FINITE DIFFERENCE APPROXIMATIONS FOR TWO-SIDED SPACE-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS*

MARK M. MEERSCHAERT† AND CHARLES TADJERAN‡

Abstract. Fractional order partial differential equations are generalizations of classical partial differential equations. Increasingly, these models are used in applications such as fluid flow, finance and others. In this paper we examine some practical numerical methods to solve a class of initial-boundary value fractional partial differential equations with variable coefficients on a finite domain. We examine the case when a left-handed or a right-handed fractional spatial derivative may be present in the partial differential equation. Stability, consistency, and (therefore) convergence of the methods are discussed. The stability (and convergence) results in the fractional PDE unify the corresponding results for the classical parabolic and hyperbolic cases into a single condition. A numerical example using a finite difference method for a two-sided fractional PDE is also presented and compared with the exact analytical solution.

Key words. finite difference approximation, stability, backward Euler method, implicit Euler method, two-sided fractional partial differential equation, left-handed fractional flow, right-handed fractional flow, fractional derivative, fractional PDE, numerical fractional PDE

AMS subject classifications. 65M06, 65N12, 26A33

1. Introduction. In this paper, we examine some finite difference numerical methods to solve the fractional partial differential equation (FPDE) of the form

$$\frac{\partial u(x,t)}{\partial t} = c_+(x,t)\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} + c_-(x,t)\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} + s(x,t) \tag{1.1}$$

on a finite domain $L < x < R$, $0 \le t \le T$. Here, we consider the case $1 \le \alpha \le 2$, where the parameter α is the fractional order (fractor) of the spatial derivative. The function $s(x,t)$ is a source/sink term. The functions $c_+(x,t) \ge 0$ and $c_-(x,t) \ge 0$ may be interpreted as transport related coefficients. We also assume an initial condition $u(x, t = 0) = F(x)$ for $L < x < R$ and zero Dirichlet boundary conditions. For the case $1 < \alpha \le 2$, the addition of a classical advective term $-v(x,t)\,\partial u(x,t)/\partial x$ on the right-hand side of (1.1) does not impact the analysis peformed in this paper, and has been omitted to simplify the notation.

The left-handed (+) and the right-handed (–) fractional derivatives in (1.1) are the Riemann-Liouville fractional derivatives of order α [11, 9] defined by

$$(D^\alpha_{L+} f)(x) = \frac{d^\alpha f(x)}{d_+ x^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_L^x \frac{f(\xi)}{(x-\xi)^{\alpha+1-n}}d\xi \tag{1.2}$$

and

$$(D^\alpha_{R-} f)(x) = \frac{d^\alpha f(x)}{d_- x^\alpha} = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^R \frac{f(\xi)}{(\xi-x)^{\alpha+1-n}}d\xi \tag{1.3}$$

where n is an integer such that $n - 1 < \alpha \le n$. If $\alpha = m$, where m is an integer, then the above definitions give the standard integer derivatives, that is

$$(D^m_{L+} f)(x) = \frac{d^m f(x)}{dx^m}; \qquad (D^m_{R-} f)(x) = (-1)^m \frac{d^m f(x)}{dx^m} = \frac{d^m f(x)}{d(-x)^m}$$

*This work was supported by NSF grant DMS-0139927.

†Department of Mathematics and Statistics, University of Nevada, Reno, Nevada 89557 USA (mcubed@unr.edu).

‡Department of Mathematics and Statistics, University of Nevada, Reno, Nevada 89557 USA (tadjeran@unr.edu).

When $\alpha = 2$, and setting $c(x,t) = c_+(x,t) + c_-(x,t)$, equation (1.1) becomes the following classical parabolic PDE

$$\frac{\partial u(x,t)}{\partial t} = c(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + s(x,t) \tag{1.4}$$

Similarly, when $\alpha = 1$, and setting $c(x,t) = c_+(x,t) - c_-(x,t)$, equation (1.1) reduces to the following classical hyperbolic PDE

$$\frac{\partial u(x,t)}{\partial t} = c(x,t)\frac{\partial u(x,t)}{\partial x} + s(x,t) \tag{1.5}$$

The case $1 < \alpha < 2$ represents a super-diffusive process, where particles diffuse faster than the classical model (1.4) predicts. For some applications to physics and hydrology, see [1, 2, 4, 13].

