Recurrence of extreme events with power-law interarrival times

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[1] In various geophysical applications, power-law interarrival times are observed between extreme events. Classical extreme value theory is based on exponentially distributed interarrivals and can not be applied to these processes. We solve for the density of the maxima of a sequence of random extreme events with any distribution of random interarrivals by applying a continuous time random max model, similar to a random walk model. The equation is exact when the distributions of the exceedances and the interarrivals are known. If only the tail properties of the exceedances and interarrivals can be estimated, then limiting extreme value distributions governing the maximum observation or exceedance are used. The general extreme value densities are obtained by transforming the classical extreme value distributions via subordination. This new class of extreme value densities can be used to obtain recurrence intervals for extreme events with power-law interarrivals. **Citation:** Benson, D. A., R. Schumer, and M. M. Meerschaert (2007), Recurrence of extreme events with power-law interarrival times, *Geophys. Res. Lett.*, *34*, L16404, doi:10.1029/ 2007GL030767.

1. Introduction

[2] Extreme value (EV) theory has been used to predict the recurrence of extreme geologic processes since the 1940s [*Gumbel*, 1941; *Nordquist*, 1945]. Today, quantification of the stochastic behavior of the largest floods, earthquakes, storms, volcanic eruptions, air pollution levels, and wave heights is a key factor in risk assessment and mitigation.

[3] Stochastic representation of both annual and partial duration (peak-over-threshold) series is typically based on the Poisson process. Maxima, minima, and threshold exceedances are represented by a best-fit probability distribution but are assumed to be separated by fixed or exponentially distributed interarrival times. These models are robust because exceedances in a stochastic process without long memory will take on a Poisson character as the number of observations becomes large [*Leadbetter et al.*, 1983]. Limiting distributions governing the maximum (or minimum) value for a properly normalized set of independent and identically distributed (iid) random variables are the max-stable extreme value distributions. The appropriate type of EV distribution can be distinguished by the tail characteristics of the observations. Generalizations of this theory focus on violations of independence or stationarity criteria [*Galambos*, 1978; *Leadbetter et al.*, 1983].

[4] While techniques for treating dependence and nonstationarity have been developed and successfully applied in geologic applications [*Enzel et al.*, 2002; *Jain and Lall*, 2001; *McCuen and Beighley*, 2003; *Singh et al.*, 2005], the Poisson assumption is almost always maintained and existing EV models cannot be used to determine recurrence intervals for processes with heavy-tailed, infinite-mean interarrivals. This property has been observed in storm origins [*Salim and Pawitan*, 2003], raindrop release and arrival on the ground [*Lavergnat and Gole*, 1998, 2006], alluvial events [*Mazzarella and Diodato*, 2002] and earthquakes [*Musson et al.*, 2002]. These studies justify the application of a renewal process model for calculating EV statistics, as suggested by *Smith and Karr* [1983]. However, there is little theory on extremal processes in the context of heavy-tailed random waiting times between events. This extension is critical, because the application of a Poisson model for a geologic process that has heavy-tailed interarrivals will result in a significant misrepresentation of the risk.

[5] To demonstrate the effect of heavy-tailed waiting times on recurrence intervals, we simulated one process with independent and identically distributed (iid) exponential observations (with mean = 75) and iid exponential interarrival times (with mean = 0.95 years), and a second point process with the same distribution for the observations, but with iid Pareto (power law) interarrivals with tail parameter $\alpha = 0.9$, and scale chosen to ensure that the first three quartiles of both waiting time distributions are similar. For each model, the simulation identified the maximum observation every year. The heavy-tailed interarrivals lead to smaller probabilities for a given event size. Hence, the maximum event size expected for a given recurrence interval of a heavy-tailed interarrival process will be significantly smaller than that of the corresponding compound Poisson process (Figure 1).

