7.1- 4 Confidence Intervals (CIs)

Suppose that the distribution of a random variable X depends on an unknown parameter Θ and let $X_1, X_2, \ldots, X_n$ be a random sample from this distribution. The idea is to find two statistics $L(X_1,X_2,\ldots,X_n)$ and $U(X_1,X_2,\ldots,X_n)$ such that the estimated parameter Θ lies between L and U with a prescribed probability (called the **confidence level** or the **confidence coefficient**), that is such that

$$P(L(X_1,X_2,\ldots,X_n) \leq \Theta \leq U(X_1,X_2,\ldots,X_n)) = 1 - \alpha$$

where α is a small positive number.

α	$1-\alpha$	confidence level $(1-\alpha)100\%$	$\alpha/2$
0.01	.99	99%	0.005
0.05	.95	95%	0.025
0.10	.90	90%	0.05

If we take a sample, so that $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, and compute $l = L(x_1,x_2,\ldots,x_n)$ and $u = U(x_1,x_2,\ldots,x_n)$, then we can say that Θ is between l and u with confidence $100(1-\alpha)\%$. The interval $[l, u]$ is called a **confidence interval (CI).**

The phrase "with confidence $100(1-\alpha)\%$" means that if the process of estimation was repeated many times in approximately $100(1-\alpha)\%$ cases the obtained confidence interval would contain an estimated parameter.

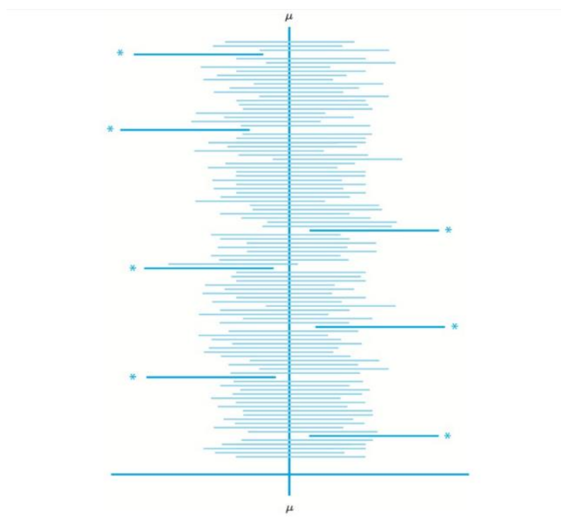


Figure 7.3 One hundred 95% CIs (asterisks identify intervals that do not include μ).

Problem: Suppose that $X_1, X_2, \dots, X_n$ is a random sample from normal distribution with unknown mean μ and known standard deviation σ. Find a 100(1 - α)% interval for μ.

Solution: The statistics $\bar{X}$ has a normal distribution with mean μ and standard deviation $\sigma/\sqrt{n}$ and hence $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ has the standard normal distribution. Given α, let $z_{\alpha/2}$ be a number such that $P(Z > z_{\alpha/2}) = \alpha/2$ [Table A.5 row ∞, or invNorm(1- $\alpha/2$)]. Then

$$P\left(-z_{\alpha/2} < \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < z_{\alpha/2}\right) = 1-\alpha$$

Solving the double inequality for μ gives that

$$P\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

Answer: A (1- α)100% confidence interval (CI) for μ is $\left[\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$ or $\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

General construction: Suppose that $X_1, X_2, \dots, X_n$ is a random sample from a certain distribution with unknown parameter θ. In order to construct a (1- α)100% confidence interval (CI) for θ we need to

1. Find a statistic $h(X_1, X_2, \dots, X_n; \theta)$ that depends on a random sample and θ, and such that we know its distribution
2. Find numbers a < b such that $P(a < h(X_1, X_2, \dots, X_n; \theta) < b) = 1-\alpha$
3. Solve the inequality $a < h(X_1, X_2, \dots, X_n; \theta) < b$ for θ

