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A bivariate infinitely divisible distribution with exponential and Mittag–Leffler marginals

Tomasz J. Kozubowski [a,*], Mark M. Meerschaert [b]

[a] *Department of Mathematics and Statistics, University of Nevada, Reno NV 89557, USA*

[b] *Department of Statistics and Probability, Michigan State University, A416 Wells Hall, East Lansing, MI 48824, USA*

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ABSTRACT

We introduce a bivariate distribution supported on the first quadrant with exponential, and heavy tailed Mittag–Leffer, marginal distributions. Although this distribution belongs to the class of geometric operator stable laws, it is a rather special case that does not follow their general theory. Our results include the joint density and distribution function, Laplace transform, conditional distributions, joint moments, and tail behavior. We also establish infinite divisibility and stability properties of this model, and clarify its connections with operator stable and geometric operator stable laws.

1. Introduction

Suppose that X_1 is a stable subordinator with scale parameter $\sigma > 0$ and Laplace transform (LT)

$$\mathbb{E}e^{-tX_1} = e^{-\sigma^\alpha t^\alpha}, \quad t \in \mathbb{R}_+, \tag{1.1}$$

and let E be a standard exponential random variable, independent of X_1. In this paper we consider a bivariate distribution defined through the stochastic representation

$$Y = (Y_1, Y_2) \stackrel{d}{=} (E^{1/\alpha} X_1, \eta E), \quad \eta > 0. \tag{1.2}$$

Clearly, the marginal distribution of Y_2 is exponential with mean η and the probability density function (PDF)

$$f(x) = \frac{1}{\eta} e^{-x/\eta}, \quad x > 0. \tag{1.3}$$

On the other hand, Y_1 has the *Mittag–Leffler* distribution (see, e.g., Pillai, 1990) with the LT

$$\mathbb{E}e^{-tY_1} = \frac{1}{1+\sigma^\alpha t^\alpha}, \quad t \in \mathbb{R}_+, \tag{1.4}$$

and the PDF

$$f_1(y) = \frac{\sin \pi\alpha}{\sigma\pi} \int_0^\infty \frac{u^\alpha e^{-xu/\sigma} du}{1+u^{2\alpha}+2u^\alpha \cos \pi\alpha} = \frac{1}{\sigma} \int_0^\infty z^{-1/\alpha} s_\alpha \left(\frac{y}{\sigma z^{1/\alpha}} \right) e^{-z} dz, \tag{1.5}$$

* Corresponding author. Tel.: +1 775 7846643; fax: +1 775 7846378.
E-mail address: tkozubow@unr.edu (T.J. Kozubowski).

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where s_α is the density of the standard stable subordinator ($\sigma = 1$ in (1.1)). We shall refer to the above distribution as BEML distribution (**b**ivariate with **e**xponential and **M**ittag-**L**effler marginals), denoting it by $\mathcal{BEML}_\alpha(\sigma, \eta)$.

The exponential distribution is one of the most important models of applied probability, with many applications in almost any area of applied research. Its generalization and younger sibling – the Mittag–Leffler distribution, also known as positive Linnik law (see, e.g., Christoph and Schreiber, 2000; Huillet, 2000; Lin, 2001) – has gained popularity in recent years due to its connections with the stochastic solution of the Cauchy problem for PDEs with fractional derivatives (see, e.g, Fulger et al., 2008, and the references therein) and relaxation theory in complex, disordered systems (see, e.g., Kotulski, 1995; Weron and Kotulski, 1996). This model was studied in the context of random stability (see, e.g., Bunge, 1996; Jayakumar and Suresh, 2003; Klebanov et al., 2006; Kozubowski, 1994) as a special case of *geometric stable* laws (see, e.g., Klebanov et al., 2006, and the references therein) and extended to time series (see, e.g., Jayakumar, 2003; Jayakumar and Pillai, 1993a), and its theoretical properties and applications has attracted the attention of numerous researchers (see, e.g, Jayakumar, 2003; Jayakumar and Pillai, 1993b, 1996; Jayakumar and Suresh, 2003; Kozubowski, 2001; Lin, 1998; Pillai, 1990). The bivariate model that links the two distributions should also be valuable for modeling data with both power and exponential tail behavior of one dimensional components. In particular, as we shall see below, this bivariate distribution provides a useful approximation of the *magnitude* X and *duration* N connected with hydro-climatic episodes (see, e.g., Biondi et al., 2005), where