2. Master Equation for Continuous Time Random Max Model

[6] The continuous time random walk (CTRW), or renewal-reward process, is a useful and flexible model that has become popular in geophysics, especially for modeling processes with heavy tails and/or irregular (possibly heavy-tailed) waiting times. The CTRW is often used to model diffusion or dispersion, in which particles perform a random walk with random waiting times between particle jumps. Here, we develop the continuous time random max model by considering the magnitude of exceedances (events) rather than particle jump lengths and record the maximum of n events rather than the sum. Suppose that we are

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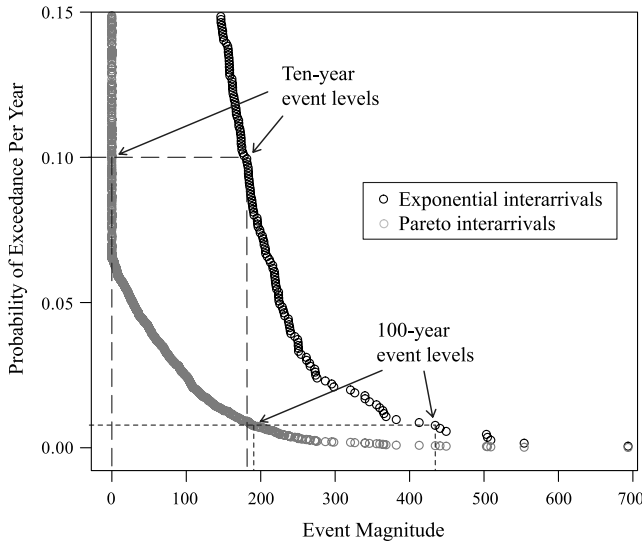
Figure 1. Plot showing the empirical quantile functions for maxima in a process with iid exponential observations separated by a choice of exponential or Pareto inter-arrival times.

observing exceedances over a given threshold for a detrended and declustered dataset. Define the random variables:

Y_n = iid exceedance event magnitudes with cumulative distribution function $F(x)$,

J_n = iid duration of interarrivals (time between exceedance events) with density $\psi(t)$,

T_n = $J_1 + J_2 + \cdots + J_n$ = time of the nth event,

N_t = $\max\{n : T_n \le t\}$ = number of events by time t,

M_n = $\max(Y_1, Y_2, \cdots, Y_n)$ = maximum of the first n observed events, and

M_t = M_{N_t} = $\max(Y_1, Y_2, \cdots, Y_{Nt})$ = maximum observed event by time t.

[7] We derive the probability distribution of the maximum of a number of iid events separated by iid waiting times following an analogous derivation by D. Benson et al. (Review of continuous time random walks: Derivation, approximation, limits, and limitations, submitted to Water Resources Research, 2007) for the distribution of the sum of events. The distribution of $M_t = M_{N_t}$ can be computed by conditioning on $N_t = 0, 1, 2, \ldots$. Write

$$P(M_{N_t} \le x) = \sum_{n=0}^{\infty} P(M_{N_t} \le x | N_t = n) P(N_t = n), \quad (1)$$

and note that

$$P(M_{N_t} \le x | N_t = n) = [F(x)]^n, \quad (2)$$

since $Y_1, Y_2, Y_3, \ldots$ are iid. Because T_n and N_t are mutually inverse processes, we have $\{N_t \ge n\} = \{T_n \le t\}$, i.e., we have at least n observations by time $t > 0$ if and only if the nth observation occurs by time t (Figure 2). The time of the nth event $T_n = J_1 + \cdots + J_n$ has a density that is the convolution of the individual inter-event periods, and hence its probability density is the n-fold convolution $\psi_1(t) \star$

$\psi_2(t) \star \cdots \star \psi_n(t)$. Denote this n-fold convolution $P(T_n \in (t, t + dt)) = \psi_{n\star}(t)dt$, and to obtain the probability $P(T_n \le t)$, we integrate the density from zero to t:

$$P(N_t \ge n) = P(T_n \le t) = \int_0^t \psi_{n\star}(u)du,$$

with Laplace transform

$$\mathcal{L}[P(N_t \ge n)] = \int_0^{\infty} e^{-st}P(N_t \ge n)dt = \frac{(\tilde{\psi}(s))^n}{s},$$

where $\tilde{\psi}(s) = \int_0^{\infty} e^{-st}\psi(t)dt$ is the Laplace transform.