We also note that the left-handed fractional derivative of $f(x)$ at a point x depends on all function values to the left of the point x, i.e., this derivative is a weighted average of such function values. Similarly, the right-handed fractional derivative of $f(x)$ at a point x depends on all function values to the right of this point. In general, the left-handed and right-handed derivatives are not equal unless α is an even integer, in which case, these derivatives become localized and equal. When α is an odd integer, these derivatives become localized and opposite in sign.

The Grünwald definitions for the right-handed and left handed fractional derivatives are respectively

$$\frac{d^\alpha f(x)}{d_+ x^\alpha} = \lim_{M_+ \to \infty} \frac{1}{h^\alpha} \sum_{k=0}^{M_+} g_k \cdot f(x - kh) \tag{1.6}$$

$$\frac{d^\alpha f(x)}{d_- x^\alpha} = \lim_{M_- \to \infty} \frac{1}{h^\alpha} \sum_{k=0}^{M_-} g_k \cdot f(x + kh) \tag{1.7}$$

where M_+, M_- are positive integers, $h_+ = (x - L)/M_+$, $h_- = (R - x)/M_-$, $\Gamma(\cdot)$ is the gamma function, and the normalized Grünwald weights are defined by

$$g_0 = 1 \quad \text{and} \quad g_k = (-1)^k \frac{(\alpha)(\alpha-1)\cdots(\alpha-k+1)}{k!} \quad \text{for } k = 1,2,3,\ldots \tag{1.8}$$

Note that these normalized weights only depend on the order α and the index k. The analytic definitions given by (1.2) and (1.3) are used in the formulation of the fractional PDE, while the Grüwald definitions (1.6) and (1.7) may be used to discretize the FPDE to obtain a numerical solution. For more details on fractional derivative concepts and definitions, see [8, 9, 11]. Reference [11] provides a more detailed treatment of the right-handed fractional derivatives, as well as a substantial treatment of left-handed fractional derivatives.

Published papers on the numerical solution of fractional partial differential equations are scarce. A different method for solving the fractional partial differential equation (1.1) is pursued in the recent paper of Liu, Ahn and Turner [6]. They transform this partial differential equation into a system of ordinary differential equations (Method of Lines), which is then solved using backward differentiation formulas. In another very recent paper, Fix and Roop [3] develop a finite element method for a two-point boundary value problem. We are unaware of any other published work on numerical solutions of fractional partial differential equations.

2. Approximating the left-handed fractional PDE. In this section we will examine the case of (1.1) when only the more common left-handed spatial fractional derivative appears. In the next section, we will use a similar approach to consider the general equation and results regarding its discretization.

If equation (1.1) only contains the left-handed fractional derivative, we will omit the directional sign notation (+ or -) and write the corresponding fractional PDE in the following form

$$\frac{\partial u(x,t)}{\partial t} = c(x,t)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + s(x,t)\ . \tag{2.1}$$

We will also assume that $c(x,t) \geq 0$ over the region $L \leq x \leq R,\ 0 \leq t \leq T$.

Define $t_n = n\Delta t$ to be the integration time $0 \leq t_n \leq T$, $\Delta x = h > 0$ to be a grid size in spatial dimension where $h = (R-L)/K$, $x_i = L + ih$ for $i = 0, \ldots, K$ so that $L \leq x_i \leq R$. Let u_i^n be the numerical approximation to $u(x_i, t_n)$. Similarly, define $c_i^n = c(x_i, t_n)$ and $s_i^n = s(x_i, t_n)$. A variable time step Δt_n may also be used. In this case $t_n = t_{n-1} + \Delta t_n$, and the results discussed in this section will be essentially unchanged.

If the differential equation (2.1) is discretized in time using an explicit (Euler) method, then one obtains

$$\frac{u(x,t_{n+1}) - u(x,t_n)}{\Delta t} = c(x,t_n)\ \frac{\partial^{\alpha} u(x,t_n)}{\partial x^{\alpha}} + s(x,t_n) + O(\Delta t)\ \ . \tag{2.2}$$

If the fractional derivatives in (2.2) is then discretized by the standard Grünwald estimates resulting from (1.6), we obtain a finite difference approximations to the equation (2.1) which stability analysis [7] shows to be unstable, therefore the numerical solution does not converge to the exact solution.

