Semi-Markov approach to continuous time random walk limit processes

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Mark M. Meerschaert
Michigan State University
East Lansing, Michigan, USA

Peter Straka
University of New South Wales
Sydney, Australia

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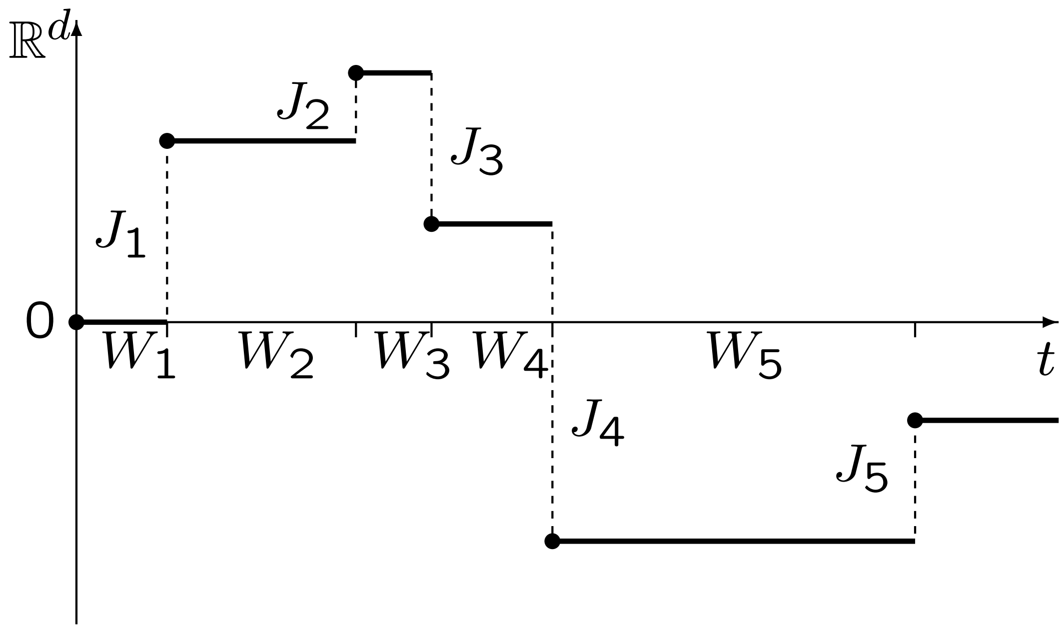
Abstract

Continuous time random walks (CTRWs) are versatile models for anomalous diffusion processes that have found widespread application in the quantitative sciences. Their scaling limits are typically non-Markovian, and the computation of their finite-dimensional distributions is an important open problem. This paper develops a general semi-Markov theory for CTRW limit processes in d-dimensional Euclidean space with infinitely many particle jumps (renewals) in finite time intervals. The particle jumps and waiting times can be coupled and vary with space and time. By augmenting the state space to include the scaling limits of renewal times, a CTRW limit process can be embedded in a Markov process. Explicit analytic expressions for the transition kernels of these Markov processes are then derived, which allow the computation of all finite dimensional distributions for CTRWlimits. Two examples illustrate the proposed method.

Continuous time random walks



The CTRW is a Markov process in space-time:

$$(S_n, T_n) = (J_1, W_1) + \cdots (J_n, W_n).$$

CTRW limit process

Assume that the CTRW at scale $c > 0$ converges

$$(S^c_{[cu]}, T^c_{[cu]}) = (A_0, D_0) + \sum_{k=1}^{[cu]} (J^c_k, W^c_k) \Rightarrow (A_u, D_u)$$

in $\mathbb{D}([0,\infty), \mathbb{R}^{d+1})$ as $c \to \infty$. If $N^c_t = \max\{k \geq 0 : T^c_k \leq t\}$ then

$$X^c_t = S^c_{N^c_t}$$

is the location of a randomly selected particle at time $t \geq 0$. Assume D_u strictly increasing and unbounded. Then

$$X^c_t \Rightarrow X_t := (A_{E_t -})^+ \quad \text{in } \mathbb{D}([0,\infty), \mathbb{R}^d) \text{ as } c \to \infty,$$

where $E_t = \inf\{u > 0 : D_u > t\}$ is the first passage time of D_u.

Two examples

Example 1: Take $\mathbb{P}[W_n^c > t] = c^{-1}t^{-\beta}/\Gamma(1-\beta)$ iid, $J_n^c = c^{-1}$.

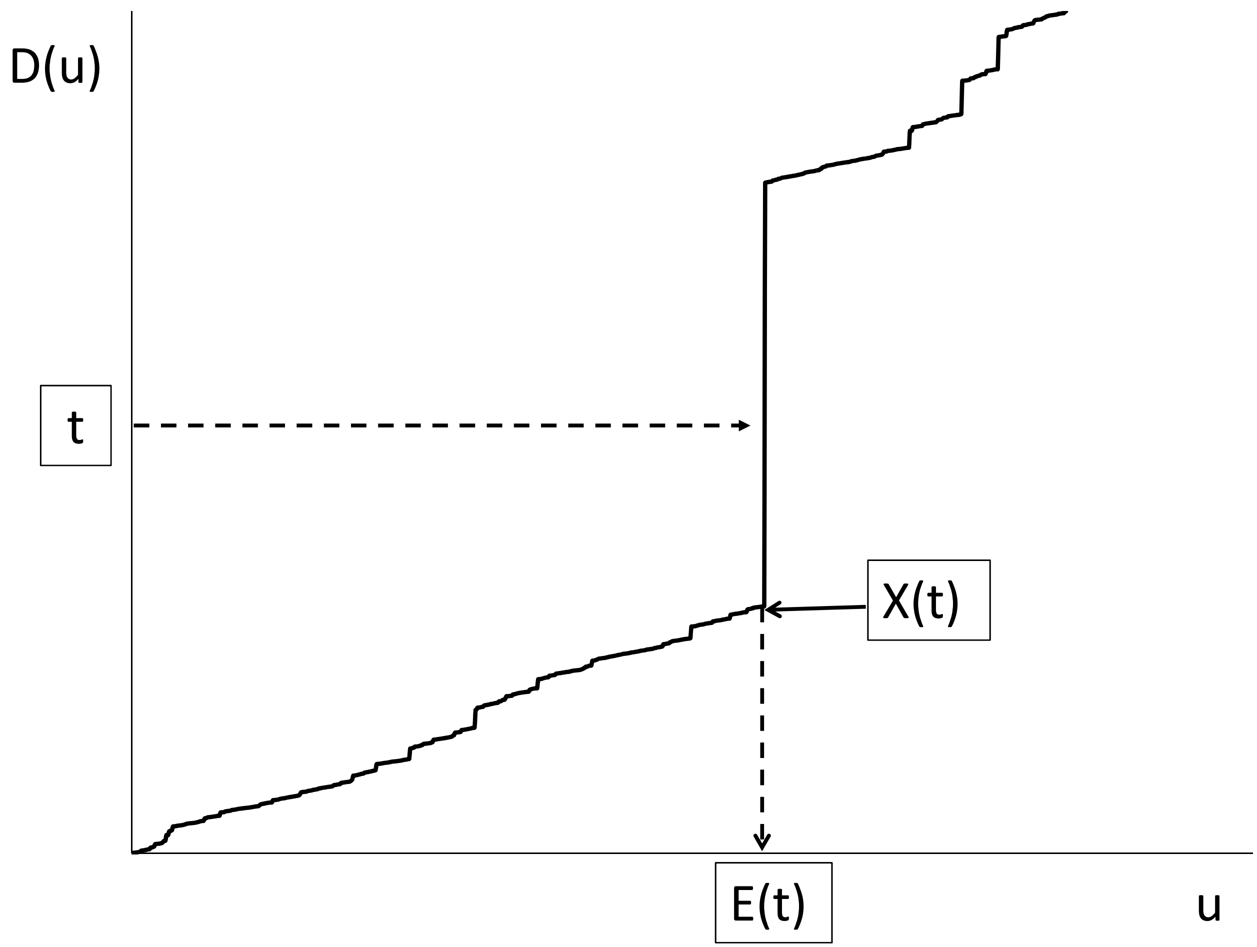
Then $(A_u, D_u) = (u, D_u)$, D_u is the β-stable subordinator with $\mathbb{E}[e^{-sD_u}] = e^{-us^\beta}$, and $X_t = A_{E_t} = E_t$, the inverse subordinator.

