Chapter for Handbook of Fractional Calculus with Applications

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Continuous time random walks and space-time fractional differential equations

DOI ..., Received ...; accepted ...

Abstract: The continuous time random walk is a model from statistical physics that elucidates the physical interpretation of the space-time fractional diffusion equation. In this model, each step in the random walk is separated by a random waiting time. The long-time limit of this model is governed by a fractional diffusion equation. If the step length of the random walk follows a power law, we get a space-fractional diffusion equation. If the waiting times also follow a power law, we get a space-time fractional diffusion equation. The index of the power law equals the order of the fractional derivative. If the waiting times and jumps are dependent random variables, the governing equation involves coupled space-time fractional derivatives.

Keywords: Continuous time random walks, governing equations, stable laws, extended central limit theorem

2010 Mathematics Subject Classification: Primary 60G52, 60J50; Secondary 26A33, 60J22.

1 Introduction

The continuous time random walk (CTRW) is a model from statistical physics, introduced by Montroll and Weiss [35] and developed further by Scher and Lax [39], Klafter and Silbey [14], and Hilfer and Anton [12]. Start with a random walk $S(n) = Y_1 + \cdots + Y_n$ where the independent and identically distributed (iid) random variables $\{Y_n\}$ represents the particle jumps. Now assume a sequence of iid positive random variables $\{J_n\}$, and suppose that the waiting time J_n separates the $n-1$st and the nth jumps. Then $T(n) = J_1 + \cdots + J_n$ is the time of the nth jump. The number of jumps by time $t \geq 0$ is $N(t) = \max\{n \geq 0 : T(n) \leq t\}$, and the CTRW

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$X(t) = S(N(t))$ gives the particle location at time $t \geq 0$. If the waiting times J_n and the jumps Y_n are independent, this is called an *uncoupled* CTRW.

Metzler and Klafter [32, 33] survey a wide variety of CTRW applications in biology, geophysics, geomorphology, finance, material science, particle physics, and turbulence. Berkowitz et al. [6] review CTRW models in hydrology. Scalas [37] reviews applications of the CTRW model in finance. Sokolov [46] reviews the physical assumptions behind applications of the CTRW. Zaburdaev, Denisov and Klafter [50] review the “Lévy walk” model, a strongly coupled CTRW where the step length is proportional to the waiting time. Metzler et al. [34] review applications of the CTRW model to single particle tracking. Scher and Montroll [40] apply the CTRW model to transient photocurrent in amorphous materials. Uchaikin and Sibatov [48] develop CTRW theory for fractional kinetics in solids. A CTRW model for the migration of cancer cells was presented in Fedotov and Iomin [9]. Schumer and Jerolmack [41] develop an interesting CTRW model for sediment deposition in the geological record. Ganti et al. [10] propose a CTRW model for gravel bed load transport in rivers. Benson and Meerschaert [4] outline a CTRW model for contaminant transport that segregates the mobile and immobile phases. Schulz et al. [42] apply the CTRW model to cell movements. Meerschaert et al. [29] propose a CTRW model for sound transmission in complex media.

2 Uncoupled CTRW

Now suppose that the iid jumps have a probability density function (pdf) $f(x)$, and the waiting times have a pdf $\psi(t)$. Montroll and Weiss [35] compute the exact pdf of the uncoupled CTRW using transforms. Using the Fourier transform (FT)

$$\hat{f}(k) = \mathbb{E}[e^{ikY_n}] = \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx \tag{1}$$

and the Laplace transform (LT)

$$\tilde{\psi}(s) = \mathbb{E}[e^{-sJ_n}] = \int_0^{\infty} e^{-st}\psi(t)\,dt, \tag{2}$$

we apply a simple conditioning argument. First note that

$$p(x,t) = \mathbb{P}[S(N(t)) = x] = \sum_{n=0}^{\infty} \mathbb{P}[S(N(t)) = x | N(t) = n]\mathbb{P}[N(t) = n]. \tag{3}$$

If $N(t) = n$, then $S(N(t)) = S(n)$ is the sum of n iid random variables, so its FT is

$$\mathbb{E}[e^{ik(Y_1+\cdots+Y_n)}] = \mathbb{E}[e^{ikY_1}]\cdots\mathbb{E}[e^{ikY_n}] = \hat{f}(k)^n.$$

Clearly

$$q(0,t) = \mathbb{P}[N(t)=0] = \mathbb{P}[J_1 > t] = 1 - \mathbb{P}[J_1 \le t] = 1 - \int_0^t \psi(u)\,du,$$

and hence $\tilde{q}(0,s) = s^{-1}(1-\tilde{\psi}(s))$, using the fact that $s^{-1}\tilde{g}(s)$ is the LT of $\int_0^t g(u)\,du$. More generally, for $N(t) = n > 0$ we require that $T(n) = u \le t$ and $J_{n+1} > t-u$, and hence

$$q(n,t) = \mathbb{P}[N(t)=n] = \int_0^t \psi^{n*}(u)\Psi(t-u)\,du,$$

where ψ^{n*} is the n-fold convolution, and $\Psi(t) = \mathbb{P}[J_{n+1} > t]$. Taking LT we see that $\tilde{q}(n,s) = \tilde{\psi}(s)^n s^{-1}(1-\tilde{\psi}(s))$ for all $n \ge 0$. Taking FT and LT in (3) leads to the Montroll-Weiss formula

$$\begin{aligned}\bar{p}(k,s) &= \int_{-\infty}^{\infty} e^{ikx} \int_0^{\infty} e^{-st} p(x,t)\,dt\,dx \\ &= \sum_{n=0}^{\infty} \hat{f}(k)^n \tilde{\psi}(s)^n s^{-1}(1-\tilde{\psi}(s)) = \frac{1}{s}\frac{1-\tilde{\psi}(s)}{1-\hat{f}(k)\tilde{\psi}(s)} \end{aligned} \tag{4}$$

that gives the exact Fourier-Laplace transform (FLT) for the pdf of the uncoupled CTRW.

