4.4 The Exponential and Gamma Distributions

The Exponential Distribution

DEFINITION

X is said to have an **exponential distribution** with parameter λ ($\lambda > 0$) if the pdf of X is

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (4.5)$$

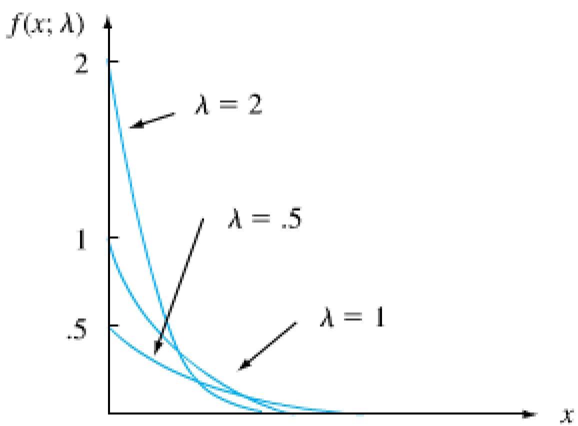

Figure 4.26 Exponential density curves

$$\mu = E(X) = \frac{1}{\lambda}$$

$$\sigma^2 = V(X) = \frac{1}{\lambda^2}$$

The cdf: $$F(x;\lambda) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

In particular $P(X > t) = 1 - F(t;\lambda) = e^{-\lambda t}$

Example 1. Suppose the response time X at a certain on-line computer terminal (the elapsed time between the end of a user's inquiry and the beginning of the system's response to that inquiry) has an exponential distribution with expected response time equal to 5 sec. Find the probability that the response time is between 5 and 10 sec.

$E(X) = \frac{1}{\lambda} = 5 \;\; \Rightarrow \;\; \lambda = 0.2$

$P(5 \le X \le 10) = \int_5^{10} 0.2\, e^{-0.2x}\, dx = -e^{-0.2x}\big|_5^{10} = e^{-1} - e^{-2} = 0.233$ or

$P(5 \le X \le 10) = F(10) - F(5) = (1 - e^{-(0.2)(10)}) - (1 - e^{-(0.2)(5)}) = 0.233$

Memoryless property: $P(X \ge t + t_0 \mid X \ge t_0) = P(X \ge t)$

Proof: $P(X \ge t + t_0 \mid X \ge t_0) = \frac{P[(X \ge t + t_0) \cap (X \ge t_0)]}{P(X \ge t_0)} = \frac{P(X \ge t + t_0)}{P(X \ge t_0)} = e^{-\lambda t}$

Relation to Poisson Process:

PROPOSITION

Suppose that the number of events occurring in any time interval of length t has a Poisson distribution with parameter αt (where α, the rate of the event process, is the expected number of events occurring in 1 unit of time) and that numbers of occurrences in nonoverlapping intervals are independent of one another. Then the distribution of elapsed time between the occurrence of two successive events is exponential with parameter $\lambda = \alpha$.

Proof: Although a complete proof is beyond the scope of the text, the result is easily verified for the time X_1 until the first event occurs:

$$P(X_1 \le t) = 1 - P(X_1 > t) = 1 - P[\text{no events in } (0, t)]$$

$$= 1 - \frac{e^{-\alpha t} \cdot (\alpha t)^0}{0!} = 1 - e^{-\alpha t}$$

which is exactly the cdf of the exponential distribution.

The Gamma Function

DEFINITION

For $\alpha > 0$, the **gamma function** $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx \tag{4.6}$$

The most important properties of the gamma function are the following:

1. For any $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1)\cdot\Gamma(\alpha - 1)$ [via integration by parts]
2. For any positive integer, n, $\Gamma(n) = (n - 1)!$
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Example 2. $\Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2} - 1\right)\Gamma\left(\frac{3}{2} - 1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

The Gamma Distribution

DEFINITION

A continuous random variable X is said to have a **gamma distribution** if the pdf of X is

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta} & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

where the parameters α and β satisfy $\alpha > 0$, $\beta > 0$. The **standard gamma distribution** has $\beta = 1$, so the pdf of a standard gamma rv is given by (4.7).

$$f(x;\alpha) = \begin{cases} \dfrac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \qquad (4.7)$$