Example 2: Take $\mathbb{P}[W_n^c > t] = c^{-1}t^{-\beta}/\Gamma(1-\beta)$ iid, $J_n^c = W_n^c$.

Then $(A_u, D_u) = (D_u, D_u)$, and $X_t = (D_{E_t-})^+$ is the height of D_u just before it jumps over level $t > 0$. Note $\mathbb{P}(X_t < t) = 1$.

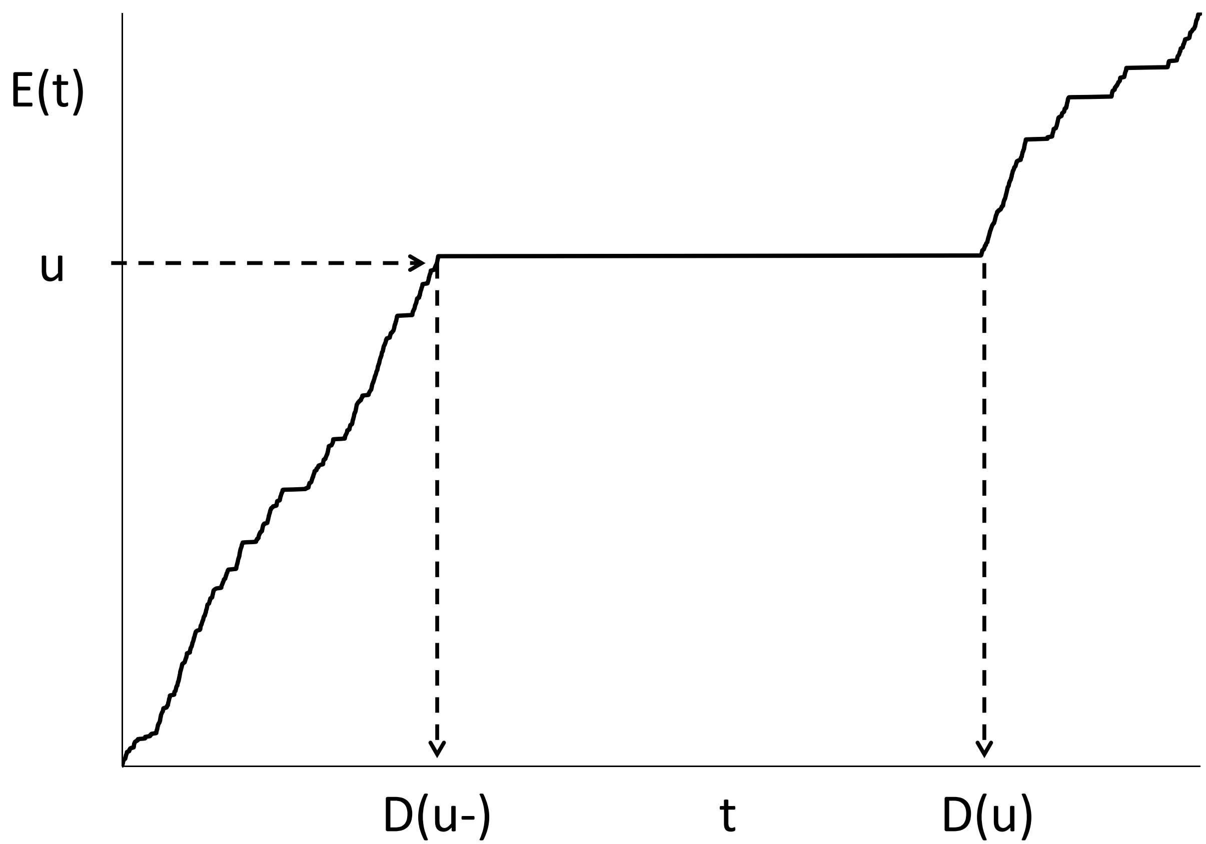
One sample path

Sample path of D_u (a pure jump process) showing E_t and X_t. Both E_t and X_t are non-Markov processes.