θ

Statistics and Their Distributions

Sample mean $\bar{X} = \frac{X_1+X_2+\cdots+X_n}{n} = \frac{\sum_{i=1}^{n} X_i}{n}$

Sample variance $S^2 = \frac{(X_1-\bar{X})^2 + (X_2-\bar{X})^2+\cdots+(X_n-\bar{X})^2}{n-1} = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1}$

Sample proportion $\hat{p} = \frac{number\ of\ sucecsses\ in\ n\ trials}{n} = \frac{X}{n}$

1. THEOREM (CLT, Section 5.4). Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from **any distribution** with mean μ and standard deviation σ. If n **is large** then both

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \qquad and \qquad Z = \frac{\bar{X}-\mu}{S/\sqrt{n}}$$

have **approximately standard normal distribution** (this follows from CLT)

2. THEOREM. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from **a dichotomous distribution** that takes only two values

$$\begin{cases} 1 & with\ probability\quad p = P(success) \\ 0 & with\ probability\ q = 1-p = P(failure) \end{cases}$$

Let $X = X_1 + X_2 + \ldots + X_n$ (number of successes in n trials) and $\hat{p} = \frac{X}{n}$. If n **is large** then the statistic

$$Z = \frac{\hat{p}-p}{\sqrt{p(1-p)/n}}$$

has **approximately standard normal distribution** (Sections 3.4 and 6.1)

3. THEOREM (Section 5.5). Suppose that $X_1, X_2, \dots, X_n$ is a random sample from **normal distribution** with mean μ and standard deviation σ. **For any *n*** the statistics

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

has **standard normal distribution**

4. THEOREM (Section 7.3). Suppose that $X_1, X_2, \dots, X_n$ is a random sample from **normal distribution** with mean μ and standard deviation σ. **For any *n*** the statistics

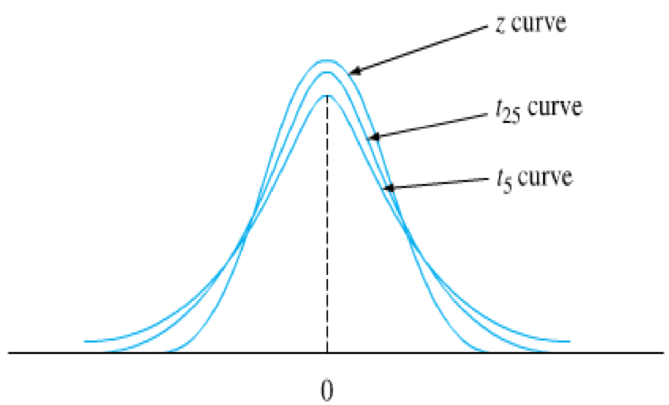
$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has a so called **(Student's) t distribution** with ν = n - 1 degrees of freedom (df).

Properties of *t* Distributions

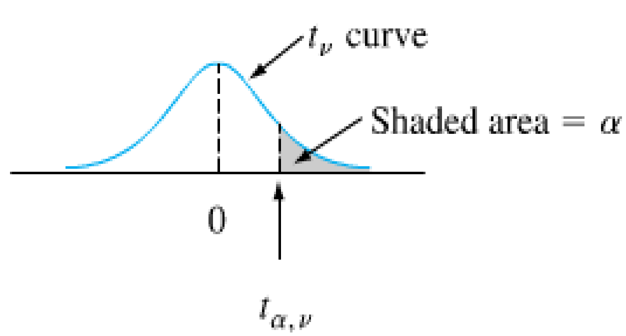
Let t_ν denote the density function curve for ν df.

1. Each t_ν curve is bell-shaped and centered at 0.
2. Each t_ν curve is more spread out than the standard normal (z) curve.
3. As ν increases, the spread of the corresponding t_ν curve decreases.
4. As $\nu \to \infty$, the sequence of t_ν curves approaches the standard normal curve (so the z curve is often called the t curve with df $= \infty$).