Rewrite (4) in the form

$$\bar{p}(k,s) = \hat{f}(k)\tilde{\psi}(s)\bar{p}(k,s) + \frac{1}{s}\left(1-\tilde{\psi}(s)\right),$$

invert the FT, and then invert the LT to obtain the master equation from Klafter and Silbey [14]:

$$p(x,t) = \int_0^t \psi(t-u) \int_{-\infty}^{\infty} f(x-y)p(y,u)\,dy\,du + \delta(x)\int_t^{\infty} \psi(u)\,du. \tag{5}$$

If $\psi(t) = \lambda e^{-\lambda t}$ for $t > 0$, then $\tilde{\psi}(s) = \lambda/(\lambda+s)$, and the Montroll-Weiss equation reduces to

$$\bar{p}(k,s) = \frac{1}{s+\lambda(1-\hat{f}(k))}.$$

Inverting the LT yields

$$\hat{p}(k,t) = e^{-\lambda t(1-\hat{f}(k))},$$

which is the well-known formula for the compound Poisson pdf with jump pdf $f(x)$ (e.g., see [28, Example 3.3]). The compound Poisson is a special case of the CTRW with exponential waiting times. Because the exponential distribution has no memory, this CTRW is a Markov process: Once the value $X(t) = S(N(t))$ is known, the pdf of $X(t+s)$ has no further dependence on the past history of $X(u)$ for $0 \le u < t$. In fact, it is even a Lévy process: The pdf of $X(s)$ is the same as that of $X(t+s) - X(t)$ (stationary increments), and the random variables $X(t)$ and $X(t+s) - X(t)$ are independent (independent increments). However, a CTRW without exponential waiting times is not a Lévy process, or even a Markov process. The influence of the memory can be seen in the master equation (5).

Next we give a heuristic explanation of the connection between CTRW and fractional calculus (e.g., see Scalas, Gorenflo and Mainardi [38]). Suppose that $\mathbb{P}[X_n > x] = Ax^{-\alpha}$ for some $A > 0$ and some $1 < \alpha < 2$. Then $\mu = \mathbb{E}[X_n]$ exists, and we can take $Y_n = X_n - \mu$. Suppose also that $\mathbb{P}[J_n > t] = Bt^{-\beta}$ for some $B > 0$ and some $0 < \beta < 1$. A calculation [28, Proposition 1.7] shows that $\hat{f}(k) = 1 + D(-ik)^\alpha + O(k^2)$ where $D = A\Gamma(2-\alpha)/(\alpha - 1)$. A similar calculation [28, Theorem 3.37] shows that $\tilde{\psi}(s) = 1 - s^\beta + O(s)$ when $B = 1/\Gamma(1-\beta)$. Now in order to obtain a limit pdf, replace Y_n by $c^{-1/\alpha} Y_n$ and J_n by $c^{-1/\beta} J_n$. Then the particle jumps have FT $\hat{f}(c^{-1/\alpha}k) = 1 + Dc^{-1}(-ik)^\alpha + O(c^{-2/\alpha}k^2)$ and the waiting times have LT $\tilde{\psi}(c^{-1/\beta}s) = 1 - c^{-1}s^\beta + O(c^{-1/\beta}s)$. Plug into the Montroll-Weiss formula, multiply by c on top and bottom, and let $c \to \infty$ to get the CTRW scaling limit

$$\begin{aligned} \bar{p}_c(k,s) &= \frac{1}{s}\frac{c^{-1}s^\beta + O(c^{-1/\beta}s)}{c^{-1}s^\beta - Dc^{-1}(-ik)^\alpha + \cdots} \\ &\to \frac{s^{\beta-1}}{s^\beta - D(-ik)^\alpha} = \bar{p}_\infty(k,s). \end{aligned} \tag{6}$$

Rewrite as $s^\beta \bar{p}_\infty(k,s) = D(-ik)^\alpha \bar{p}_\infty(k,s) + s^{\beta-1}$, and invert the FT and LT to get the space-time fractional diffusion equation

$$\partial_t^\beta p_\infty(x,t) = D\partial_x^\alpha p_\infty(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{7}$$

that governs the pdf of the long-time CTRW scaling limit in terms of Riemann-Liouville fractional derivatives. Here we use the fact that $\partial_t^\beta g(t)$ has LT $s^\beta \tilde{g}(s)$, $\partial_x^\alpha f(x)$ has FT $(-ik)^\alpha \hat{f}(k)$, and $t^{-\beta}/\Gamma(1-\beta)$ has LT $s^{\beta-1}$ [28, Example 2.9]. This statistical physics argument illustrates how the fractional derivative in space codes long particle jumps, and the fractional derivative in time represents long waiting times. The argument is not completely rigorous because the LT inversion requires more assumptions, e.g., see the proof of [26, Theorem 3.1].

Remark 2.1. Mainardi [18] computes a solution to the time fractional diffusion equation (7) with $\alpha = 2$ using Wright functions. Mainardi and Gorenflo [19] solve time-fractional differential equations using the Mittag-Leffler function. Hilfer and

Anton [12] and Mainardi, Luchko and Pagnini [20] use Mittag-Leffler functions to solve the general equation (7). The Mittag-Leffler function also has a special role in CTRW modeling: Mainardi, Gorenflo and Scalas[21] note that for Mittag-Leffler waiting times $\mathbb{P}[J_n > t] = E_\beta(-\lambda t^\beta)$, the time process is already in its asymptotic form, see also Meerschaert, Nane, and Vellaisamy [27].