All the numerical methods discussed in this paper are consistent. The consistency proof for the spatial Grüwald estimates are facilitated by assuming zero Dirichlet boundary conditions, so that the solution may be zero-extended beyond the interval $L \leq x \leq R$. Thus the Riemann definition and the Liouville definition for the fractional derivative become equivalent, and the spatial discretizations have been shown to be $O(\Delta x)$. See [12] for the standard Grüwald estimates, and [7] for the shifted Grüwald estimates. The temporal discretization for the Euler methods are also $O(\Delta t)$. In view of Lax's equivalence theorem, these methods converge if and only if they are stable. Therefore, we only examine and refer to the stability of the numerical methods discussed in this paper.

To obtain a stable explicit Euler method when $1 \leq \alpha \leq 2$, we define the shifted Grünwald formula

$$\frac{d^{\alpha} f(x)}{dx^{\alpha}} = \lim_{M\to\infty} \frac{1}{h^{\alpha}} \sum_{k=0}^{M} g_k\ \cdot f[x-(k-1)h] \tag{2.3}$$

which defines the following shifted Grüwald estimate to the left-handed fractional derivative (see Ref. [7])

$$\frac{d^{\alpha} f(x)}{dx^{\alpha}} = \frac{1}{h^{\alpha}} \sum_{k=0}^{M} g_k\ \cdot f[x-(k-1)h] + O(h)\ . \tag{2.4}$$

The shifted Grünwald estimate defined by (2.4) generally provides a more accurate approximation than the standard (unshifted) Grünwald finite sum estimates obtained from (1.6).

With the shifted Grünwald estimate, the discretized (2.1) takes the following form

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\frac{c_i^n}{h^\alpha}\sum_{k=0}^{i+1} g_k\ u_{i-k+1}^n\ +s_i^n \tag{2.5}$$

for $i=1,2,..K-1$. The resulting equation can be explicitly solved for u_i^{n+1} to give

$$u_i^{n+1}=\beta\ c_i^n\ g_0\ u_{i+1}^n+(1+\beta\ c_i^n g_1)u_i^n+\beta\ c_i^n\sum_{k=2}^{i+1} g_k\ u_{i-k+1}^n\ +s_i^n\ \Delta t \tag{2.6}$$

where $\beta=\Delta t/h^\alpha$. The Dirichlet boundary condition(s) and the initial condition are also discretized accordingly. We now prove that this method is conditionally stable.

PROPOSITION 2.1. *The explicit Euler method* (2.6) *is stable if* $\Delta t/h^\alpha\le 1/(\alpha\ c_{max})$, *where* c_{max} *is the maximum value of* $c(x,t)$ *over the region* $L\le x\le R,\ 0\le t\le T$.

Proof. At each time step, we will apply a matrix stability analysis to the linear system of equations arising from the finite difference equations defined by (2.6), and will use the Greschgorin Theorem [5] to determine a stability condition.

The difference equations defined by (2.6), together with the Dirichlet boundary conditions, result in a linear system of equations of the form $\underline{U}^{n+1}\ =\underline{A}\ \underline{U}^n+\Delta t\ \underline{S}^n$ where

$$\underline{U}^n=[u_0^n,u_1^n,u_2^n,\ldots,u_K^n]^T$$
$$\underline{S}^n=[0,s_1^n,s_2^n,\ldots,s_{K-1}^n,0]^T$$

$\underline{A}$ is the matrix of coefficients, and is the sum of a lower triangular matrix and a super-diagonal matrix. The matrix entries $A_{i,j}$ for $i=1,\ldots,K-1$ and $j=1,\ldots,K-1$ are defined by

$$A_{i,j}=\begin{cases}0 & \text{, when } j\ge i+2\\ 1+g_1\ c_i^n\ \beta & \text{, when } j=i\\ g_{i-j+1}\ c_i^n\ \beta & \text{otherwise}\end{cases}$$

while $A_{0,0}=1$, $A_{0,j}=0$ for $j=1,\ldots,K$, $A_{K,K}=1$, and $A_{K,j}=0$ for $j=0,\ldots,K-1$. Note that $g_1=-\alpha$, and for $1\le\alpha\le 2$ and $i\ne 1$ we have $g_i\ge 0$ (the strict inequality holds for non-integer values of α. We also have $-g_1\ge\sum_{k=0,k\ne 1}^{k=N} g_i$, which follows from the well-known equality $\sum_{k=0}^{\infty} g_i=0$.