[8] Because we require the probability that the number of events by time t is exactly n, we take the difference between the probability of $n + 1$ and n events:

$$P(N_t = n) = P(N_t \ge n) - P(N_t \ge n + 1),$$

with Laplace transform:

$$\mathcal{L}[P(N_t = n)] = \frac{(\tilde{\psi}(s))^n}{s} - \frac{(\tilde{\psi}(s))^{n+1}}{s} = \frac{1 - \tilde{\psi}(s)}{s}(\tilde{\psi}(s))^n. \quad (3)$$

Replacing the terms on the right-hand side of (1) with (2) and (3), we find the Laplace transform:

$$\begin{aligned}
\mathcal{L}[P(M_{N_t} \le x)] &= \int_0^{\infty} e^{-st}P(M_{N_t} \le x)dt \\
&= \sum_{n=0}^{\infty} \frac{1 - \tilde{\psi}(s)}{s}(\tilde{\psi}(s))^n (F(x))^n \quad (4) \\
&= \frac{1 - \tilde{\psi}(s)}{s} \sum_{n=0}^{\infty} (\tilde{\psi}(s))^n (F(x))^n.
\end{aligned}$$

Because the sum of a geometric series $\sum_{i=0}^{\infty} r^i = 1/(1 - r)$, provided that $|r| < 1$, we have

$$\mathcal{L}[P(M_{N_t} \le x)] = \frac{(1 - \tilde{\psi}(s))}{s} \frac{1}{1 - \tilde{\psi}(s)F(x)}. \quad (5)$$

[9] Equation (5) resembles the complete solution for uncoupled CTRWs [*Montroll and Weiss*, 1965; *Scher and Lax*, 1973] and is the corresponding master equation for continuous time random max models. Identifying the distribution of the observations without regard to their timing, and the distribution of the interarrival times, allows us to calculate the distribution of the maximum observation in any time period using (5). If the continuous time random max is a Poisson process, then interarrival times are exponentially distributed $\psi(t) = \lambda e^{-\lambda t}$ with Laplace transform $\tilde{\psi}(s) = \lambda/(\lambda + s)$. Then relation (5) yields

$$\begin{aligned}
\int_0^{\infty} e^{-st}P(M_{N_t} \le x)dt &= \frac{1 - \lambda/(\lambda + s)}{s} \frac{1}{1 - F(x)\frac{\lambda}{s + \lambda}} \\
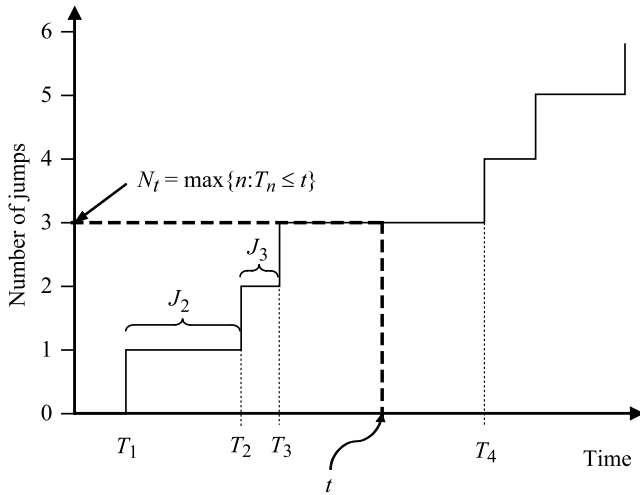&= \frac{1}{s + \lambda(1 - F(x))}. \quad (6)
\end{aligned}$$

Figure 2. Definition of interarrival times J_n, time of the *n*th observation T_n, and number of observations by time *t*.

This is the Laplace transform (in *t*) of:

$$P(M_{N_t} \leq x) = e^{-\lambda t(1 - F(x))}, \qquad (7)$$

which is known to be the distribution of the maximum for a compound Poisson process [*Reiss and Thomas*, 2001; *Rosbjerg et al.*, 1992; *Smith*, 1987].