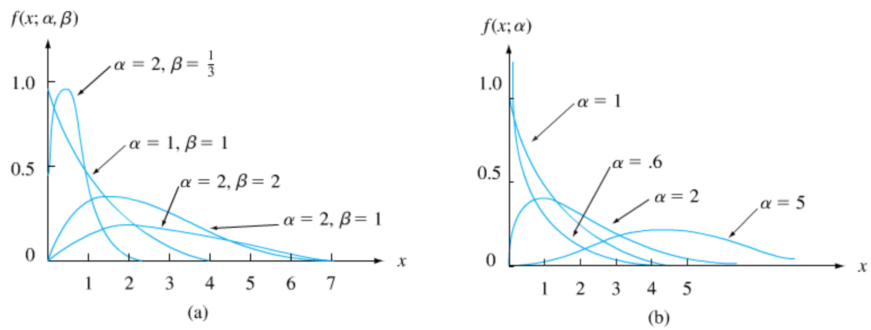

Figure 4.27 (a) Gamma density curves; (b) standard gamma density curves

Mean: $\mu = E(X) = \alpha\beta$ **Variance:** $\sigma^2 = V(X) = \alpha\beta^2$

Cdf: $P(X \le x) = F(x;\ \alpha,\beta) = \int_0^x \frac{1}{\beta^\alpha \Gamma(\alpha)}\ t^{\alpha-1} e^{-\frac{t}{\beta}}\ dt = \int_0^{x/\beta} \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}\ dy = F(\frac{x}{\beta};\alpha)$

where $F(x;\alpha) = \int_0^x \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}\, dy \qquad x > 0$

Let X have a gamma distribution with parameters α and β. Then for any $x > 0$, the cdf of X is given by

$$P(X \le x) = F(x;\alpha,\beta) = F\left(\frac{x}{\beta};\alpha\right)$$

where $F(\cdot;\alpha)$ is the incomplete gamma function.

Example 1. (4.4 Exercise 66) Suppose the time spent by a randomly selected student who uses a terminal connected to a local time-sharing computer facility has a gamma distribution with mean 20 min and variance 80 min^2.

a. What are the values of α and β? $\alpha\beta = 20 \;\; and \;\; \alpha\beta^2 = 80 \;\; \Rightarrow \;\; \alpha = 5 \; and \; \beta = 4$
b. What is the probability that a student uses the terminal for at most 24 min?

$$P(X \leq 24) = F(24; 5, 4) = F(24/4; 5) = F(6; 5) = 0.715$$

The Chi-Squared Distribution

DEFINITION

Let ν be a positive integer. Then a random variable X is said to have a **chi-squared distribution** with parameter ν if the pdf of X is the gamma density with $\alpha = \nu/2$ and $\beta = 2$. The pdf of a chi-squared rv is thus

$$f(x; \nu) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu/2)-1} e^{-x/2} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad (4.10)$$

The parameter ν is called the **number of degrees of freedom** (df) of X. The symbol χ^2 is often used in place of "chi-squared."

Mean: $\mu = E(X) = \alpha\beta = \nu$ **Variance:** $\sigma^2 = V(X) = \alpha\beta^2 = 2\nu$

The Erlang Distribution

The special case of the gamma distribution in which α is a positive integer n is called an Erlang distribution. If we replace β by $1/\lambda$ in Expression (4.8), the Erlang pdf is

$$f(x; \lambda, n) = \begin{cases} \dfrac{\lambda(\lambda x)^{n-1}e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

It can be shown that if the times between successive events are independent, each with an exponential distribution with parameter λ, then the total time X that elapses before all of the next n events occur has pdf $f(x; \lambda, n)$.

Exercise 68. Suppose that X has an Erlang distribution.

a. What is the expected value of X? If the time (in minutes) between arrivals of successive customers is exponentially distributed with $\lambda = .5$, how much time can be expected to elapse before the tenth customer arrives?

b. If customer interarrival time is exponentially distributed with $\lambda = .5$, what is the probability that the tenth customer (after the one who has just arrived) will arrive within the next 30 min?

c. The event $\{X \leq t\}$ occurs iff at least n events occur in the next t units of time. Use the fact that the number of events occurring in an interval of length t has a Poisson distribution with parameter λt to write an expression (involving Poisson probabilities) for the Erlang cdf $F(t; \lambda, n) = P(X \leq t)$.