According to the Greschgorin Theorem (cf. [5] pp. 135-136) the eigenvalues of the matrix $\underline{A}$ lie in the union of the circles centered at $A_{i,i}$ with radius $r_i=\sum_{k=0,k\ne i}^{K} A_{i,k}$. Here we have $A_{i,i}=1+g_1\ c_i^n\ \beta=1-\alpha\ c_i^n\ \beta$ and

$$r_i=\sum_{k=0,k\ne i}^{K} A_{i,k}=\sum_{k=0,k\ne i}^{i+1} A_{i,k}=c_i^n\ \beta\sum_{k=0,k\ne i}^{i+1} g_i\le\alpha\ c_i^n\ \beta$$

and therefore $A_{i,i}+r_i\le 1$. We also have $A_{i,i}-r_i\ge 1-2\ \alpha\ c_i^n\ \beta\ge 1-2\ \alpha\ c_{max}\ \beta$. Therefore for the spectral radius of the matrix $\underline{A}$ to be at most one, it suffices to have $(1-2\ \alpha\ c_{max}\ \beta)\ge -1$, which yields the following condition on β

$$\beta=\frac{\Delta t}{h^\alpha}\le\frac{1}{\alpha\ c_{max}}. \tag{2.7}$$

So, under the condition on β defined by (2.7) the spectral radius of matrix $\underline{A}$ is bounded by one. With the spectral radius so bounded, the numerical errors do not grow, and therefore the explicit Euler method defined above is conditionally stable. □

The explicit Euler method defined by (2.5) is consistent with order $O(\Delta t) + O(h^{[\alpha]})$, where $[\alpha]$ denotes the largest integer that is less than or equal to α. This consistency of the finite difference method together with the above result on the stability imply that the explicit Euler method is convergent if the condition (2.7) is met. Note that in most applications, this condition may impose a severe bound on the size of the time step to meet the stability condition.

The convergence analysis of the explicit Euler methods for the classical parabolic PDE ($\alpha = 2$) and hyperbolic PDE ($\alpha = 1$) are special cases of the above results. For ease of illustration here, assume that the coefficient function in (2.1) is given by $c(x,t) = 1$. Then (2.7) may be written

$$\beta = \frac{\Delta t}{h^\alpha} \leq \frac{1}{\alpha}.$$

For the classical parabolic PDE, that is $\alpha = 2$ in (2.1), the resulting explicit Euler method from (2.5) is the classical finite difference equation given by ($g_0 = 1,\ g_1 = -2,\ g_2 = 1,$ and $g_3 = g_4 = \cdots = 0$)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}$$

And the stability requirement for this parabolic PDE becomes the classical step size constraint given by

$$\frac{\Delta t}{h^2} \leq \frac{1}{2}$$

For the classical hyperbolic PDE, that is $\alpha = 1$ in (2.1), the resulting explicit Euler method is the classical finite difference equation given by ($g_0 = 1,\ g_1 = -1,$ and $g_2 = g_3 = g_4 = \cdots = 0$)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^n - u_i^n}{h}$$

And the stability requirement for this hyperbolic PDE becomes the classical step size constraint given by

$$\frac{\Delta t}{h} \leq 1$$

Refer to [10] for the stability analysis of the explicit Euler methods for the classical parabolic and hyperbolic PDEs. Therefore, the fractional PDE 'unifies' the stability (and convergence) results for the explicit (and as shown below also implicit) Euler approximation methods for the classical parabolic and the classical hyperbolic problems.

3. Finite Difference approximations to the two-sided fractional PDE. We now examine the implicit Euler approximation to the two-sided fractional PDE (1.1). Using the shifted Grüwald estimate, the resulting discretization takes the following form

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{h^\alpha}\left[\sum_{k=0}^{i+1} g_k\, c_{+,i}^{n+1} u_{i-k+1}^{n+1} + \sum_{k=0}^{K-i+1} g_k\, c_{-,i}^{n+1}\, u_{i+k-1}^{n+1}\right] + s_i^{n+1} \qquad (3.1)$$

with $h = (R-L)/K$ for $i = 1, 2, ..K-1$, together with the Dirichlet boundary conditions that define u_0^{n+1} and u_K^{n+1} as appropriate. We now show that this yields a convergent numerical solution, by examining the stability of this consistent method.