Another way to derive the CTRW scaling limit uses the extended central limit theorem. Since $T(n)$ has LT

$$\mathbb{E}\big[e^{-sT(n)}\big] = \mathbb{E}\big[e^{-s(J_1+\cdots+J_n)}\big] = \mathbb{E}\big[e^{-sJ_1}\big]\cdots\mathbb{E}\big[e^{-sJ_n}\big] = \tilde{\psi}(s)^n \tag{8}$$

the rescaled sum $c^{-1/\beta}T([ct])$ has LT

$$\begin{aligned}\mathbb{E}\big[e^{-sc^{-1/\beta}T([ct])}\big] &= \tilde{\psi}(sc^{-1/\beta})^{[ct]}\\ &= \left(1 - \frac{s^\beta}{c} + o(c^{-1})\right)^{[ct]} \to e^{-ts^\beta}\end{aligned} \tag{9}$$

as $c \to \infty$, using the fact that $(1+a/c+o(1/c))^c \to e^a$. The limit is the LT of a stable Lévy process $D(t)$ with index β, and the continuity theorem for the LT implies that $c^{-1/\beta}T([ct]) \Rightarrow D(t)$ in distribution. A similar argument [28, Section 1.2] shows that $c^{-1/\alpha}S([ct]) \Rightarrow A(t)$, an α-stable Lévy process with $\mathbb{E}[e^{ikA(t)}] = \exp(Dt(-ik)^\alpha)$.

Since the renewal process $N(t)$ is the inverse of the random walk $T(n)$, it has an inverse limit [24, Theorem 3.2]: Observe that $\{N(t) \ge u\} = \{T(\lceil u\rceil) \le t\}$, where $\lceil u\rceil$ is the smallest integer $n \ge u$. Define the inverse time process $E(t) = \inf\{u > 0 : D(u) > t\}$, and note that $\{E(t) \le u\} = \{D(u) \ge t\}$. Then

$$\begin{aligned}\mathbb{P}[c^{-\beta}N(ct) \le u] &= \mathbb{P}[N(ct) \le c^\beta u]\\ &= \mathbb{P}[T(\lceil c^\beta u\rceil) \ge ct]\\ &= \mathbb{P}[c^{-1}T(\lceil c^\beta u\rceil) \ge t]\\ &= \mathbb{P}[(c^\beta)^{-1/\beta}T(\lceil c^\beta u\rceil) \ge t] \to \mathbb{P}[D(u) \ge t] = \mathbb{P}[E(t) \le u]\end{aligned}$$

so that $c^{-\beta}N(ct) \Rightarrow E(t)$. Then an argument [24, Theorem 4.2] using the continuous mapping theorem (e.g., see Billingsley [8] or Whitt [49] shows that

$$c^{-\beta/\alpha}S(N(ct)) = (c^\beta)^{-1/\alpha}S(c^\beta\, c^{-\beta}N(ct)) \approx (c^\beta)^{-1/\alpha}S(c^\beta\, E(t)) \Rightarrow A(E(t)).$$

Remark 2.2. Technically, the continuous mapping argument in [24, Theorem 4.2] requires *process convergence*: Not only does $c^{-1/\alpha}S([ct]) \Rightarrow A(t)$ for a single $t > 0$, but also

$$(c^{-1/\alpha}S([ct_1]), \ldots, c^{-1/\alpha}S([ct_n])) \Rightarrow (A(t_1), \ldots, A(t_n))$$

for any $0 \le t_1 < \cdots < t_n$ as random vectors. This is called convergence of finite dimensional distributions. To extend to all $t \ge 0$, one considers $\{A(t) : t \ge 0\}$ as a random element of the space of right-continuous functions from $[0, \infty)$ to the real line. Then [24, Theorem 4.1] establishes the convergence $c^{-1/\alpha}S([ct]) \Rightarrow A(t)$ for all $t \geq 0$ in the Skorokhod J_1 topology on that space of functions. In this setting, [24, Theorem 4.2] establishes CTRW process convergence in the Skorokhod M_1 topology. Straka and Henry [47, Theorem 3.6] establish CTRW process convergence in the stronger J_1 topology. For more details, see [28, Chapter 4].

The pdf $g(t, u)$ of $t = D(u)$ has LT $\tilde{g}(s, u) = e^{-us^\beta}$ by (9), and it follows easily that $D(u)$ has the same pdf as $u^{1/\beta}D(1)$. Hence $g(t, u) = u^{-1/\beta}g_\beta(u^{-1/\beta}t)$ where $g_\beta(t) = g(t, 1)$ is the density of $D(1)$. Then

$$\begin{aligned}\mathbb{P}[E(t) \leq u] &= \mathbb{P}[D(u) \geq t] \\ &= \mathbb{P}[u^{1/\beta}D(1) \geq t] = \mathbb{P}[D(1) \geq tu^{-1/\beta}] = 1 - \int_0^{tu^{-1/\beta}} g_\beta(u)\,du. \qquad (10)\end{aligned}$$

Differentiate (10) to see that $u = E(t)$ has density

$$h(u, t) = \frac{t}{\beta}u^{-1-1/\beta}g_\beta(tu^{-1/\beta}). \qquad (11)$$

Next we compute the LT of this pdf. Since

$$\mathbb{P}[E(t) \leq u] = \mathbb{P}[D(u) \geq t] = \int_t^\infty g(w, u)\,dw$$

the inner process $E(t)$ has density

$$h(u, t) = \frac{d}{du}\mathbb{P}[E(t) \leq u] = \frac{d}{du}\left[1 - \int_0^t g(w, u)\,dw\right]$$

with LT

$$\tilde{h}(u, s) = -\frac{d}{du}\left[s^{-1}\tilde{g}(s, u)\right] = -\frac{d}{du}\left[s^{-1}e^{-us^\beta}\right] = s^{\beta-1}e^{-us^\beta}$$

using the fact that integration corresponds to multiplication by s^{-1} in LT space.

Now a simple conditioning argument, similar to (3), shows that the CTRW limit $A(E(t))$ has pdf

$$p_\infty(x, t) = \int_0^\infty q(x, u)h(u, t)\,du \approx \sum_u \mathbb{P}(A(u) = x|E(t) = u)\mathbb{P}(E(t) = u),$$

where $q(x, u)$ is the pdf of $x = A(u)$. Since $\hat{q}(k, u) = e^{Dt(-ik)^\alpha}$, the CTRW limit density has FLT

$$\bar{p}_\infty(k, s) = \int_0^\infty e^{Du(-ik)^\alpha} s^{\beta-1} e^{-us^\beta}\, du = \frac{s^{\beta-1}}{s^\beta - D(-ik)^\alpha}$$

which agrees with (6). Hence the CTRW limit pdf $p_\infty(x, t)$ solves the space-time fractional diffusion equation (7), see Becker-Kern et al. [3, Example 5.1] for more details. This probabilistic argument provides a rigorous connection between the CTRW and the space-time fractional diffusion equation, as well as a stochastic model for the long-time CTRW limit. The method extends naturally to vector particle jumps, see [28, Chapter 6].