Notation

Let $t_{\alpha,\nu}$ = the number on the measurement axis for which the area under the t curve with ν df to the right of $t_{\alpha,\nu}$ is α; $t_{\alpha,\nu}$ is called a ***t* critical value.**



Critical values $t_{\alpha,\nu}$ for a *t* distribution are in Table A.5

5. THEOREM (Section 7.4). Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from **normal distribution** with mean μ and standard deviation σ. **For any *n*** the statistics

$$\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2}$$

has **a chi-squared (χ^2) distribution** with ν = n - 1 degrees of freedom (df).

Recall from section 4.4 that

Let ν be a positive integer. Then a random variable X is said to have a **chi-squared distribution** with parameter ν if the pdf of X is the gamma density with $\alpha = \nu/2$ and $\beta = 2$. The pdf of a chi-squared rv is thus

$$f(x;\nu) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu/2)-1} e^{-x/2} & x \ge 0 \\ 0 & x < 0 \end{cases} \quad (4.10)$$

The parameter ν is called the **number of degrees of freedom** (df) of X. The symbol χ^2 is often used in place of "chi-squared."

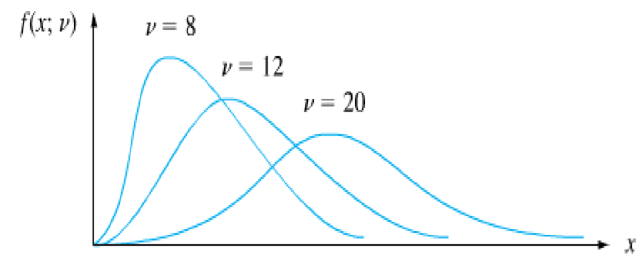
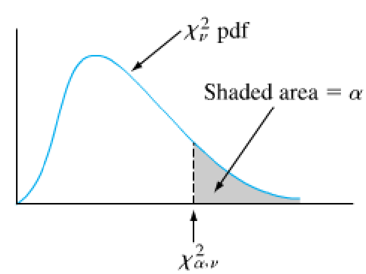


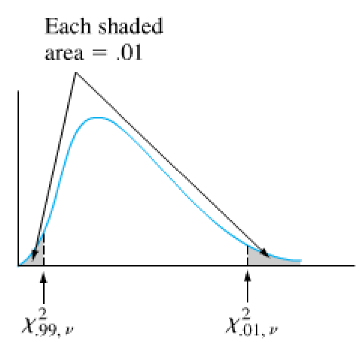
Figure 7.9 Graphs of chi-squared density functions

Notation

Let $\chi^2_{\alpha,\nu}$, called a **chi-squared critical value,** denote the number on the measurement axis such that α of the area under the chi-squared curve with ν df lies to the right of $\chi^2_{\alpha,\nu}$.

Critical values for a chi-squared distribution are in Table A.5





(1-α)100% Confidence Intervals

ARBITRARY POPULATION; LARGE SAMPLE

1. For the mean μ of any population when σ is known and the sample size n is large

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right)$$

or, equivalently, by $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$.

2. For the mean μ of any population when σ is not known and the sample size n is large

$$\bar{x} \pm z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

DICHOTOMOUS POPULATION; LARGE SAMPLE

3. For the population proportion p when the sample size is large

Let $\tilde{p} = \dfrac{\hat{p} + z_{\alpha/2}^2/2n}{1 + z_{\alpha/2}^2/n}$. Then a **confidence interval for a population proportion *p*** with confidence level approximately $100(1 - \alpha)\%$ is

$$\tilde{p} \pm z_{\alpha/2} \frac{\sqrt{\hat{p}\hat{q}/n + z_{\alpha/2}^2/4n^2}}{1 + z_{\alpha/2}^2/n} \qquad (7.10)$$

where $\hat{q} = 1 - \hat{p}$ and, as before, the $-$ in (7.10) corresponds to the lower confidence limit and the $+$ to the upper confidence limit.