[10] If the interarrival distribution is chosen judiciously in (5), the limiting governing distribution can be obtained in closed form. In the classic case of exponentially distributed exceedances $F(x) = 1 - e^{-\lambda_x x}$ and exponentially distributed interarrival times $\varphi(t) = \lambda_t e^{-\lambda_t t}$, we find $P(M_{N_t} \leq x) = \exp(-\lambda_t t e^{-\lambda_x x})$. With the addition of generalized Pareto (with three subtypes corresponding to the exponential, Pareto, and beta distributions) event magnitudes $F(x)$, the distribution of the maximum in time (7) is GEV (with subtypes I, II, and III corresponding to the Gumbel, Frechet, and Weibull distributions [*Reiss and Thomas*, 2001]).

[11] The master equation is free of any distributional assumptions and can accept any type of interarrival distribution. The max-distribution can be obtained numerically for datasets where interarrival distributions are neither exponential nor power-law using methods described by *Abate and Whitt* [1995].

3. Limit Processes

[12] Extreme value theory provides a framework for extrapolating a finite event history to a long time scale. The standard EV models are based on asymptotic arguments that allow us to approximate the behavior of the maximum event by letting the number of events go to infinity and using limit theorems to obtain the extreme value models that can be calibrated using observations data. Some well-known results for the maximum limit process preface the discussion of the interarrival/counting time process for context.

3.1. Maximum Limit Process

[13] As the number of events *n* becomes large, the distribution of the maximum $[F(x)]^n$ tends toward a "fixed point," if a proper rescaling (renormalization) is performed. In terms of random variables, after a proper rescaling by coefficients that depend on sample size, the maximum M_n converges to a "max-stable" random variable: $a_n(M_n - b_n) \Rightarrow A$, where $\Rightarrow$ denotes convergence in distribution. The random variable *A* is max-stable with CDF $P(A \leq x) = G(x)$. In the distributional sense, the CDF of the individual events moves to the fixed point $[F(a_n^{-1}x + b_n)]^n \to G(x)$. The two max-stable distributions listed below are especially useful for large samples sizes (i.e., large times) of events with no fixed upper bound, because they depend only on the behavior of the upper tail of $F(x)$, which can be estimated without regard for the entire distribution. *Gnedenko* [1943] was the first to show that the tail of the parent distribution determines which type of extreme value distribution will occur in the limit. Restricting our attention to the maxima of events with no upper bound, the scaling coefficients and the corresponding max-stable distributions are of form:

[14] Type I. All moments of $F(x)$ are finite, then $G_1(x) = \exp(-e^{-x})$, and the coefficients a_n and b_n depend on the distribution of individual events $F(x)$ [*Leadbetter et al.*, 1983]. For example, it can be shown that exponential events with $F(x) = 1 - e^{-\lambda x}$ have $a_n = \lambda$, $b_n = (\ln n)/\lambda$.

[15] Type II. The tail for large values follows $F(x) \sim 1 - Cx^{-\alpha}$ and some moments diverge, then $a_n \sim (Cn)^{-1/\alpha}$, $b_n \sim 0$, and $G_2(x) = \exp(-x^{-\alpha})$ for $x \geq 0$.

[16] We can define the process $A(t)$, which is the max-stable corresponding to *t* events. In this context, $A = A(1)$. The distribution of *t* max-stable events follows $A(t) \sim [G(x)]^t$. In the case that the events are max-stable, it is simple to find the scaling coefficients. For example, the Type II max-stable follows $[G_2(x)]^t = \exp(-tx^{-\alpha}) = G_2(t^{-1/\alpha}(x - 0)) = G_2(a_t(x - b_t))$. A Type I max-stable has $a_t = 1$ and $b_t = \ln(t)$, so that $[G_1(x)]^t = \exp(-te^{-x}) = G_1(x - \ln(t)) = G_1(a_t(x - b_t))$.