Remark 2.3. Since the CTRW scaling limit $X(t) = A(E(t))$ is not a Markov process, its transition density $p(x, t)$ does not completely characterize the process. Meerschaert and Straka [30, 31] develop a method for computing the joint pdf of $X(t)$ at multiple times. This method is based on a semi-Markov representation of the CTRW limit, where the memory is explicitly included, see also Germano et al.[11]. Krüsemann, Schwarz and Metzler [17] demonstrate how the non-Markovian nature (ageing, or memory) can be observed in Scher-Montroll experiments on transient photocurrent in amorphous materials. Barkai and Cheng [2] develop a theory of ageing CTRW.

Remark 2.4. A closely related model called the continuous time random maximum (CTRM) describes the biggest jump, rather than the sum. Using the same setup as before, let $M(n) = \max(Y_1, \ldots, Y_n)$ and consider the CTRM $M(N(t))$ that describes the biggest jump by time $t \geq 0$. Letting $F(x) = \int_{-\infty}^x f(y)\, dy$ denote the cumulative distribution function (cdf) of the jumps, note that

$$\mathbb{P}[M(n) \leq x] = \mathbb{P}[Y_1 \leq x, \ldots, Y_n \leq x] = \mathbb{P}[Y_1 \leq x] \cdots \mathbb{P}[Y_n \leq x] = F(x)^n$$

and argue in exactly the same way as before that the CTRM has cdf

$$P(x, t) = \mathbb{P}[M(N(t)) \leq x] = \sum_{n=0}^\infty \mathbb{P}[M(N(t)) \leq x | N(t) = n]\mathbb{P}[N(t) = n] \tag{12}$$

with LT

$$\tilde{P}(x, s) = \sum_{n=0}^\infty F(x)^n \tilde{\psi}(s)^n s^{-1}(1 - \tilde{\psi}(s)) = \frac{1}{s} \frac{1 - \tilde{\psi}(s)}{1 - F(x)\tilde{\psi}(s)}. \tag{13}$$

If $\mathbb{P}[Y_n > x] = Dx^{-\alpha}$ for some $\alpha > 0$, then $c^{-1/\alpha} M([ct])$ has cdf

$$F^{[ct]}(c^{1/\alpha} x) = \left(1 - \frac{Dx^{-\alpha}}{c}\right)^{[ct]} \to e^{-Dtx^{-\alpha}}$$

as $c \to \infty$. It follows that $c^{-1/\alpha}M([ct]) \Rightarrow Z(t)$, a max-stable process with cdf $G(x,t) = e^{-Dtx^{-\alpha}}$ for all $x > 0$. Then $c^{-\beta/\alpha}M(N(ct)) \Rightarrow Z(E(t))$ as $c \to \infty$. The CTRM limit has cdf

$$P_\infty(x,t) = \int_0^\infty G(x,u)h(u,t)\,du$$

with LT

$$\tilde{P}_\infty(x,s) = \int_0^\infty e^{-Dux^{-\alpha}} s^{\beta-1} e^{-us^\beta}\,du = \frac{s^{\beta-1}}{s^\beta + Dx^{-\alpha}}$$

for all $s > 0$. Rewrite as $s^\beta \tilde{P}_\infty(x,s) = -Dx^{-\alpha}\tilde{P}_\infty(x,s) + s^{\beta-1}$ and invert the LT to see that the cdf of the CTRW limit solves the time-fractional ordinary differential equation

$$\partial_t^\beta P_\infty(x,t) = -Dx^{-\alpha}P_\infty(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{14}$$

See Benson et al. [5] for more details, and an application to rainfall data.

3 Coupled CTRW

In Section 2, we considered the uncoupled CTRW, where the waiting times are independent of the particle jumps. Now we consider the more general *coupled* CTRW, where the length of the particle jump can depend on the waiting time. This model extension is useful to bound particle velocity, the ratio of jump length over waiting time. Let (Y_i, J_i) be iid with (Y, J) on $\mathbb{R}\times\mathbb{R}_+$, where Y_i models the i-th jump of a walker and J_i is the waiting time before or after the i-th jump. Set

$$T(n) = J_1 + \cdots + J_n \quad \text{and} \quad S(n) = Y_1 + \cdots + Y_n,$$

so that $(S(n), T(n))$ is a space-time random walk on $\mathbb{R}\times\mathbb{R}_+$. Let

$$N(t) = \max\{n \ge 0 : T(n) \le t\} \tag{15}$$

denote the number of jumps by time $t \ge 0$. For $t \ge 0$ we define the continuous time random walk (CTRW)

$$S(N(t)) = Y_1 + \cdots + Y_{N(t)} \tag{16}$$

and the overshooting continuous time random walk (OCTRW)

$$S(N(t)+1) = Y_1 + \cdots + Y_{N(t)} + Y_{N(t)+1} \tag{17}$$

which involves one additional jump. Observe that the CTRW corresponds to the "first wait, then jump" scenario, whereas the OCTRW corresponds to the "first jump, then wait" picture. That is, in the CTRW we begin with a waiting time, then jump, then repeat. In the OCTRW we begin with a jump, then wait, then repeat. See Figures 1 and 2 for an illustration.

Since Y_i and J_i can be dependent, $S(n)$ and $N(t)$ can be dependent, which makes the analysis of the long-time limiting behavior of the coupled CTRW process in (16) and the coupled OCTRW process in (17) more involved than the uncoupled case. Hence the analysis of the limit process and the governing equation is more delicate than the special case discussed in Section 2.