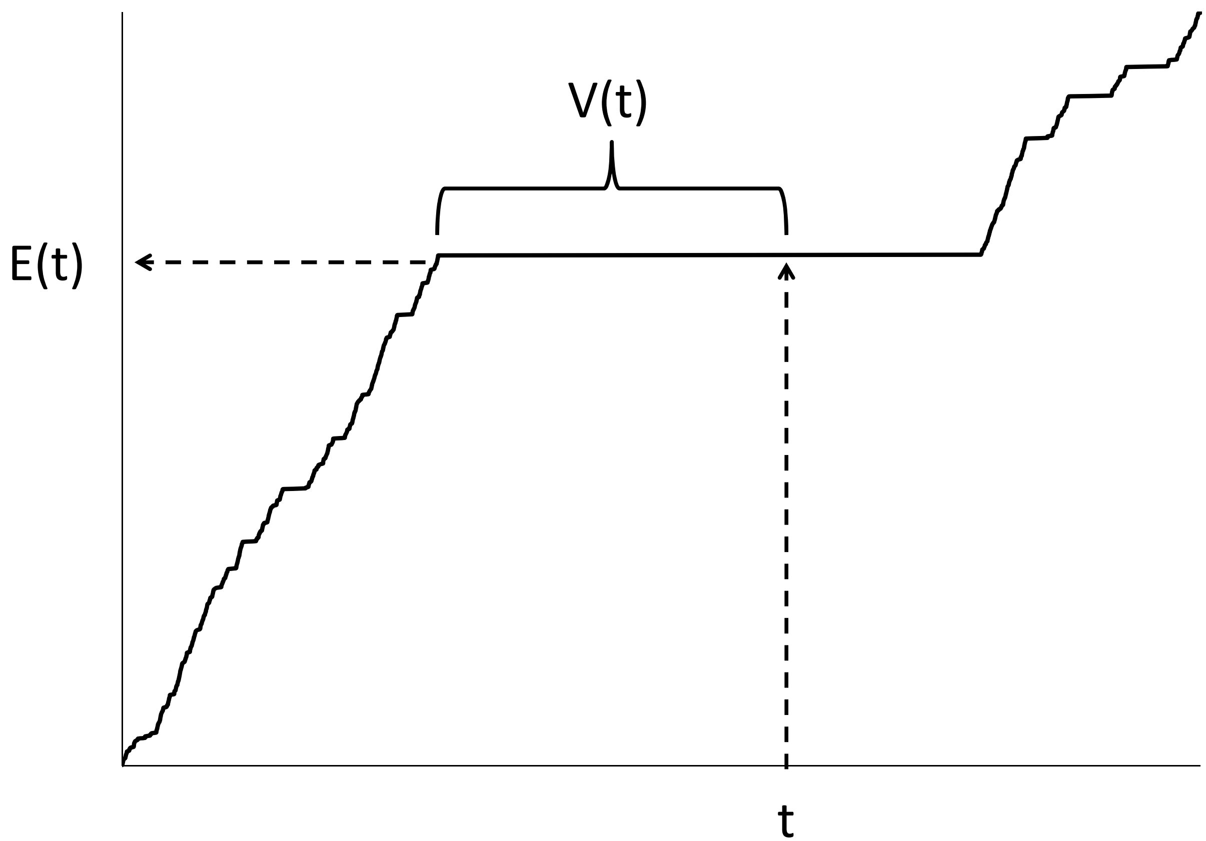
Inverse processes

Graph of E_t is just the graph of D_u with the axes flipped.



Markov embedding

Let V_t denote the spent waiting time for E_t. Extend an idea from renewal theory: Any (X_{t-}, V_{t-}) is an $\mathcal{F}_{E_{t-}}$ Markov process.



Assumptions

The semigroup $T_u f(x,t) = \mathbb{E}^{x,t}[f(A_u, D_u)]$ on $C_0(\mathbb{R}^{d+1})$ has generator in jump-diffusion form, as in Applebaum Eq. (6.42):

$$\begin{aligned}\mathcal{A}f(x,t) = &\sum_{i=1}^{d} b_i(x,t)\partial_{x_i} f(x,t) + \gamma(x,t)\partial_t f(x,t) \\ &+ \frac{1}{2}\sum_{1\le i,j\le d} a_{ij}(x,t)\partial^2_{x_i x_j} f(x,t)\end{aligned}$$
$$+\int \left[f(x+y, t+w) - f(x,t) - \sum_{i=1}^{d} h_i(y,w)\partial_{x_i} f(x,t) \right] K(x,t;dy,dw)$$

where $h_i(x,t) = x_i \mathbf{1}\{(x,t) \in [-1,1]^{d+1}\}$, (a_{ij}) is non-negative definite, jump kernel $K(x,t;dy,dw)$ on $(dy,dw) \in \mathbb{R}^d \times [0,\infty)$, $\gamma(x,t) \ge 0$, and

$$\int \left[1 \wedge (\|y\|^2 + |w|) \right] K(x,t;dy,dw) < \infty \ \forall \ (x,t) \in \mathbb{R}^{d+1}.$$

Main Result: Finite dimensional distributions

Time-invariant case: Transition probabilities of (X_{t-}, V_{t-}) are

$$\begin{aligned} P_t(x_0, 0; dx, dv) &= \gamma(x,t) u^{x_0,t_0}(x,t)\, dx\, \delta_0(dv) \\ &\quad + K(x_0; \mathbb{R}^d \times [v,\infty)) U^{x_0,t_0}(dx, t-dv) \mathbf{1}\{0 \le v \le t\}, \\ P_t(x_0, v_0; dx, dv) &= \delta_{x_0}(dx) \delta_{v_0+t}(dv) K_{v_0}(x_0; \mathbb{R}^d \times [v_0+t, \infty)) \\ &\quad + \int_{y \in \mathbb{R}^d} \int_{w \in [v_0, v_0+t)} P_{v_0+t-w}(x_0+y, 0; dx, dv) K_{v_0}(x_0; dy, dw), \end{aligned}$$

where the 0-potential (mean occupation measure)

$$\int f(x,v) U^{x_0,t_0}(dx, dv) = \mathbb{E}^{x_0,t_0}\left[\int_0^\infty f(A_u, D_u) du\right]$$

has density $u^{x_0,t_0}(x,v)$, and the conditional jump intensity

$$K_t(x_0; dx, dv) = \frac{K\left(x_0; dx, dv \mathbf{1}\{v > t\}\right)}{K\left(x_0; \mathbb{R}^d \times [t, \infty)\right)},$$

Proof: Sample path analysis, compensation formula.

Example 1: FDD of E_t

Space drift $b = 1$, time drift $\gamma = 0$, diffusion $a = 0$.

D_u has PDF $g(t, u)$, Lévy measure $\phi[y, \infty) = y^{-\beta}/\Gamma(1-\beta)$.