$$(X, N) \stackrel{d}{=} \left(\sum_{i=1}^{N_p} u^{(i)}, N_p \right) = \sum_{i=1}^{N_p} (u^{(i)}, 1) \tag{1.6}$$

and the parameter $p \in (0, 1)$ is close to zero. Here, the $\{u^{(i)}\}$ are independent and identically distributed (IID) positive quantities with *infinite mean* (representing a climatic variable such as precipitation) while the N_p is independent of the $\{u^{(i)}\}$ geometric random variables with distribution

$$\mathbb{P}(N_p = n) = p(1-p)^{n-1}, \quad n \in \mathbb{N}, \tag{1.7}$$

representing the number of climatic events (such as rainfall events) in a given time interval. This provides a *heavy-tail* generalization of the case where the $\{u^{(i)}\}$ are *light tailed* (e.g., exponential), where the BEG model of Kozubowski and Panorska (2005) and its generalizations (see Kozubowski et al., 2008) are useful in approximating the distribution (1.6). This new model should also play an important role in other areas where events of random magnitudes occur independently in time, and the joint distribution of their number and the total magnitude is of concern. Examples include insurance claims in actuarial science, or magnitudes of earthquakes in geophysics.

In Section 2 we present basic properties of the BEML model, including the joint density and distribution function, Laplace transform, conditional distributions, joint moments, and tail behavior, as well as its divisibility and stability properties. In Section 3 we clarify its connections with operator stable and geometric operator stable laws, and briefly mention possible applications.

2. Definition and basic properties

Suppose that X_1 is a stable subordinator with the LT (1.1). Then the random vector $X = (X_1, X_2)$, where $X_2 = \eta > 0$ (constant with probability one), has the LT of the form

$$\phi(t, s) = \mathbb{E}e^{-tX_1 - sX_2} = e^{-\sigma^\alpha t^\alpha - \eta s}, \quad (t, s) \in \mathbb{R}^2_+. \tag{2.1}$$

This infinitely divisible distribution is operator stable (see, e.g., Jurek and Mason, 1993; Meerschaert and Scheffler, 2001), in the sense that

$$A_n \sum_{i=1}^{n} X^{(i)} \stackrel{d}{=} X, \quad n \in \mathbb{N}, \tag{2.2}$$

where the $\{X^{(i)}\}$ are IID copies of X and

$$A_n = \begin{bmatrix} n^{-1/\alpha} & 0 \\ 0 & n^{-1} \end{bmatrix} := \mathrm{diag}(n^{-1/\alpha}, n^{-1}). \tag{2.3}$$

Note that we can also write $A_n = n^{-B} = \exp(-B \log n)$, where $B = \mathrm{diag}(1/\alpha, 1)$ and $\exp(A) = I + A + A^2/2! + \cdots$ is the usual matrix exponential. The matrix B is called an exponent of the operator stable random vector X, see Meerschaert and Scheffler (2001). It follows that the random vector Y with the LT

$$\psi(t, s) = \frac{1}{1 - \log \phi(t, s)} = \frac{1}{1 + \sigma^\alpha t^\alpha + \eta s}, \quad (t, s) \in \mathbb{R}^2_+, \tag{2.4}$$

is (strictly) *operator geometric stable* (cf. Kozubowski et al., 2005), that is

$$A_p \sum_{i=1}^{N_p} Y^{(i)} \stackrel{d}{=} Y, \quad p \in (0, 1). \tag{2.5}$$

Here, $Y, Y^{(1)}, Y^{(2)}, \ldots$ are IID random vectors with the LT (2.4), N_p is a geometric variable (1.7), independent of the $\{Y^{(i)}\}$, and $A_p = p^B$. Moreover, we have $Y \stackrel{d}{=} E^B X$, that is we have the stochastic representation (1.2). The following definition summarizes this discussion.