3.2. Interarrival/Counting Time Processes

[17] Exceedances in a stochastic process without long memory will take on a Poisson character as the number of observations becomes large [*Leadbetter et al.*, 1983]. We are concerned with the more general limiting process that can accommodate infinite mean (power-law, with exponent $\gamma < 1$) interarrival distributions. The special case where the interarrivals have finite mean, but infinite variance, will be treated elsewhere.

[18] In the case of heavy tailed interarrival times, the events are separated by a wide distribution of times, and the number of events by time *t*, N_t, can also grow more slowly than linear with time. To calculate the rescaling of N_t, we use known limiting properties of T_n and the inverse relationship between N_t and T_n. As with the event maxima, we rescale the time between events with power law tails and infinite mean: $P(J > t) \sim \frac{C}{\Gamma(1-\gamma)} t^{-\gamma}$ for large interarrival times *t*, where *C* is an arbitrary constant and $\gamma < 1$. Taking the scaling limit is equivalent to shrinking the time axis and the counting axis of Figure 2 at the correct rates so that the graph remains invariant. A functional central limit theorem shows that the rescaled sum of interarrival periods converges to a Lévy process: $c^{-1/\gamma}T_{[cu]} = c^{-1/\gamma}(J_1 + \ldots + J_{[cu]}) \Rightarrow W(u)$ [*Meerschaert and Scheffler*, 2004].

[19] The random variable $W(u)$ corresponds to the random interarrival time associated with the rescaled number of extreme events *u*. $W(u)$ is called the stable subordinator; it is
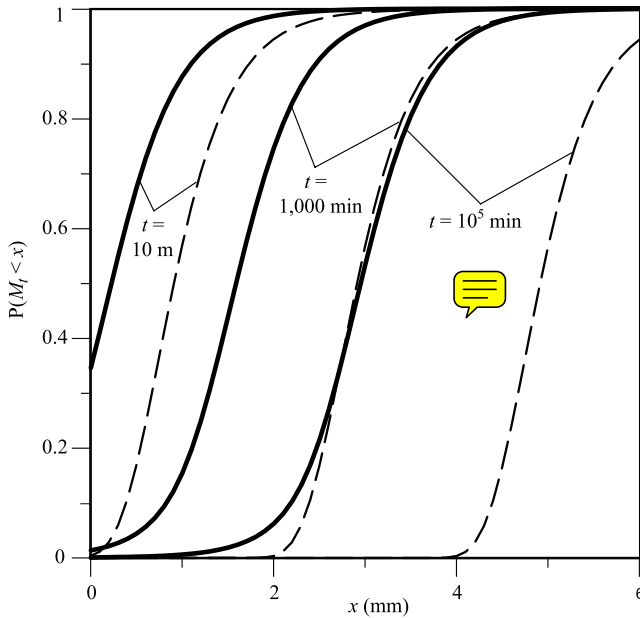
Figure 3. Cumulative distributions of maximum drop size at 10, 1,000, and 100,000 s. Dashed lines assume Poisson arrival process, solid lines use the observed power-law interarrivals. The Poisson process overestimates the maxima due to an overestimate of the number of arrivals, especially at late time.

strictly positive, has a stable limit distribution, and subordinates the time variable [*Feller*, 1971]. The Laplace transform of the density ($t \rightarrow s$) is given by $E[e^{-sW(u)}] = e^{-Cus^\gamma}$, with inverse $(Cu)^{-1/\gamma}g_\gamma(t(Cu)^{-1/\gamma})$, where $g_\gamma(t)$ is the standard stable density. The density shows the probability of various time t required to accumulate a given number (u) of events. The rescaling allows non-integer values of u.

[20] For a given u, the time $W(u)$ is random. We can also fix a time, and the rescaled counting process keeps track of the random number of events that have occurred. This random process $U(t)$ must be accounted for in a non-Poisson process. The relation between the counting process and the interarrival time process $\{N_t \geq n\} = \{T_n \leq t\}$ leads to the rescaled counting process $c^{-\gamma}N_{ct} \Rightarrow U(t) = \inf\{u: W(u) > t\}$. The density of the counting process is related to the waiting time density by $h(u|t) = \frac{t}{u\gamma}(Cu)^{-1/\gamma} g_\gamma(t(Cu)^{-1/\gamma})$ [*Meerschaert and Scheffler*, 2004].