In order to prove limit theorems for these processes, we need to make an assumption on the joint distribution of Y and J, i.e., the distribution of the random vector (Y, J). In order to make this exposition as simple as possible, we assume that for some $0 < \alpha < 2$ and $0 < \beta < 1$ we have

$$(n^{-1/\alpha} S(n), n^{-1/\beta} T(n)) \Rightarrow (A, D) \tag{18}$$

as $n \to \infty$, where A and D are nondegenerate. It follows from (18) by projecting on either coordinate that A has a strictly α-stable distribution and D has a β-stable distribution. It follows [3, Eq. (2.18)] that for any $t > 0$ we have

$$(c^{-1/\alpha} S(ct), c^{-1/\beta} T(ct)) \Rightarrow (A(t), D(t)) \tag{19}$$

as $c \to \infty$, where $\{(A(t), D(t))\}_{t \geq 0}$ is a Lévy process on $\mathbb{R} \times \mathbb{R}_+$ with $(A(1), D(1)) = (A, D)$. Observe again, that $A(t)$ and $D(t)$ can and in general will be dependent.

The characteristic function of $(A(u), D(u))$ for $u > 0$ is characterized by a variant of the well known Lévy-Khinchine formula. Namely, in the present case, under assumption (18), we have that [3, Lemma 2.1]

$$\mathbb{E}[e^{-sD(u)+ikA(u)}] = \exp(-u\psi(k, s)), \tag{20}$$

for all $(k, s) \in \mathbb{R} \times \mathbb{R}_+$, where the symbol is given by

$$\psi(k, s) = iak + \int_{\mathbb{R} \times \mathbb{R}_+ \setminus \{(0,0)\}} \left(1 - e^{ikx} e^{-st} + \frac{ikx}{1 + x^2}\right) \phi(dx, dt) \tag{21}$$

for some $a \in \mathbb{R}$. The so-called Lévy measure $\phi(dx, dt)$ is finite outside every neighborhood of the origin and satisfies

$$\int_{0 < x^2 + t \leq 1} (x^2 + t)\, \phi(dx, dt) < \infty.$$

Let $\phi_A(dx) = \phi(dx, \mathbb{R}_+)$ denote the Lévy measure of the Lévy process $\{A(u)\}_{u \geq 0}$. By setting $s = 0$ in (21) we have

$$\mathbb{E}[e^{ikA(u)}] = e^{-u\psi_A(k)}$$

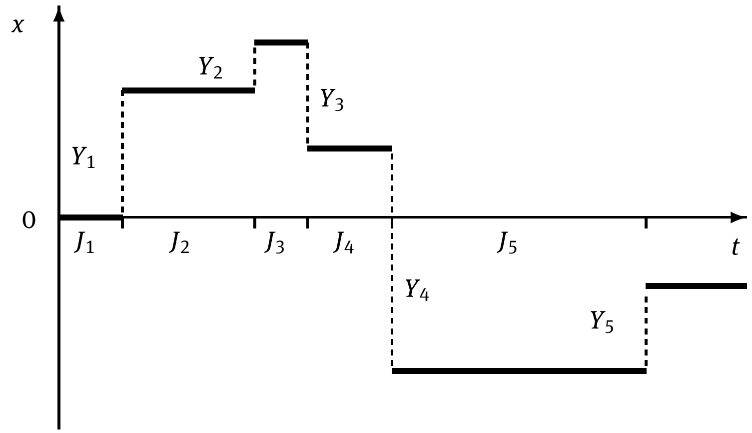

Fig. 1: The CTRW model (16). Each random waiting time J_i is followed by a random jump Y_i. In the coupled CTRW, the pdf of the particle jump Y_i can depend on the previous waiting time J_i.

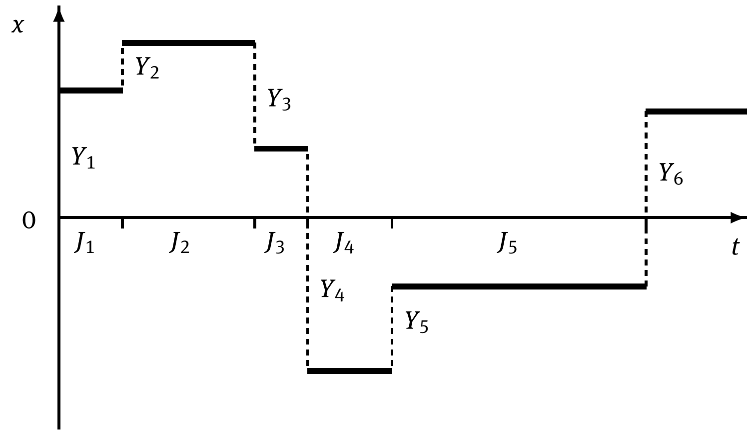

Fig. 2: The OCTRW model (17). Each random waiting time J_i follows a random jump Y_i. In the coupled OCTRW, the pdf of the waiting time J_i can depend on the previous particle jump Y_i.

where

$$\psi_A(k) = iak + \int\limits_{\mathbb{R}\setminus\{0\}} \left(1 - e^{-ikx} + \frac{ikx}{1+x^2}\right) \phi_A(dx) \tag{22}$$

is the symbol of $A = A(1)$.
Moreover, let $\phi_D(dt) = \phi(\mathbb{R}, dt)$ denote the Lévy measure of the Lévy process $\{D(u)\}_{u\geq 0}$. If we let $k = 0$ in (21) we get

$$\mathbb{E}\left[e^{-sD(u)}\right] = e^{-u\psi_D(s)}$$

where

$$\psi_D(s) = \int\limits_0^\infty \left(1 - e^{-sv}\right) \phi_D(dv) \tag{23}$$

is the Laplace symbol of $\{D(u)\}_{u\geq 0}$. Since D is β-stable it is well known that $\psi_D(s) = cs^\beta$ for some constant $c > 0$. Furthermore, observe [3, Corollary 2.3] that A and D and hence the Lévy processes $\{A(u)\}_{u\geq 0}$ and $\{D(u)\}_{u\geq 0}$ are independent, so that the CTRW is uncoupled, if and only if

$$\phi(dx, dt) = \delta_0(dx)\phi_D(dt) + \phi_A(dx)\delta_0(dt) \tag{24}$$

where δ_0 denotes the point mass at zero.