Approximate formula: $\hat{p} \pm z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n}$

NORMAL POPULATION

4. For the mean μ of <u>normal population</u> when <u>σ is known; any sample size</u>

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right)$$

or, equivalently, by $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$.

5. For the mean μ of <u>normal population</u> when <u>σ is not known; any sample size</u>

$$\left(\bar{x} - t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}\right)$$

or, more compactly, $\bar{x} \pm t_{\alpha/2,n-1} \cdot s/\sqrt{n}$.

6. For the variance σ^2 of <u>normal population</u> when the <u>mean μ is not known; any sample size</u>

$$\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}$$

NOTE: We do not cover one sided confidence intervals (confidence bounds), prediction intervals, or tolerance intervals

EXERCISES

4. A CI is desired for the true average stray-load loss μ (watts) for a certain type of induction motor when the line current is held at 10 amps for a speed of 1500 rpm. Assume that stray-load loss is normally distributed with $\sigma = 3.0$.

c. Compute a 99% CI for μ when $n = 100$ and $\bar{x} = 58.3$.

d. Compute an 82% CI for μ when $n = 100$ and $\bar{x} = 58.3$.

e. How large must n be if the width of the 99% interval for μ is to be 1.0?

14. The article "Evaluating Tunnel Kiln Performance" (*Amer. Ceramic Soc. Bull.*, Aug. 1997: 59–63) gave the following summary information for fracture strengths (MPa) of $n = 169$ ceramic bars fired in a particular kiln: $\bar{x} = 89.10$, $s = 3.73$.

a. Calculate a (two-sided) confidence interval for true average fracture strength using a confidence level of 95%. Does it appear that true average fracture strength has been precisely estimated?

b. Suppose the investigators had believed a priori that the population standard deviation was about 4 MPa. Based on this supposition, how large a sample would have been required to estimate μ to within .5 MPa with 95% confidence?

23. The Pew Forum on Religion and Public Life reported on Dec. 9, 2009, that in a survey of 2003 American adults, 25% said they believed in astrology.

a. Calculate and interpret a confidence interval at the 99% confidence level for the proportion of all adult Americans who believe in astrology.

b. What sample size would be required for the width of a 99% CI to be at most .05 irrespective of the value of $\hat{p}$?

25. A state legislator wishes to survey residents of her district to see what proportion of the electorate is aware of her position on using state funds to pay for abortions.

a. What sample size is necessary if the 95% CI for p is to have a width of at most .10 irrespective of p?

b. If the legislator has strong reason to believe that at least 2/3 of the electorate know of her position, how large a sample size would you recommend?

33. The article "Measuring and Understanding the Aging of Kraft Insulating Paper in Power Transformers" (*IEEE Electrical Insul. Mag.*, 1996: 28–34) contained the following observations on degree of polymerization for paper specimens for which viscosity times concentration fell in a certain middle range:

418	421	421	422	425	427	431
434	437	439	446	447	448	453
454	463	465				

a. Construct a boxplot of the data and comment on any interesting features.

b. Is it plausible that the given sample observations were selected from a normal distribution?

c. Calculate a two-sided 95% confidence interval for true average degree of polymerization (as did the authors of the article). Does the interval suggest that 440 is a plausible value for true average degree of polymerization? What about 450?

44. The amount of lateral expansion (mils) was determined for a sample of $n = 9$ pulsed-power gas metal arc welds used in LNG ship containment tanks. The resulting sample standard deviation was $s = 2.81$ mils. Assuming normality, derive a 95% CI for σ^2 and for σ.

ANSWERS:

4. c. (57.5, 58.1); d. (57.9,58.7); e. 239

14. a. (88.54,89.66); b. 246

23. a. (22.5%, 27.5%); b. 2655

25. a. 358; b. 342

33. a. b. – use TI-83; c. (430.51, 446,08)

44. for σ^2 (360, 28.98); for σ (1.9, 5.4)