0-potential density $u^{x_0,t_0}(x, t) = g(t - t_0, x - x_0)\mathbf{1}\{t > t_0, x > x_0\}$

Jump intensity $K(x, t; dy, dw) = \delta_0(dy)\beta w^{-\beta-1}dw/\Gamma(1-\beta)$

Conditional jump intensity

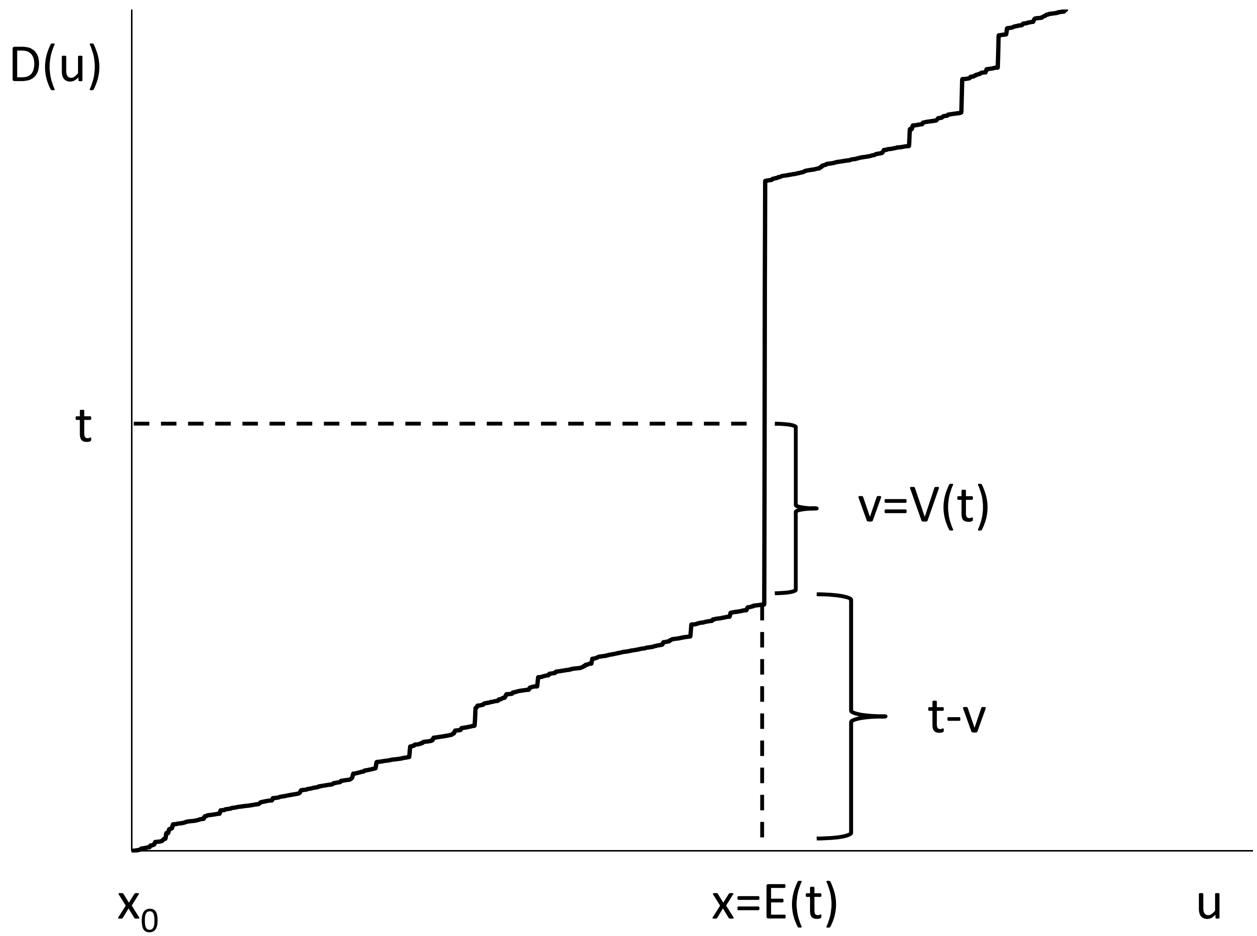
$$
\begin{aligned}
K_v(x, t; dy, dw) &= \frac{\delta_0(dy)\beta w^{-\beta-1}dw/\Gamma(1-\beta)}{v^{-\beta}/\Gamma(1-\beta)} \\
&= \delta_0(dy)\beta(v/w)^{\beta}dw/w
\end{aligned}
$$

Transition probability (for $t_0 = 0$ and $v_0 = 0$)

$$
P_t(x_0, 0; dx, dv) = \frac{v^{-\beta}}{\Gamma(1-\beta)} g(t - v, x - x_0)\, dx\, dv
$$

Example 1: Case $v_0 = 0$ (point of increase of E_t)

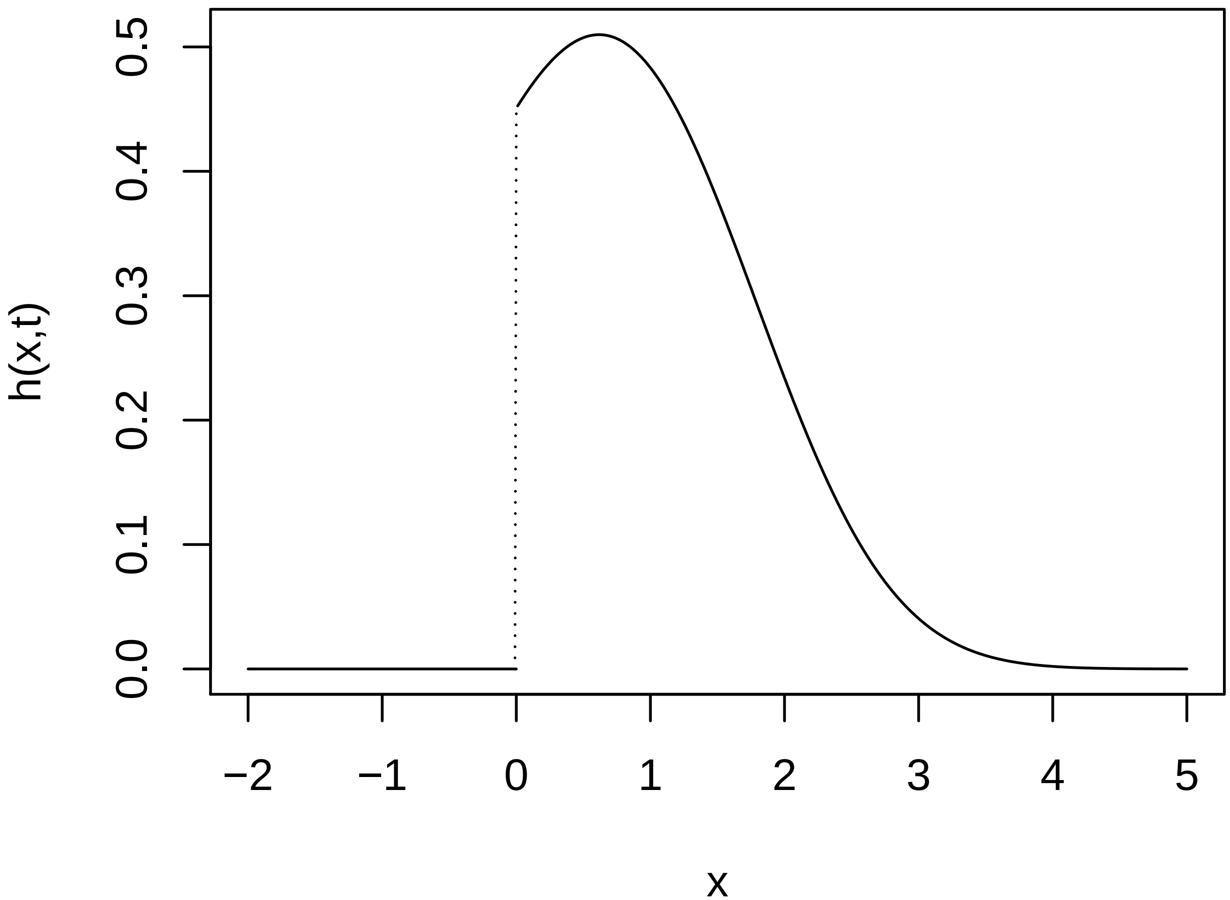
Transition probability $P_t(x_0, 0; dx, dv) = \frac{v^{-\beta}}{\Gamma(1-\beta)} g(t-v, x-x_0)\, dx\, dv$