Define $t_n = n\Delta t$ to be the integration time $0 \le t_n \le T$, $\Delta x = h > 0$ to be a grid size in spatial dimension where $h = (R-L)/K$, $x_i = L + ih$ for $i = 0, \ldots, K$ so that $L \le x_i \le R$. Let u_i^n be the numerical approximation to $u(x_i, t_n)$. Similarly, define $c_{-,i}^n = c_-(x_i, t_n)$, $c_{+,i}^n = c_+(x_i, t_n)$ and $s_i^n = s(x_i, t_n)$. A variable time step Δt_n may also be used. In this case $t_n = t_{n-1} + \Delta t_n$, and again the results discussed in this section will be essentially unchanged.

Proposition 3.1. *The implicit Euler method approximation defined by* (3.1) *to the fractional partial differential equation* (1.1) *with* $1 \le \alpha \le 2$ *is unconditionally stable.*

Proof. Define $\xi_i = c_{+,i}^{n+1}\ \Delta t/h^\alpha$, and $\eta_i = c_{-,i}^{n+1}\ \Delta t/h^\alpha$. The system of equations defined by (3.1), together with the Dirichlet boundary conditions, define a linear system $\underline{A}\ \underline{U}^{n+1} = \underline{U}^n + \Delta t\ \underline{S}^{n+1}$, where $\underline{U}^n = [u_0^n, u_1^n, u_2^n, \ldots, u_K^n]^T$, and $\underline{S}^n = [0, s_1^n, s_2^n, \ldots, s_{K-1}^n, 0]^T$. Note that this matrix $\underline{A}$ is a non-sparse matrix. To illustrate this matrix pattern, we list the corresponding first three equations for the rows $i = 1, 2$ and 3:

$$\begin{aligned}
u_1^n + s_1^{n+1}\Delta t =& (\xi_1 g_2 + \eta_1 g_0)u_0^{n+1} + [1 - (\xi_1 + \eta_1)g_1]u_1^{n+1} - (\xi_1 g_0 + \eta_1 g_2)u_2^{n+1} \\
& - \eta_1 g_3 u_3^{n+1} \cdots - \eta_1 g_K u_K^{n+1} \\
u_2^n + s_2^{n+1}\Delta t =& - \xi_2 g_3 u_0^{n+1} - (\xi_2 g_2 + \eta_2 g_0)u_1^{n+1} + [1 - (\xi_2 + \eta_2)g_1]u_2^{n+1} \\
& - (\xi_2 g_0 + \eta_2 g_2)u_3^{n+1} - \eta_2 g_3 u_4^{n+1} \cdots - \eta_2 g_{K-1} u_K^{n+1} \\
u_3^n + s_3^{n+1}\Delta t =& - \xi_3 g_4 u_0^{n+1} - \xi_3 g_3 u_1^{n+1} - (\xi_3 g_2 + \eta_3 g_0)u_2^{n+1} + [1 - (\xi_3 + \eta_3)g_1]u_3^{n+1} \\
& - (\xi_3 g_0 + \eta_3 g_2)u_4^{n+1} \cdots - \eta_3 g_{K-2} u_K^{n+1}
\end{aligned}$$

The matrix entries $A_{i,j}$ for $i = 1, \ldots, K-1$ and $j = 1, \ldots, K-1$ are defined by

$$A_{i,j} = \begin{cases} 1 - (\xi_i + \eta_i)g_1 & \text{for } j = i \\ -(\xi_i g_2 + \eta_i g_0) & \text{for } j = i-1 \\ -(\xi_i g_0 + \eta_i g_2) & \text{for } j = i+1 \\ -\xi_i g_{i-j+1} & \text{for } j < i-1 \\ -\eta_i g_{j-i+1} & \text{for } j > i+1 \end{cases}$$

while $A_{0,0} = 1$, $A_{0,j} = 0$ for $j = 1, \ldots, K$, $A_{K,K} = 1$, and $A_{K,j} = 0$ for $j = 0, \ldots, K-1$.