Definition 2.1. A random vector $Y = (Y_1, Y_2)$ given by the LT (2.4) or the stochastic representation (1.2), where E is standard exponential and X_1 is a stable subordinator with LT (1.1), independent of E, is said to have a BEML distribution with tail parameter $\alpha \in (0, 1)$ and scale parameters $\sigma, \eta > 0$. This distribution is denoted by $\mathcal{BEML}_\alpha(\sigma, \eta)$.

It is clear from the above representation, that the conditional distribution of Y_1 given $Y_2 = y > 0$ coincides with that of $\sigma_y W$, where W is a standard stable subordinator (with scale parameter equal to 1) and

$$\sigma_y = \frac{\sigma y^{1/\alpha}}{\eta^{1/\alpha}}. \tag{2.6}$$

Since the marginal density of Y_2 is the exponential PDF (1.3), we immediately obtain the following result concerning the PDF of a BEML random vector.

Theorem 2.2. *The distribution function and density of $Y = (Y_1, Y_2) \sim \mathcal{BEML}_\alpha(\sigma, \eta)$ are, respectively,*

$$F(y_1, y_2) = \int_0^{y_2/\eta} S_\alpha\left(\frac{y_1}{\sigma z^{1/\alpha}}\right) \mathrm{e}^{-z} \mathrm{d}z, \quad (y_1, y_2) \in \mathbb{R}_+^2, \tag{2.7}$$

and

$$f(y_1, y_2) = \frac{\eta^{1/\alpha - 1} \mathrm{e}^{-y_2/\eta}}{\sigma y_2^{1/\alpha}} s_\alpha\left(\frac{\eta^{1/\alpha} y_1}{\sigma y_2^{1/\alpha}}\right), \quad (y_1, y_2) \in \mathbb{R}_+^2, \tag{2.8}$$

where S_α and s_α are, respectively, the distribution function and the PDF of the standard stable subordinator.

Dividing the joint PDF (2.8) by the marginal PDF of Y_1 given by (1.5) leads to the conditional PDF of Y_2 given $Y_1 = y > 0$, described in our next result.

Theorem 2.3. *Let $Y = (Y_1, Y_2) \sim \mathcal{BEML}_\alpha(\sigma, \eta)$.*
(i) *The conditional distribution of $Y_2 | Y_1 = y > 0$ is weighted exponential with the PDF*

$$f_{Y_2|Y_1=y}(y_2) = \frac{\omega(y_2)(1/\eta)\mathrm{e}^{-y_2/\eta}}{\int_0^\infty \omega(x)(1/\eta)\mathrm{e}^{-x/\eta} dx}, \quad y_2 > 0. \tag{2.9}$$

The weight function in (2.9) is

$$\omega(x) = x^{-1/\alpha} s_\alpha\left(\frac{y\eta^{1/\alpha}}{\sigma x^{1/\alpha}}\right), \quad x > 0, \tag{2.10}$$

where s_α is the density of the standard stable subordinator.
(ii) *The conditional distribution of $Y_1 | Y_2 = y > 0$ is the same as that of $\sigma_y W$, where W is a standard stable subordinator and σ_y is given by (2.6).*

The following two results deal with tail behavior and joint moments of BEML variables. In the first result, which is straightforward to prove by a Tauberian argument (compare with Samorodnitsky and Taqqu (1994, Property 1.2.15)), we present the exact tail behavior of linear combinations of the components of an BEML random vector, showing that they are heavy tailed with the same tail index α.