Inverse stable subordinator density

Since $E_0 = 0$, E_t has density $h(x,t) = \int_0^t \frac{v^{-\beta}}{\Gamma(1-\beta)} g(t-v,x)\,dv$.

Here $\beta = 0.6$ and $t = 1$. Can show $h(0+,t) = t^{-\beta}/\Gamma(1-\beta)$.



Example 1: Case $v_0 > 0$ (resting point of E_t)

Depends on $v_0 = 0$ case. We compute

$$P_t(x_0, v_0; dx, dv) = \delta_{x_0}(dx)\delta_{v_0+t}(dv)\left(\frac{v_0 + t}{v_0}\right)^{-\beta} \mathbf{1}\{v_0 > 0\}$$

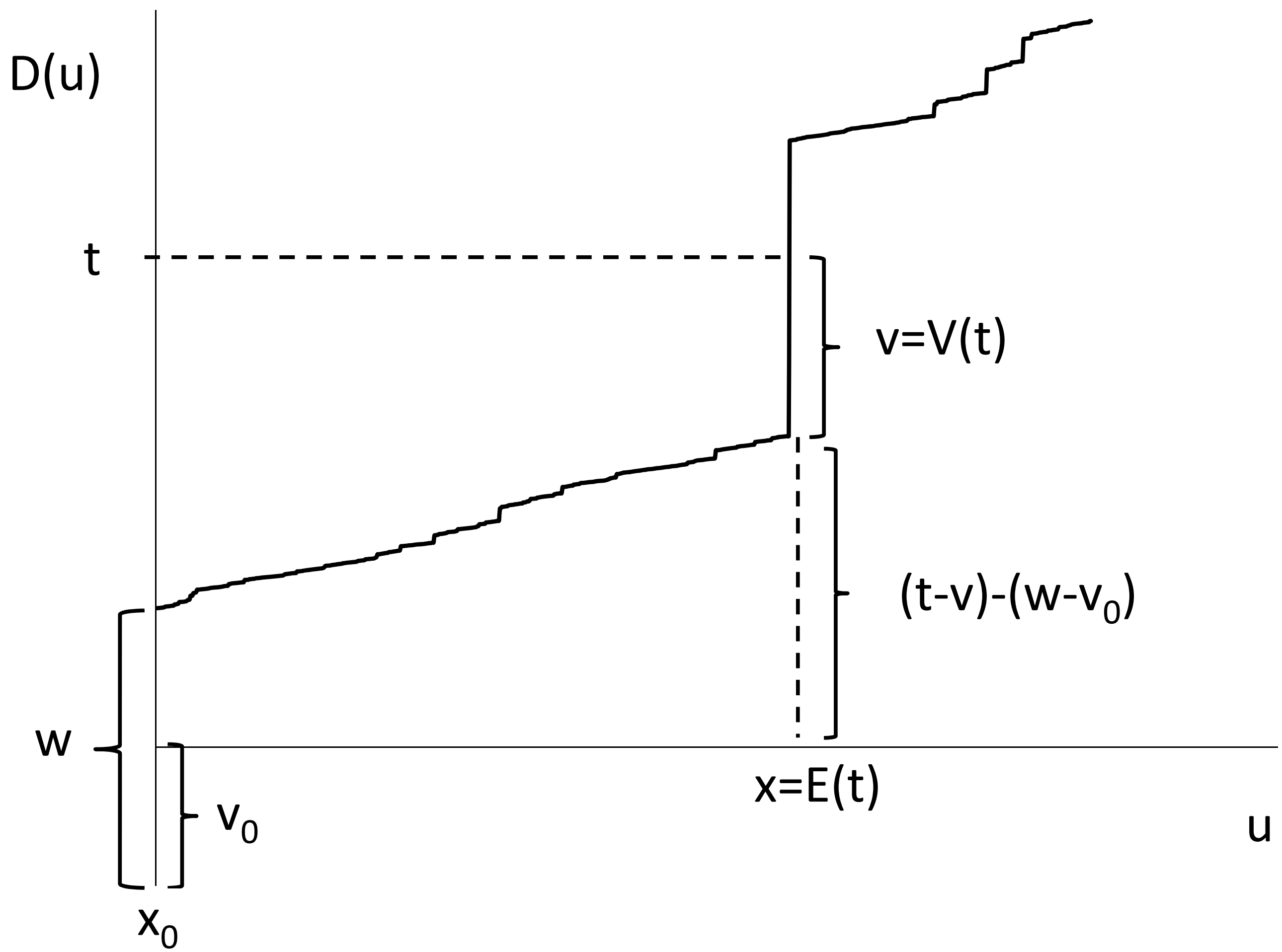
$$+ \left(\frac{v}{v_0}\right)^{-\beta} \int_{s=v_0}^{v_0+t-v} g((t-v) - (w - v_0), x - x_0)\frac{\beta w^{-\beta-1}}{\Gamma(1-\beta)} dw\, dx\, dv.$$

The first term represents the chance that $\Delta D_{x_0} > v_0 + t$ given $\Delta D_{x_0} > v_0$, in which case $x = x_0$ and $v = v_0 + t$.