According to the Greschgorin theorem, the eigenvalues of the matrix $\underline{A}$ are in the disks centered at $A_{i,i} = 1 - (\xi_i + \eta_i)g_1 = 1 + (\xi_i + \eta_i)\ \alpha$, with radius

$$r_i = \sum_{k=0, k\neq i}^{K} A_{i,k} = \sum_{k=0, k\neq i}^{i+1} \xi_i g_k + \sum_{k=0, k\neq i}^{K-i+1} \eta_i g_k \le (\xi_i + \eta_i)\ \alpha$$

with strict inequality holding true when α is not an integer. This implies that the eigenvalues of the matrix $\underline{A}$ are all no less than 1 in magnitude. Hence the spectral radius of the inverse matrix $\underline{A}^{-1}$ is less than or equal to 1. Thus any error in $\underline{U}^n$ is not magnified, and therefore the method is stable. ▯

If equation (1.1) is instead discretized by an explicit Euler method, we have the following result regarding the stability (and therefore the convergence) of the resulting method.

PROPOSITION 3.2. *The explicit Euler method approximation using the shifted Grüwald estimates to the fractional partial differential equation* (1.1) *with* $1 < \alpha \le 2$ *is stable if*

$$\frac{\Delta t}{h^\alpha} \le \frac{1}{\alpha(c_{+max} + c_{-max})} \tag{3.2}$$

Proof. The proof is similar to Propositions 3.1 and 2.1. The explicit Euler method, with shifted Grünwald estimate, to discretize (1.1) can be written

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{h^\alpha}\left[c_{+,\,i}^n \sum_{k=0}^{i+1} g_k\, u_{i-k+1}^n + c_{-,\,i}^n \sum_{k=0}^{K-i+1} g_k\, u_{i+k-1}^n\right] + s_i^n \tag{3.3}$$

with $h = (R-L)/K$, for $i = 1, 2, ..K-1$. Define $\xi_i = c_{+,\,i}^n\ \Delta t/h^\alpha$, and $\eta_i = c_{-,\,i}^n\ \Delta t/h^\alpha$. The equation defined by (3.3) can be written in the explicit matrix form $\underline{U}^{n+1} = \underline{B}\ \underline{U}^n + \Delta t\ \underline{S}^n$. To illustrate the matrix $\underline{B}$ pattern, we list the corresponding first two equations for $i = 1$ and 2:

$$\begin{aligned} u_1^{n+1} =& (\xi_1 g_2 + \eta_1 g_0)u_0^n + (1 + \xi_1 g_1 + \eta_1 g_1)u_1^n + (\xi_1 g_0 + \eta_1 g_2)u_2^n \\ &+ \xi_1 g_3 u_3^n \cdots + \xi_1 g_K u_K^n + s_1^n \Delta t \\ u_2^{n+1} =& \xi_2 g_3 u_0^n + (\xi_2 g_2 + \eta_2 g_0)u_1^n + (1 + \xi_2 g_1 + \eta_2 g_1)u_2^n \\ &+ (\xi_2 g_0 + \eta_2 g_2)u_3^n \cdots + \xi_2 g_{K-1} u_K^n + s_2^n \Delta t \end{aligned}$$

The matrix entries $B_{i,j}$ for $i = 1, \ldots, K-1$ and $j = 1, \ldots, K-1$ are defined by

$$B_{i,j} = \begin{cases} 1 + (\xi_i + \eta_i)g_1 & \text{for } j = i \\ \xi_i g_2 + \eta_i g_0 & \text{for } j = i-1 \\ \xi_i g_0 + \eta_i g_2 & \text{for } j = i+1 \\ \xi_i g_{i-j+1} & \text{for } j < i+1 \\ \eta_i g_{j-i+1} & \text{for } j > i+1 \end{cases}$$

while $B_{0,0} = 1$, $B_{0,j} = 0$ for $j = 1, \ldots, K$, $B_{K,K} = 1$, and $B_{K,j} = 0$ for $j = 0, \ldots, K-1$.

Again, by the Greschgorin theorem, the eigenvalues of this matrix $\underline{B}$ are located in the union of disks centered at $B_{i,i} = 1 + \xi_i g_1 + \eta_i g_1 = 1 - (\xi_i + \eta_i)\alpha$ and radius

$$r_i = \sum_{k=0, k\neq i}^{K} B_{i,k} = \xi_i \sum_{k=0,k\neq i}^{i+1} g_k + \eta_i \sum_{k=0,k\neq i}^{K-i+1} g_k \le \xi_i \alpha + \eta_i \alpha$$

Therefore, to constrain the spectral radius of matrix $\underline{B}$ to achieve convergence, it suffices to require $1 - 2(\xi_i + \eta_i)\alpha \ge -1$, for all $i's$. Hence, the explicit Euler method is stable under the condition specified by (3.2). □