Theorem 2.4. *Let $Y = (Y_1, Y_2) \sim \mathcal{BEML}_\alpha(\sigma, \eta)$ and let $(a, b) \in \mathbb{R}_+^2$ with $a^2 + b^2 > 0$. Then, as $x \to \infty$, we have*

$$P(aY_1 + bY_2 > x) \sim \begin{cases} \dfrac{(a\sigma)^\alpha}{\Gamma(1-\alpha)} x^{-\alpha} & \text{for } a \neq 0, \\ \mathrm{e}^{-\frac{x}{b\eta}} & \text{for } a = 0. \end{cases} \tag{2.11}$$

In the second result we give conditions for the existence of joint moments of BEML random vectors.

Theorem 2.5. *Let $Y = (Y_1, Y_2) \sim \mathcal{BEML}_\alpha(\sigma, \eta)$ and let $\alpha_1, \alpha_2 \geq 0$. Then the joint moment $\mathbb{E}|Y_1|^{\alpha_1}|Y_2|^{\alpha_2}$ exists if and only if $\alpha_1 < \alpha$, in which case we have*

$$\mathbb{E}|Y_1|^{\alpha_1}|Y_2|^{\alpha_2} = \frac{\sigma^{\alpha_1}\eta^{\alpha_2}\Gamma\left(\frac{\alpha_1}{\alpha} + \alpha_2 + 1\right)\Gamma\left(1 - \frac{\alpha_1}{\alpha}\right)}{\Gamma(1-\alpha_1)}. \tag{2.12}$$

Proof. The result follows from (1.2) and well-known moment conditions and formulas for standard exponential and standard α-stable subordinator variables E and W, respectively,

$$\mathbb{E}E^p = \Gamma(1+p) \quad \text{for } p > 0, \quad \mathbb{E}W^p = \frac{\Gamma(1-p/\alpha)}{\Gamma(1-p)} \quad \text{for } 0 < p < \alpha. \quad \square$$

Remark 2.6. Note that for $\alpha_2 = 0$ we obtain absolute moments of the Mittag–Leffler distribution (see, e.g., Kozubowski and Panorska, 1996) while for $\alpha_1 = 0$ we get fractional moments of the exponential distribution with mean η. The above formula may be useful in estimating the parameters of BEML laws.

2.1. Divisibility and stability properties

A random vector Y (and its probability distribution) is said to be *geometric infinitely divisible* (GID) if for all $p \in (0, 1)$ we have

$$Y \stackrel{d}{=} \sum_{i=1}^{N_p} Y^{(pi)}, \tag{2.13}$$

where N_p is geometrically distributed random variable (1.7), the variables $\{Y^{(pi)}\}$ are IID for each p, and N_p and $\{Y^{(pi)}\}$ are independent (see, e.g., Klebanov et al., 1984). It is well known that both exponential and ML distributions are GID. Our next result shows that the same property is shared by the BEML distributions.

Proposition 2.7. *Let Y be $\mathcal{BEML}_\alpha(\sigma, \eta)$ with LT (2.4). Then Y is GID and the relation (2.13) holds where the $\{Y^{(pi)}\}$ have the $\mathcal{BEML}_\alpha(p^{1/\alpha}\sigma, p\eta)$ distribution.*

Proof. Let ψ and ψ_p be the LTs of Y and the $\{Y^{(pi)}\}$, respectively. Then, the relation (2.13) takes the form

$$\psi(t, s) = \frac{p\psi_p(t, s)}{1-(1-p)\psi_p(t, s)},$$

which is easily shown to hold when ψ and ψ_p are the LTs corresponding to the $\mathcal{BEML}_\alpha(\sigma, \eta)$ and $\mathcal{BEML}_\alpha(p^{1/\alpha}\sigma, p\eta)$ distributions, respectively. $\square$

Remark 2.8. We note that the BEML distributions are infinitely divisible in the classical sense as well. Indeed, for each integer $n \geq 1$ their LT (2.4) can be expressed as the nth power of $\psi_{1/n}(t, s)$, where

$$\psi_u(t, s) = \left(\frac{1}{1+\sigma^\alpha t^\alpha + \eta s}\right)^u, \quad (t, s) \in \mathbb{R}_+^2, \quad u > 0, \tag{2.14}$$

is the LT of a random vector given by (1.2) with X_1 as before and E having a standard gamma distribution with shape parameter u. In this context, the BEML distribution arises as the distribution of $Y(1)$, where $\{Y(u), u > 0\}$ is a bivariate Lévy process with marginal distributions of $Y(u)$ given by the LT (2.14).