Then E_t continues to rest throughout the interval of length t.

Example 1: Case $v_0 > 0$ (resting point of E_t)

$$\text{2nd term } \left(\frac{v_0}{v}\right)^{\beta} \int_{s=v_0}^{v_0+t-v} g((t-v)-(w-v_0), x-x_0)\frac{\beta w^{-\beta-1}}{\Gamma(1-\beta)} dw\, dx\, dv$$



Example 1: Finite dimensional distributions

Two time points: Put together cases $v_0 = 0$ and $v_0 > 0$ to get

$$\begin{aligned}
&\mathbb{P}^{0,0}[E_{t_1} \in dx, V_{t_1} \in dv, E_{t_2} \in dy, V_{t_2} \in dw] \\
&= P_{t_1}(0,0; dx, dv) P_{t_2 - t_1}(x, v; dy, dw) \\
&= \frac{v^{-\beta}}{\Gamma(1-\beta)} g(t_1 - v, x)\, dx\, dv \mathbf{1}\{x > 0, 0 < v < t_1\} \\
&\times \Bigg[\delta_x(dy) \delta_{v+t_2-t_1}(dw) \left(\frac{v + t_2 - t_1}{v} \right)^{-\beta} \\
&+ \left(\frac{v}{w} \right)^{\beta} \int_{s=v}^{v+t_2-t_1-w} g((t_2 - t_1 - w) - (s - v), y - x) \frac{\beta s^{-\beta-1}}{\Gamma(1-\beta)} ds\, dy\, dw \Bigg]
\end{aligned}$$

Integrate out v, w to get joint PDF of E_{t_1}, E_{t_2}.

Example 2: FDD of X_t

Recall $X_t = (D_{E_t-})^+$, height of D_u before jump over level t.

Space drift $b = 0$, time drift $\gamma = 0$, diffusion $a = 0$.

(D_u) has PDF $g(t, u)$, Lévy measure $\phi[y, \infty) = y^{-\beta}/\Gamma(1-\beta)$.

Jump Intensity is $K(x, t; dy, dw) = \delta_w(dy)\beta w^{-\beta-1}dw/\Gamma(1-\beta)$

0-potential density $U^{x_0,0}(dx, dt) = \delta_{x_0+t}(dx)t^{\beta-1}dt/\Gamma(\beta)$

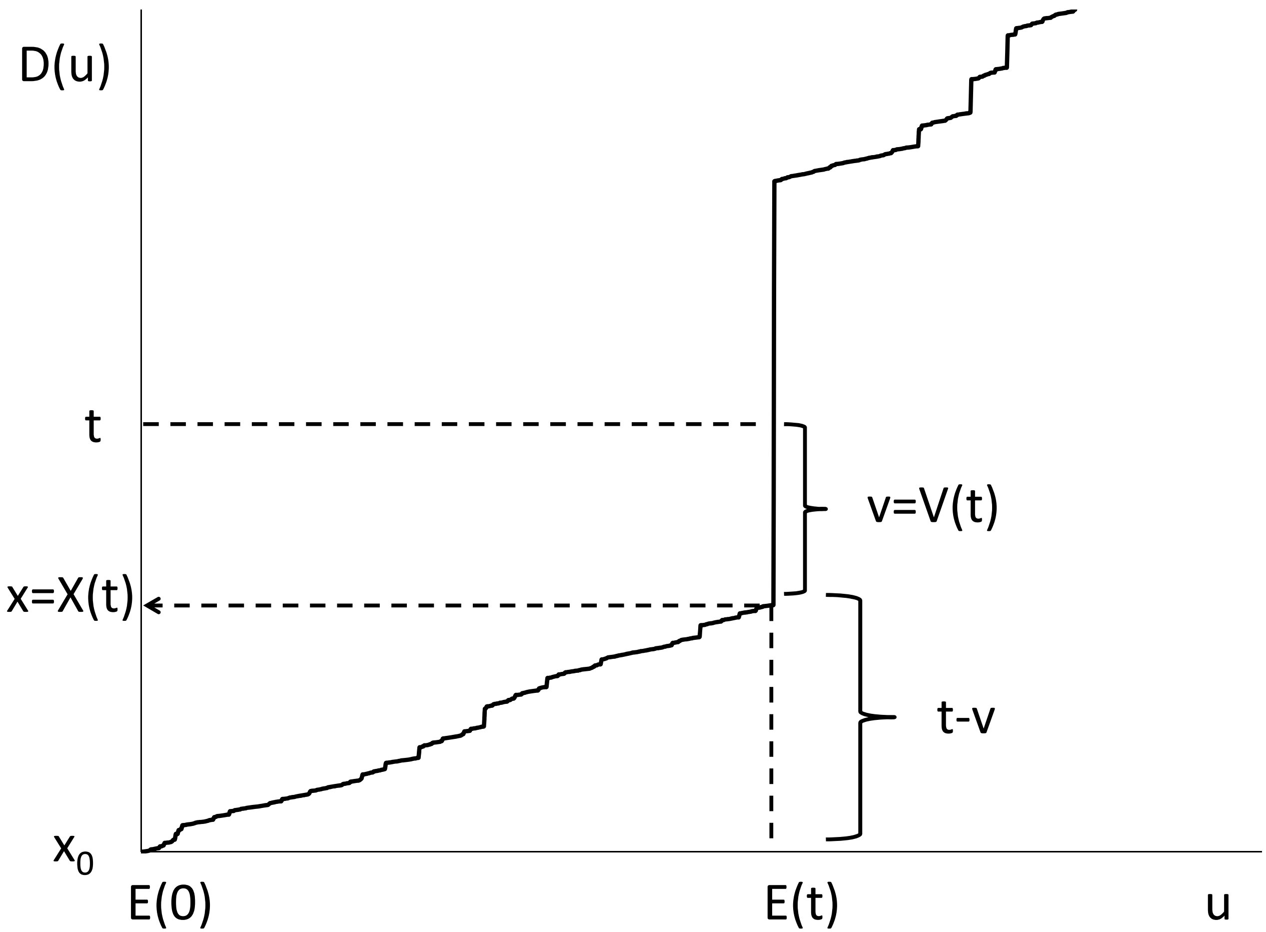
Conditional jump intensity $K_v(x, t; dy, dw) = \delta_w(dy)\beta(v/w)^\beta dw/w$.

