\right)}}$$

o. marginal density for X defined by $f_X(x) = \int_1^2 f(x,\ y)\,dy$

$f_X(x) = \int_1^2 \frac{4}{7}\frac{x}{y^2} + \frac{4}{7}y\,dy = \left[-\frac{4}{7}\frac{x}{y} + \frac{4}{14}y^2\right]_1^2 = \left(-\frac{4}{14}x + \frac{8}{14}\right) - \left(-\frac{4}{7}x + \frac{4}{14}\right) = \frac{2(3+x)}{7}$

for $0 < x <$

p. conditional density of y GIVEN x defined $f_{y|x}(y) = \frac{f(x,y)}{f_X(x)}$

$$= \frac{\frac{4}{7}\left(\frac{x}{y^2} + y\right)}{\frac{2(3+x)}{7}} = \frac{4}{2}\frac{\left(\frac{x}{y^2} + y\right)}{(3+x)}$$

q. conditional mean of y GIVEN x defined $E(Y \mid X = x) = \int y\, f_{y|x}(y)\, dy$

$$= \int_1^2 y \frac{2\left(\frac{x}{y^2}+y\right)}{3+x}\, dy = \int_1^2 \frac{1}{(3+x)} \cdot \left(\frac{2x}{y} + 2y^2\right) dy = \left[\frac{1}{(3+x)} \cdot \left(2x \ln y + \frac{2}{3}y^3\right)\right]_1^2$$

$$= \frac{1}{(3+x)}\left[\left(2x\ln 2 + \frac{16}{3}\right) - \left(\frac{2}{3}\right)\right]$$

r. using the definitions, prove that in general E Y = E (E(Y | X)), i.e.

$$= \frac{14 + x\ln 64}{9+3x}$$

$$\underbrace{\int y\, f_Y(y)\, dy}_{E\,Y} = \int \big(\underbrace{\int y\, f_{y|x}(y)\, dy}_{E(Y \mid X = x)} \big) f_X(x)\, dx$$

$$EY = \frac{4}{3} + \frac{2\ln 2}{7}$$

$$EE(Y|X) = E(Y|X=x) = \int_0^1 \left(\int_1^2 y\, f_{y|x}(y)\, dy \right) f_x(x)\, dx = \int_0^1 \left(\frac{14 + x\ln 64}{9+3x}\right)\left(\frac{2(3+x)}{7}\right) dx$$

$$\int_1^2 \int_0^1 y\, f_{x,y}(x,y)\, dx\, dy$$