We have already seen above that the BEML distributions have the stability proposition (2.5) with $A_p = \mathrm{diag}(p^{1/\alpha}, p)$. The following result, which is an extension of corresponding stability properties of univariate Mittag–Leffler and exponential distributions (see, e.g., Kotz et al., 2001) and parallels Theorem 3.11 in Kozubowski et al. (2005), shows this property provides a characterization of this class.

Theorem 2.9. *Let $Y, Y^{(1)}, Y^{(2)}, \ldots$ be IID positive bivariate random vectors whose second components have finite mean, and let N_p be a geometrically distributed random variable independent of the sequence $\{Y^{(i)}\}$. Then*

$$S_p = A_p \sum_{i=1}^{N_p} (Y^{(i)} + b_p) \stackrel{d}{=} Y, \quad p \in (0, 1), \tag{2.15}$$

with some diagonal $\{A_p\}$ and $b_p \in \mathbb{R}^2$ if and only if Y has a BEML distribution given by the LT (2.4). Moreover, we must necessarily have $b_p = 0$ and $A_p = \mathrm{diag}(p^{1/\alpha}, p)$ for each p, where $0 < \alpha \leq 1$.

Proof. This follows from Theorem 3.9 in Kozubowski et al. (2005) and similar result for geometric stable distributions (see Kozubowski, 1994, Theorem 3.2), where we take into account that the stability relation holds for each coordinate of Y. $\square$

3. Operator geometric stable laws and domains of attraction

Operator stable random vectors are the weak distributional limits of sums of independent and identically distributed random vectors. Given $U, U^{(1)}, U^{(2)} \ldots$ IID random vectors, we say that U belongs to the strict generalized domain of attraction (GDOA) of the random vector X if we have the weak convergence

$$A_n \sum_{i=1}^{n} U^{(i)} \Rightarrow X, \tag{3.1}$$

in which case we also say that X is strictly operator stable (OS). If we assume that the limit X is full (i.e., not supported on any lower dimensional affine subspace) then the convergence (3.1) has a number of consequences. The same limit can be obtained for a sequence of norming operators that is regularly varying, in the sense that $A_{[\lambda n]}A_n^{-1} \to \lambda^{-B}$ in the operator norm for every $\lambda > 0$. Here B is an exponent of X, so that if $X^{(i)}$ are IID copies of X, the random vectors $n^B X$ and $X^{(1)} + \cdots + X^{(n)}$ are identically distributed. The exponent also codes the tail behavior of U, see Meerschaert and Scheffler (2001) for details. It is customary to assume that X is full to ensure that the limit in (3.1) can serve as a useful approximation of the normalized sum (i.e., $\sum_{n=1}^{n} U^{(i)} \approx A_n^{-1} X$). For example, it ensures that A_n is invertible. However, the fullness assumption is not strictly necessary to develop a useful theory of GDOA for OS laws.

Suppose that $u^{(i)}$ are IID random variables in the strict domain of normal attraction of the stable subordinator X_1, so that

$$n^{-1/\alpha} \sum_{i=1}^{n} u^{(i)} \Rightarrow X_1$$

as $n \to \infty$. (Note: The term "normal" here refers to the norming constants, not the limit.) Define random vectors $U^{(i)} = (u^{(i)}, 1)$ where the second coordinate is non-random. Now it is easy to see that

$$n^{-B} \sum_{i=1}^{n} U^{(i)} \Rightarrow X = (X_1, 1), \tag{3.2}$$

where $B = \mathrm{diag}(1/\alpha, 1)$. Hence X is operator stable, but not full. We can also see that $n^B X \approx \sum_{i=1}^{n} U^{(i)}$ is a useful approximation, since this is equivalent to $\sum_{i=1}^{n} u^{(i)} \approx n^{1/\alpha} X_1$.