Transition probability (for $t_0 = 0$ and $v_0 = 0$) is

$$P_t(x_0, 0; dx, dv) = \delta_{x_0+t-v}(dx)\frac{v^{-\beta}}{\Gamma(1-\beta)}\frac{(t-v)^{\beta-1}}{\Gamma(\beta)}dv\,\mathbf{1}\{0 < v < t\}$$

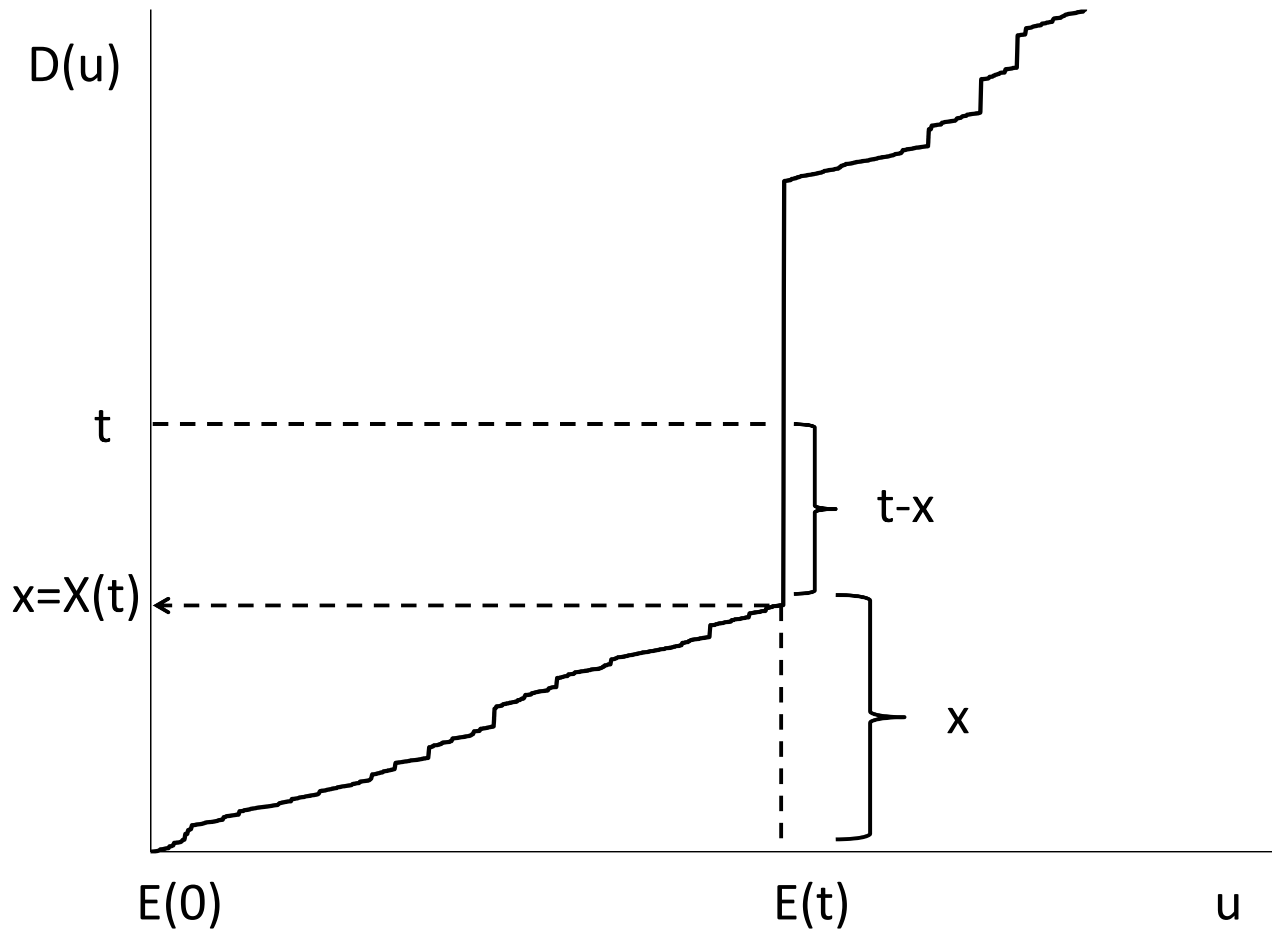
Example 2: Case $v_0 = 0$ (point of increase of E_t)

$$P_t(x_0, 0; dx, dv) = \delta_{x_0+t-v}(dx) \frac{v^{-\beta}}{\Gamma(1-\beta)} \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} dv \, \mathbf{1}\{0 < v < t\}$$



Example 2: Probability density of X_t (given $X_0 = 0$)

Probability density $f(x,t) = \dfrac{(t-x)^{-\beta}}{\Gamma(1-\beta)} \dfrac{x^{\beta-1}}{\Gamma(\beta)} \mathbf{1}\{0 < x < t\}$



Example 2: Case $v_0 > 0$ (resting point of E_t)