Note that $\{D(u)\}_{u\geq 0}$ is a β-stable subordinator and hence the sample paths of $D(u)$ are cádlág, strictly increasing and $D(u) \to \infty$ as $u \to \infty$. Define the *first passage time* process by

$$E(t) = \inf\{u \geq 0 : D(u) > t\} \tag{25}$$

for $t \geq 0$.

Finally observe that the symbol $\psi(k, s)$ in (21) induces a pseudo-differential operator $\psi(i\partial_x, \partial_t)$ which for suitable functions $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ has the representation

$$\begin{aligned}\psi(i\partial_x, \partial_t)f(x,t) &= -a\partial_x f(x,t)\\ &- \int\limits_{\mathbb{R}\times\mathbb{R}_+} \left(H(t-u)f(x-y, t-u) - f(x,t) + \frac{y\partial_x f(x,t)}{1+y^2}\right) \phi(dy, du)\end{aligned} \tag{26}$$

where $H(t) = I(t \geq 0)$ denotes the Heaviside step function. In fact, if we denote by $L^1_\omega(\mathbb{R} \times \mathbb{R}_+)$ the Banach space of measurable functions for which the norm

$$\|f\|_\omega := \int\limits_{\mathbb{R}\times\mathbb{R}_+} e^{-\omega t}|f(t,x)|\, dx\, dt$$

exists, then (26) is valid for all functions in $L^1_\omega(\mathbb{R} \times \mathbb{R}_+)$ whose weak first and second order spatial derivatives as well as weak first order time derivatives belong to $L^1_\omega(\mathbb{R} \times \mathbb{R}_+)$, see Baeumer et al. [1, Theorem 3.2].

4 Limit theorems and governing equations

In this section we derive the long-time scaling limit of the coupled CTRW and OCTRW processes. Moreover, the governing pseudo-differential equations for the densities of the limit processes are obtained. Recall the definition of the first passage time from (25) above. The following result is from Jurlewicz et al. [13, Theorem 3.1].

Theorem 4.1. *Suppose that* (Y_i, J_i) *are iid random vectors on* $\mathbb{R} \times \mathbb{R}_+$ *such that* (18) *holds.*

(a) *For the CTRW in* (16) *we have for any* $t > 0$ *that*

$$c^{-\beta/\alpha} S(N(ct)) \Rightarrow A(E(t)-) \tag{27}$$

as $c \to \infty$.

(b) *For the OCTRW in* (17) *we have for any* $t > 0$ *that*

$$c^{-\beta/\alpha} S(N(ct) + 1) \Rightarrow A(E(t)) \tag{28}$$

as $c \to \infty$.

Sketch of the proof. By projecting on the second coordinate in (19) we see that

$$c^{-1/\beta} T(ct) \Rightarrow D(t) \quad \text{as } c \to \infty.$$

Using (15) and (25) this implies that

$$c^{-\beta} N(ct) \Rightarrow E(t) \quad \text{as } c \to \infty.$$

In fact, we even get from (19) that

$$\left(c^{-\beta/\alpha} S(c^\beta t), c^{-\beta} N(ct)\right) \Rightarrow (A(t), E(t))$$

as $c \to \infty$. Now write for the CTRW

$$c^{-\beta/\alpha} S(N(ct)) = c^{-\beta/\alpha} S\left(c^\beta \, c^{-\beta} N(ct)\right)$$

and use the continuity of the composition mapping to see that (27) holds true. The proof of (28) is similar, using a result from Silvestrov [44] on randomly stopped processes. □

Example 4.2. Figure 3 illustrates the difference between CTRW limit process $A(E(t)-)$ and the OCTRW limit process $A(E(t))$ in the special case where $A(u) =$

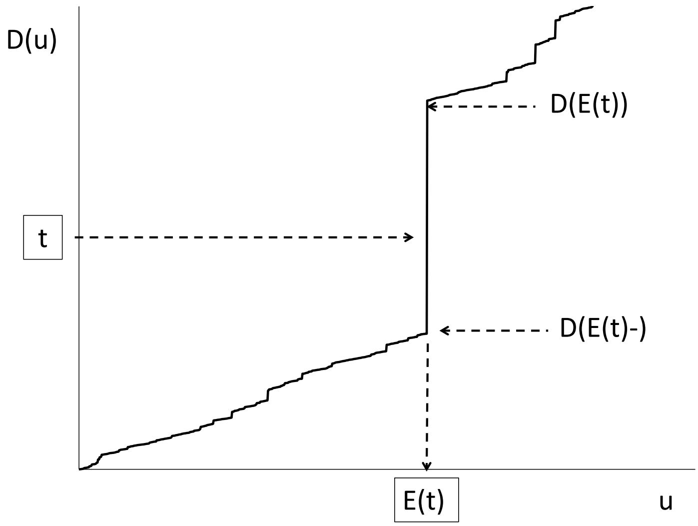

Fig. 3: Illustration of the difference between the CTRW limit process $A(E(t)-)$ and the OCTRW limit process $A(E(t))$ in the special case where $A(u) = D(u)$.

$D(u)$. This occurs when the jumps equal the waiting times, i.e., $Y_n = J_n$ for all n, see Example 5.2 for more details. At $u = E(t)$, since $D(u-) < D(u)$ at a jump, we also have $D(E(t)-) < D(E(t))$. In fact $D(E(t)-)$ is the value of the subordinator $A = D$ just before the jump, and $D(E(t))$ is the value of the subordinator after the jump. Since $D(E(t)-) < t$ and $D(E(t)) > t$ with probability one for any $t > 0$ (e.g., see Bertoin [7, III, Theorem 4]), the situation in Figure 3 is typical.

Recall that a stochastic process $\{X(t)\}_{t\geq 0}$ is called *self-similar* with index H if for any scale $c > 0$ we have $X(ct) \stackrel{d}{=} c^H X(t)$ for all $t \geq 0$, where $\stackrel{d}{=}$ denotes equality in distribution.