Operator geometric stable random vectors are the weak distributional limits of random sums of IID random vectors, with a geometrically distributed number of summands (see Kozubowski et al., 2005). Suppose that N_p has a geometric distribution (1.7). Given $U, U^{(1)}, U^{(2)} \ldots$ IID random vectors, we say that U belongs to the strict generalized geometric domain of attraction (GGDOA) of the random vector Y if

$$A_p \sum_{i=1}^{N_p} U^{(i)} \Rightarrow Y, \tag{3.3}$$

in which case we also say that Y is strictly operator geometric stable (OGS). It follows that, if $Y^{(i)}$ are IID copies of Y, then $p^B Y$ and $Y^{(1)} + \cdots + Y^{(N_p)}$ are identically distributed for any $p > 0$, where again N_p is geometric and independent of $Y, Y^{(1)}, Y^{(2)}, \ldots$ and B is some linear operator called an exponent of Y.

The basic theory of OGS laws and GGDOA is laid out in Kozubowski et al. (2005). If X is OS with exponent B, and E is standard exponential, independent of X, then $Y = E^B X$ is OGS with the same exponent. Conversely, if Y is OGS with exponent B, then it can always be written in the form $Y = E^B X$ where X is OS with the same exponent. A companion paper (see Kozubowski et al., 2003) relates GDOA and GGDOA. Theorem 3.3 in that paper implies that, if U belongs to the GDOA of X, then it also belongs to the GGDOA of Y, and vice versa. Hence the GDOA of X and the GGDOA of $Y = E^B X$ are equal. Both papers (Kozubowski et al., 2003, 2005) assume that X and Y are full. It is not hard to see that X full implies that $Y = E^B X$ is full. However, the converse is not true, as we shall soon illustrate. Hence it is useful to note that, even without assuming X full, the same arguments from Kozubowski et al. (2003, 2005) allow us to conclude that Y OGS with exponent B implies $Y = E^B X$ for some (not necessarily full) OS random vector X, and that any U in the GGDOA of Y also belongs to the GDOA of X. In fact, the argument extends immediately, since the fundamental results of Rosiński (1976) used in Kozubowski et al. (2003) do not require fullness.

If (3.2) holds for IID random vectors $U^{(i)} = (u^{(i)}, 1)$, where the $u^{(i)}$ belong to the strict domain of normal attraction of X_1, then Theorem 1 in Rosiński (1976) yields that (3.3) also holds, where $Y = E^B X$ and $B = \mathrm{diag}(1/\alpha, 1)$. It follows that the BEML random vector Y is OGS with exponent B, and the bivariate distribution of (X, N) in (1.6) can be approximated by $A_p^{-1} Y$. Note that, although Y is full, the OS random vector X is not full.

OGS laws are a special case of ν-operator stable laws (see Kozubowski et al., 2003). If (3.2) holds for IID random vectors $U^{(i)} = (u^{(i)}, 1)$, where the $u^{(i)}$ belong to the strict domain of normal attraction of X_1, and if N_n are integer valued random variables independent of $U^{(i)}$ such that $N_n/n \Rightarrow E > 0$ then Theorem 1 in Rosiński (1976) shows that (3.3) still holds with $Y = E^B X$. Even in the case where N_n are $U^{(i)}$ are dependent, the same holds under the stronger assumption that $N_n/n \to E$ in probability. This follows from Theorem 2.4 of Becker-Kern (2002). Finally, we note that if $X(t)$ is an operator stable Lévy process such that $X(1)$ is infinitely divisible (ID) with X, then Y is ID with $X(E)$, since $X(t)$ is operator self-similar with exponent B: the finite dimensional distributions of the stochastic processes $X(ct)$ and $c^B X(t)$ are equal.

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