The use of the shifted Grüwald estimates is required to stabilize these finite difference schemes. The instability inherent when the standard Grünwald estimates are used for the left-handed fractional derivatives in the fractional PDEs was demonstrated in [7]. It may be surprising to note that the addition of a 'balancing' right-handed derivative does not improve the instability of the Euler methods with the standard Grünwald estimates. To illustrate, consider the following example

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} + \frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} + s(x,t) \tag{3.4}$$

on a finite domain $0 < x < 1$, $0 \le t \le T$, with zero Dirichlet boundary conditions. Then in an analysis similar to [7], if the propagation of error at the node x_{K-1} is considered, the resulting finite difference equation at this gridpoint gives:

$$u_{K-1}^{n+1} = (1 + \xi_{K-1} + \eta_{K-1}) u_{K-1}^{n} + \xi_{K-1} \sum_{k=1}^{K-2} g_k \cdot u_{K-k-1}$$

where $\xi_i = c_{+,\,i}^n \; \Delta t / h^\alpha$, and $\eta_i = c_{-,\,i}^n \; \Delta t / h^\alpha$. The magnification factor at time t_{n+1} for this explicit scheme at the gridpoint x_{K-1} is then

$$\mu_{K-1}^{n+1} = 1 + \xi_{K-1} + \eta_{K-1} > 1 \; .$$

Hence the above numerical scheme which uses the standard (unshifted) Grünwald estimate is unconditionally unstable.

4. Numerical Example. The following two-sided fractional partial differential equation

$$\frac{\partial u(x,t)}{\partial t} = c_+(x,t) \frac{\partial^{1.8} u(x,t)}{\partial_+ x^{1.8}} + c_-(x,t) \frac{\partial^{1.8} u(x,t)}{\partial_- x^{1.8}} + s(x,t)$$

was considered on a finite domain $0 < x < 2$ and $t > 0$ with the coefficient functions

$$c_+(x,t) = \Gamma(1.2)\; x^{1.8} \quad \text{and} \quad c_-(x,t) = \Gamma(1.2)\; (2-x)^{1.8},$$

the forcing function

$$s(x,t) = -32 e^{-t} [x^2 + (2-x)^2 - 2.5\big(x^3 + (2-x)^3\big) + \;\; \frac{25}{22}(x^4 + (2-x)^4)],$$

initial condition $u(x,0) = 4x^2(2-x)^2$, and boundary conditions $u(0,t) = u(2,t) = 0$. This fractional PDE has the exact solution $u(x,t) = 4e^{-t}x^2(2-x)^2$, which can be verified by applying the fractional differential formulas

$$D_{L+}^{\alpha}(x-L)^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}(x-L)^{p-\alpha} \text{ and } D_{R-}^{\alpha}(R-x)^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}(R-x)^{p-\alpha}.$$

Figure 1 shows the numerical solution at time $t = 1.0$ obtained from the backward (implicit) Euler method discussed above. This numerical solution compares well with the exact analytic solution. The numerical result shown is with $\Delta t = \frac{1}{40}$ and $\Delta x = h = \frac{1}{20}$. The algorithm was coded using the Intel Fortran compiler on a Dell Pentium PC. All computations were performed in single precision.

To examine the performance of this finite difference method for this example problem, the maximum numerical error at time $t = 1.0$ was computed starting with $\Delta t = 0.1$ and $\Delta x = h = 0.2$. Figure 2 shows that as the number of time steps/spatial subintervals is doubled (i.e., step sizes are halved), an (almost) linear reduction in the maximum error is observed, as expected from the order $O(\Delta t) + O(\Delta x)$ of the convergence of the method.

5. Conclusions. Fractional derivatives in space are used to model anomalous diffusion, where particles spread faster than the classical models predict. A two sided fractional PDE allows modeling different flow regime impacts from either side. The implicit Euler method, based on a modified Grünwald approximation to the fractional