Corollary 4.3. *[13, Corollary 3.3] The limit processes $A(E(t)-)$ and $A(E(t))$ in Theorem 4.1 are both self-similar with index β/α.*

Proof. This follows easily since the scaling factor in both (27) and (28) is $c^{-\beta/\alpha}$. □

We now present the governing pseudo-differential equations of the CTRW limit process $A(E(t)-)$ and the OCTRW limit process $A(E(t))$ obtained in Theorem 4.1. We show that the governing equations of the CTRW and OCTRW only differ in their initial/boundary conditions. While this may seem like a minor difference, the result can be quite dramatic, as we shall see in the examples in Section 5. Recall the representation of the pseudo-differential operator $\psi(i\partial_x, \partial_t)$ from (26). Also observe that for $t > 0$ the set $\mathbb{R} \times (t, \infty)$ is bounded away from $(0, 0)$ and hence $\phi(dx, (t, \infty))$ is a finite measure.

Theorem 4.4. *[13, Theorem 4.1]*
(a) *The density $c(x, t)$ of the CTRW limit $A(E(t)-)$ is a solution to the governing equation*

$$\psi(i\partial_x, \partial_t)c(x, t) = \delta_0(dx)\phi_D(t, \infty). \tag{29}$$

(b) *The density $a(x, t)$ of the OCTRW limit $A(E(t))$ is a solution to the governing equation*

$$\psi(i\partial_x, \partial_t)a(x, t) = \phi(dx, (t, \infty)). \tag{30}$$

Remark 4.5. In the uncoupled case, where A and D are independent we get using (24) that

$$\phi(dx, (t, \infty)) = \delta_0(dx)\phi_D(t, \infty)$$

so that in the uncoupled case the CTRW limit $A(E(t)-)$ and the OCTRW limit $A(E(t))$ are identical.

Remark 4.6. The densities in (29) and (30) are the point source solutions to those equations, that is $c(x, 0) = \delta_0(x)$ and $a(x, 0) = \delta(x)$. If one has a (smooth) initial condition $p(y)$ the CTRW and OCTRW governing equations read

$$\psi(i\partial_x, \partial_t)c(x, t) = p(x)\phi_D(t, \infty)$$

and

$$\psi(i\partial_x, \partial_t)a(x, t) = \int_{-\infty}^{\infty} p(x - y)\phi(dy, (t, \infty)),$$

respectively.

Solving the governing equations (29) and (30) for the CTRW and OCTRW limit processes relies heavily on Fourier-Laplace transform (FLT) techniques. Let $\{X(t)\}_{t\geq 0}$ be a stochastic process and let $m(x, t)$ denote the density of $X(t)$. Then the FLT of $m(x, t)$ is defined as

$$\bar{m}(k, s) = \int_0^{\infty}\int_{\mathbb{R}} e^{ikx}e^{-st}m(x, t)\, dx\, dt \tag{31}$$

for $k \in \mathbb{R}$ and $s > 0$. The following result gives the FLT of the densities of the CTRW and OCTRW limit. Recall the definition of the symbols of the Lévy processes $\{(A(t), D(t))\}_{t\geq 0}$, $\{A(t)\}_{t\geq 0}$ and $\{D(t)\}_{t\geq 0}$ from (21), (22) and (23) above.

Theorem 4.7. *[13, Proposition 4.2]*

(a) *The density $c(x,t)$ of the CTRW limit $A(E(t)-)$ has FLT*

$$\bar{c}(k,s) = \frac{1}{s}\frac{\psi_D(s)}{\psi(k,s)} \tag{32}$$

for $k \in \mathbb{R}$, $s > 0$.

(b) *The density $a(x,t)$ of the OCTRW limit $A(E(t))$ has FLT*

$$\bar{a}(k,s) = \frac{1}{s}\frac{\psi(k,s) - \psi_A(k)}{\psi(k,s)} \tag{33}$$

for $k \in \mathbb{R}$, $s > 0$.

5 Examples

In this section we will present several concrete examples of coupled CTRW and OCTRW limits, and solve the corresponding governing equations. In the coupled case, these equations involve coupled space-time fractional derivative operators.

Example 5.1. (uncoupled case)
Here we revisit the uncoupled case from Section 2, to show how the same results follow from the more general coupled CTRW limit theory. If Y_n and J_n are independent, then so are $A(t)$ and $D(t)$. Then the FL-symbol is $\psi(k,s) = \psi_A(k) + \psi_D(s)$ and in view of Remark 4.5 we have $\phi(dx,(t,\infty)) = \delta_0(dx)\phi_D(t,\infty)$. Suppose that the stable Lévy motion $\{A(t)\}_{t\geq 0}$ is totally positively skewed with Fourier-symbol $\psi_A(k) = -b(-ik)^\alpha$ for some $0 < \alpha \leq 2$, $\alpha \neq 0$. Suppose further the $\{D(t)\}_{t\geq 0}$ is a standard β-stable subordinator with Laplace-symbol

$$\psi_D(s) = s^\beta = \int_0^\infty (1 - e^{-su})\,\phi_D(du). \tag{34}$$

Then in view of [23, Theorem 7.3.7] we have

$$\phi_D(t,\infty) = \frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{35}$$

Since in the uncoupled case the CTRW limit $A(E(t)-)$ and the OCTRW limit $A(E(t))$ are identical, the governing equations (29) and (30) read

$$\partial_t^\beta c_1(x,t) = b\partial_x^\alpha c_1(x,t) + \delta_0(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{36}$$

where $b < 0$ if $0 < \alpha < 1$ and $b > 0$ for $1 < \alpha \le 2$. The β-stable random variable D has a smooth density $g_\beta(u)$ supported on $u > 0$ and the stable Lévy-motion $A(t)$ has a smooth density $p(x, t)$. It follows from a simple conditioning argument, as in Section 2, that $A(E(t)) = A(E(t)-)$ has the density

$$c_1(x,t) = \int_0^\infty p(x,(t/s)^\beta) g_\beta(s)\,ds = \frac{t}{\beta}\int_0^\infty p(x,u) g_\beta(tu^{-1/\beta}) u^{-1/\beta-1}\,dx, \tag{37}$$

that solves the governing equation (36). See [22, 24] for details. Equation (36) is called space-time fractional diffusion equation. Figure 4 plots the solution for $\alpha = 2$, so that $A(t)$ is a traditional Brownian motion, and (36) reduces to the time-fractional diffusion equation. The plot compares the case of heavy tailed waiting times $\beta = 0.6$ with the case of light tailed waiting times $\beta = 1$. In the light tailed case, (36) reduces to the traditional diffusion equation. The introduction of a time-fractional derivative produces a sharper peak at the origin, and heavier tails. Both are the consequence of long waiting times between jumps. The plot was drawn in the open source programming language R [36] using the `stabledist` package. Codes for all the figures in this paper are available from the authors upon request.