3.3. Max-Renewal Process and Extreme Value Distributions

[21] The limit extremal process $A(t)$ can be used to represent the maximum value of a set of observations or exceedances. The limit distribution of M_{N_t} is obtained by simultaneously rescaling the max process and counting process, since both are linked by the number of events: $c^{-\gamma/\alpha}M_{N_{ct}} \Rightarrow M(t)$ as $c \rightarrow \infty$ in the case of heavy tailed observations (type II). M. Meerschaert and S. Stoev (Extremal limit theorems for observations separated by random waiting times, submitted to *Stochastic Processes and Their Applications*, 2007) show that the limiting max process has the convergence property $M(t) = A(U(t))$, which shows the subordination of the Markov process $A(t)$ by the

directing process $U(t)$. Since the max process and time process are independent, the density of $M(t)$ is

$$
\begin{aligned}
P(M(t) \leq x) &= \int_0^\infty P(A(s) \leq x)h(s|t)ds \\
&= \int_0^\infty G(x)^s h(s|t)ds \qquad (8) \\
&= \frac{t}{\gamma}\int_0^\infty G(x)^s s^{-1}(Cs)^{-1/\gamma}g_\gamma\left(t(Cs)^{-1/\gamma}\right)ds.
\end{aligned}
$$

where $G(x)$ is one of the subtypes of classical extreme value distributions. The continuous-time extreme value distribution (8) allows for both heavy-tailed exceedances and heavy-tailed interarrivals.

3.4. Example

[22] *Lavergnat and Gole* [1998] present an extensive data set of raindrop sizes and arrival times at a 250×40 mm counter. For timescales greater than tens to hundreds of seconds, they find a power-law interarrival distribution that follows, asymptotically, $P(J > t) \sim \frac{C}{\Gamma(1-\gamma)} t^{-\gamma}$ with $C = 3.94$ and $\gamma = 0.68$. They also remark that their fitted mean interarrival time is 1.78 s. If we were to assume a Poisson process, then we might use a rate of $\lambda_t = 1/1.78 = 0.562$ s^{-1}. Their drop size histogram [*Lavergnat and Gole*, 1998, Figure 2b] can be fit, for drops larger than roughly 0.1 mm, by an exponential density $p(x) = \lambda_x e^{-\lambda_x x}$ with $\lambda_x = 2.3$. Ours is not a rigorous fitting for lack of the exact sample size. The maximum drop from an iid series of n drops converges by $\lambda_x M_n - \ln(n) \Rightarrow A$, where A has a type I max-stable distribution $G_1(x) = e^{-e^{-x}}$. Assuming a large enough number of events that the max-stable is a fair representation of the properly scaled maximum, we have $P(M_n < x) \approx e^{-ne^{-\lambda_x x}}$. As shown by (7), a Poisson process would then have $n = t\lambda_t$ and $P(M_n < x) \approx e^{-t\lambda_t e^{-\lambda_x x}}$. For the actual heavy-tailed waiting times, we subordinate using (8). The Poisson assumption overestimates the number of drops that may occur (Figure 3), especially at large timescales, and therefore overestimates the maximum drop size probability. The heavy-tailed interarrivals, when properly accounted for, also show a much larger probability of very small maximum drop sizes, simply because of the higher likelihood of fewer drops arriving. We note that *Lavergnat and Gole* [1998] use their analysis to accurately predict cumulative rainfall. We speculate that in a similar way, our method may be used to predict the recurrence intervals of the greatest rainfall intensities.

4. Conclusions

[23] When predicting the probability of extreme events over certain time periods, one must account for 1) the distribution of the events—without respect to interarrival times—and 2) the time spacing between events. Using a continuous time random max model, we find the exact formula for the Laplace transform of the probability density if both distributions are known. If only the tails of the distributions can be estimated, we apply limit theory to find the generalized extreme value distributions. In the case of heavy-tailed interarrivals, the generalized EV is calculated using a subordination integral. These limit distributions are most useful for very long-term predictions.

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