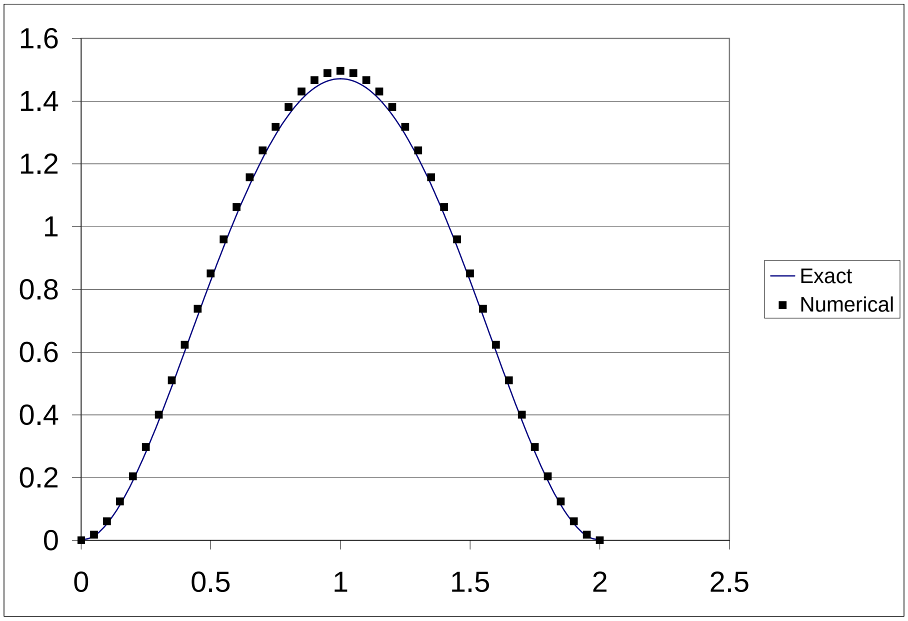

FIG. 4.1. *Comparison of exact solution for the example problem at time* $t = 1.0$ *to the numerical solution from the implicit Euler method with* $\Delta t = 1/40$ *and* $\Delta x = 1/20$.

Δt	Δx	Maximum Error
0.1000	0.200	0.1417
0.0500	0.100	0.0571
0.0250	0.050	0.0249
0.0125	0.025	0.0113

FIG. 4.2. *Maximum error behavior versus gridsize reduction for the example problem.*

derivative, is consistent and unconditionally stable. If the usual Grünwald approximation is used, the implicit Euler method is unstable. The explicit Euler method, using the shifted Grünwald method to solve the two-sided fractional PDEs, is conditionally stable. The stability results in the fractional PDE case are a generalization and unification for the corresponding results in the classical parabolic and hyperbolic PDEs.

REFERENCES

[1] D.A. Benson, S.W. Wheatcraft and M.M. Meerschaert, *The fractional-order governing equation of Lévy motion.* Water Resour. Res. 36 (2000), pp. 1413–1424.
[2] A. Chaves, *A fractional diffusion equation to describe Lévy flights.* Phys. Lett. A 239 (1998), pp. 13–16.
[3] G.J. Fix and J.P. Roop, Least squares finite element solution of a fractional order two-point boundary value problem. *Computers Math. Applic.*, to appear.
[4] R. Gorenflo, F. Mainardi, E. Scalas and M. Raberto, *Fractional calculus and continuous-time finance. III. The diffusion limit.* Mathematical Finance, (Konstanz, 2000), Trends Math., Birkhäuser, Basel, pp. 171–180.
[5] E. Isaacson and H.B. Keller, *Analysis of Numerical Methods.* Wiley, New York, 1966.

[6] F. Liu, V. Ahn and I. Turner, *Numerical solution of the fractional advection-dispersion equation*, preprint.
[7] M. Meerschaert, C. Tadjeran, *Finite difference approximations for fractional advection-dispersion flow equations*, J. Comput. Appl. Math., to appear. Available at `http://unr.edu/homepage/mcubed/frade.pdf`
[8] K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley and Sons, New York, 1993.
[9] I. Podlubny, *Fractional Differential Equations*. Academic Press, New York, 1999.
[10] R.D. Richtmyer and K.W. Morton, *Difference Methods for Initial-Value Problems*. Krieger Publishing, Malabar, Florida, 1994.
[11] S. Samko, A. Kilbas and O. Marichev, *Fractional Integrals and derivatives: Theory and Applications*. Gordon and Breach, London, 1993.
[12] V.K. Tuan and R. Gorenflo, *Extrapolation to the limit for numerical fractional differentiation*, Z. agnew. Math. Mech. 75 (1995) , pp, 646–648.
[13] C. Tadjeran, D.A. Benson, and M.M. Meerschaert, *Fractional radial flow and its application to field data*, available at `http://unr.edu/homepage/mcubed/frade_wrr.pdf`