Depends on $v_0 = 0$ case. We compute

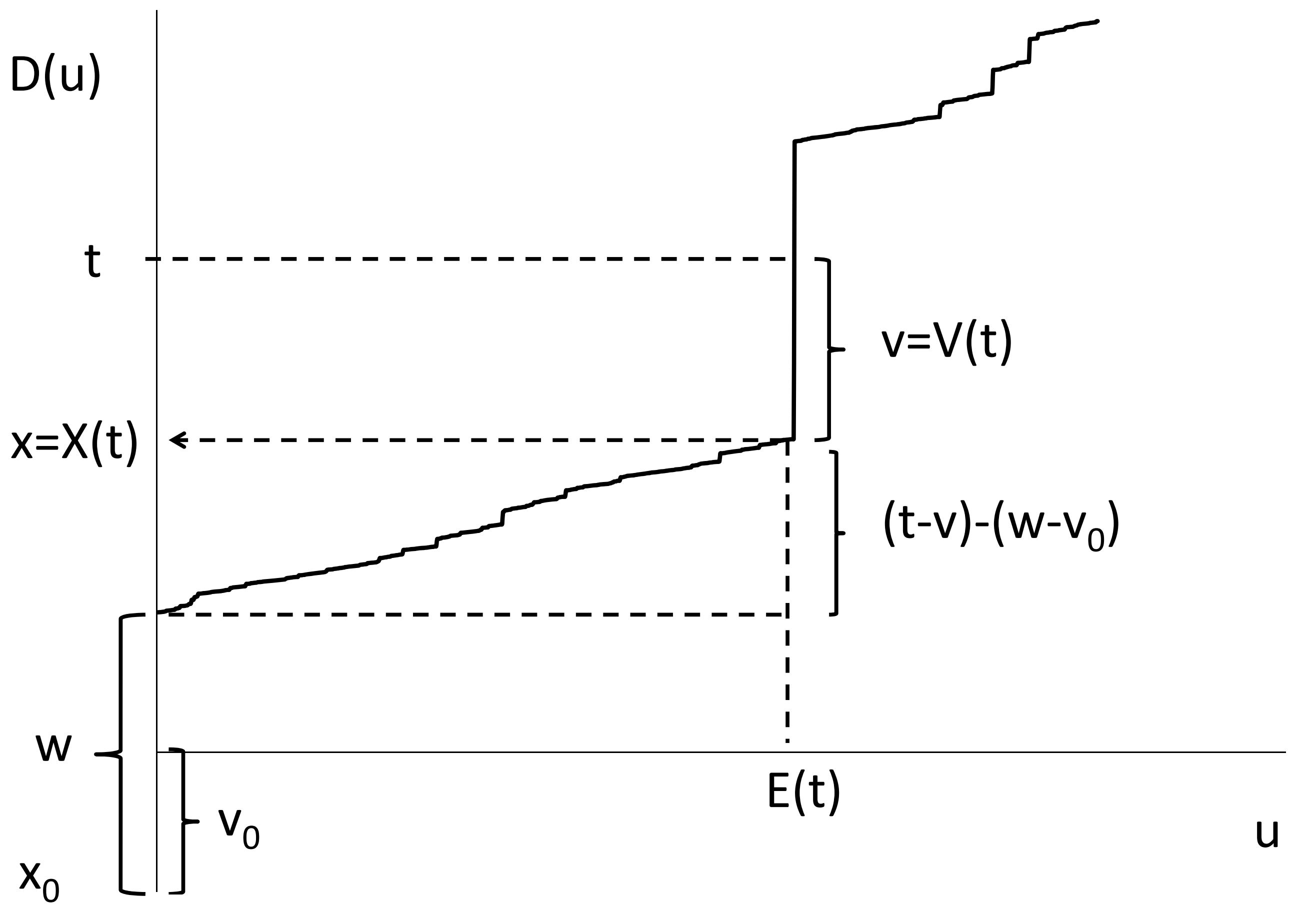
$$P_t(x_0, v_0; dx, dv) = \delta_{x_0}(dx)\delta_{v_0+t}(dv)\left(\frac{v_0+t}{v_0}\right)^{-\beta}$$
$$+ \int_{w=v_0}^{v_0+t}\left(\frac{v}{v_0}\right)^{-\beta}\delta_{x_0+v_0+t-v}(dx)\frac{(v_0+t-w-v)^{\beta-1}}{\Gamma(\beta)}$$
$$\times\, \mathbf{1}\{0 < v < v_0+t-w\}\frac{\beta w^{-\beta-1}}{\Gamma(1-\beta)}dw\, dv$$

The first term represents the chance that $\Delta D_{x_0} > v_0 + t$ given $\Delta D_{x_0} > v_0$, in which case $x = x_0$ and $v = v_0 + t$ (as in Ex. 1).

Then X_t remains constant throughout the interval of length t.

Example 2: Case $v_0 > 0$ (2nd term)

$$\delta_{x_0+v_0+t-v}(dx)\left(\frac{v_0}{v}\right)^{\beta}\int_{w=v_0}^{v_0+t}\frac{((t-v)-(w-v_0))^{\beta-1}}{\Gamma(\beta)}\frac{\beta w^{-\beta-1}}{\Gamma(1-\beta)}dw\,dv$$



Summary

- CTRW limit process is not Markov
- Embed in a Markov process
- Can include coupled space-time jumps
- Can be space-time inhomogeneous
- All FDD can be computed
- Solves an important problem in physics

References

1. D.D. Applebaum, *Lévy Processes and Stochastic Calculus.* 2nd Ed. Cambridge Studies in Advanced Mathematics **116**, Cambridge University Press, 2009.

2. **M.M. Meerschaert and P. Straka, Semi-Markov approach to continuous time random walk limit processes. The Annals of Probability 42 (2014), No. 4, pp. 1699–1723.**

3. M.M. Meerschaert and P. Straka, Inverse stable subordinators. *Mathematical Modeling of Natural Phenomena* **8** (2013), No. 2, pp. 1–16.

Example 1: Computing the 0-potential

Recall that $A_u = u$ and D_u has PDF $g(t,u)$ given $D_0 = 0$. Make a change of variables $y = u + x_0$, $w = t + t_0$ to see that

$$\begin{aligned}
\mathbb{E}^{x_0,t_0}\left[\int_0^\infty f(A_u, D_u)du\right] &= \mathbb{E}^{0,0}\left[\int_0^\infty f(A_u + x_0, D_u + t_0)du\right] \\
&= \int_0^\infty \int_0^\infty f(u + x_0, t + t_0) g(t,u)\, dt\, du \\
&= \int_{y=x_0}^\infty \int_{w=t_0}^\infty f(y,w) g(w - t_0, y - x_0)\, dw\, dy \\
&= \int_{y=x_0}^\infty \int_{w=t_0}^\infty f(y,w) u^{x_0,t_0}(y,w)\, dw\, dy
\end{aligned}$$

so that $u^{x_0,t_0}(y,w) = g(w - t_0, y - x_0)\, \mathbf{1}\{w > t_0, y > x_0\}$.

Example 2: Computing the 0-potential

Recall that $A_u = D_u$ has PDF $g(t,u)$ given $D_0 = 0$. WLOG $t_0 = 0$. Then

$$\begin{aligned}
\mathbb{E}^{x_0,0}\left[\int_0^\infty f(A_u, D_u)du\right] &= \mathbb{E}^{0,0}\left[\int_0^\infty f(D_u + x_0, D_u)du\right] \\
&= \int_0^\infty \int_0^\infty f(t + x_0, t)g(t,u)\,dt\,du \\
&= \int_{x\in\mathbb{R}} \int_0^\infty \int_0^\infty f(x,t)g(t,u)\,du\,\delta_{x_0+t}(dx)\,dt \\
&= \int_{x\in\mathbb{R}} \int_{t=0}^\infty f(x,t)U^{x_0,0}(dx,dt)
\end{aligned}$$

so that

$$U^{x_0,0}(dx,dt) = \delta_{x_0+t}(dx)\int_{u=0}^\infty g(t,u)\,du\,dt = \delta_{x_0+t}(dx)\frac{t^{\beta-1}dt}{\Gamma(\beta)}$$

since $\int_0^\infty e^{-st}\int_0^\infty g(t,u)\,du = \int_0^\infty e^{-us^\beta}du = s^{-\beta} = \mathcal{L}\left[t^{\beta-1}/\Gamma(\beta)\right]$