The remaining examples are coupled. Suppose that J_n are iid with D, a standard β-stable random variable with Laplace symbol (34) and Lévy measure (35). For any probability measure ω on $\mathbb{R}$ and any $p > \beta/2$, suppose that the conditional distribution of Y_n given $J_n = t$ is $\omega(t^{-p}dx)$. Then [3, Theorem 2.2] shows that (18) holds, and that the Lévy measure of (A, D) is given by

$$\phi(dx, dt) = (t^p\omega)(dy)\phi_D(dt). \tag{38}$$

In this case, A is stable with index $\alpha = \beta/p$.

Example 5.2. (Lévy walk)
Suppose that $Y_n = J_n$ as in Kotulski [16]. Take J_n iid with D, a standard β-stable random variable. From (38) with $p = 1$ and $\omega = \varepsilon_1$ (the point mass in one) we see that the Lévy measure of (A, D) is given by

$$\phi(dy, dt) = \varepsilon_t(dy)\phi_D(dt) \tag{39}$$

which is concentrated on the line $y = t$. It follows that $\psi_A(k) = \psi_D(-ik)$ and $\psi(k, s) = (s - ik)^\beta$. Since $A = D$, the joint distribution of $(A(s), D(s))$ is given by

$$P_{(A(s),D(s))}(dx, dt) = \varepsilon_t(dx)P_{D(s)}(dt).$$

Theorem 4.7 shows that the CTRW limit $A(E(t)-) = D(E(t)-)$ in (27) has FLT

$$\bar{c}_2(k,s) = \frac{1}{s}\frac{\psi_D(s)}{\psi(k,s)} = \frac{s^{\beta-1}}{(s-ik)^\beta}. \tag{40}$$

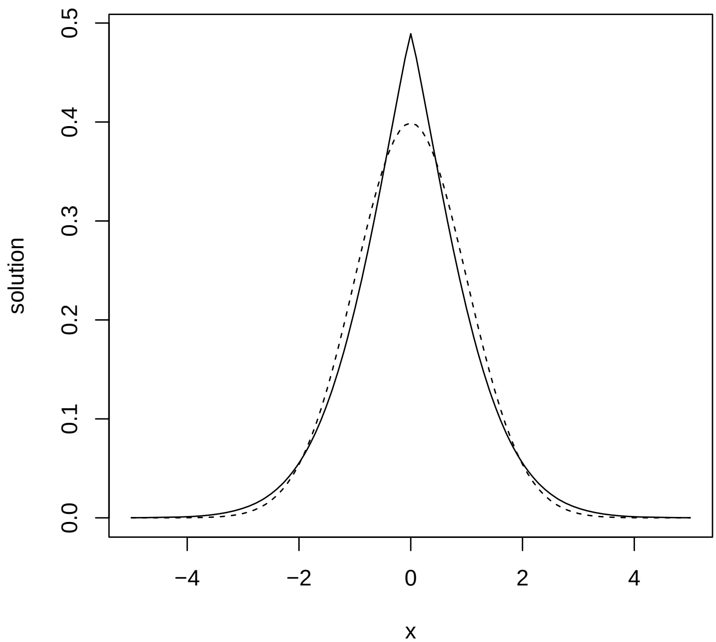

Fig. 4: Solution $c_1(x, t)$ to the uncoupled OCTRW limit equation (37) with $t = 1.0$, $\alpha = 2$, and $b = 1$ in the case $\beta = 0.6$ (solid line), compared with the solution to (37) with $t = 1.0$, $\alpha = 2$, and $b = 1$ in the traditional diffusion case $\beta = 1$ (dashed line). In the uncoupled case, the CTRW and OCTRW are governed by the same equation.

As in [3, Example 5.4] one can invert the FLT in (40) to get

$$c_2(x, t) = \frac{x^{\beta-1}(t-x)^{-\beta}}{\Gamma(\beta)\Gamma(1-\beta)} \quad 0 < x < t. \tag{41}$$

It solves the coupled governing equation (29) which can be written as

$$(\partial_t + \partial_x)^\beta c_2(x, t) = \delta_0(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{42}$$

It follows from (33) that the OCTRW limit $A(E(t)) = D(E(t))$ in (28) has FLT

$$\bar{a}_2(k, s) = \frac{1}{s}\frac{\psi(k, s) - \psi_A(k)}{\psi(k, s)} = \frac{1}{s}\frac{(s-ik)^\beta - (-ik)^\beta}{(s-ik)^\beta}. \tag{43}$$

Inverting the FLT as in [3, Example 5.4] yields that

$$a_2(x,t) = \frac{x^{-1}}{\Gamma(\beta)\Gamma(1-\beta)}\left(\frac{t}{x-t}\right)^{\beta} \quad x > t \tag{44}$$

is the density of the OCTRW limit $D(E(t))$. It solves the governing equation

$$(\partial_t + \partial_x)^{\beta} a_2(x,t) = \frac{1}{\Gamma(1-\beta)} \int_t^{\infty} \delta_0(x-u)\beta u^{-\beta-1}\, du \tag{45}$$

Both governing equations (42) and (45) involve the fractional material derivative $(\partial_t + \partial_x)^{\beta}$ considered by Sokolov and Metzler [45].

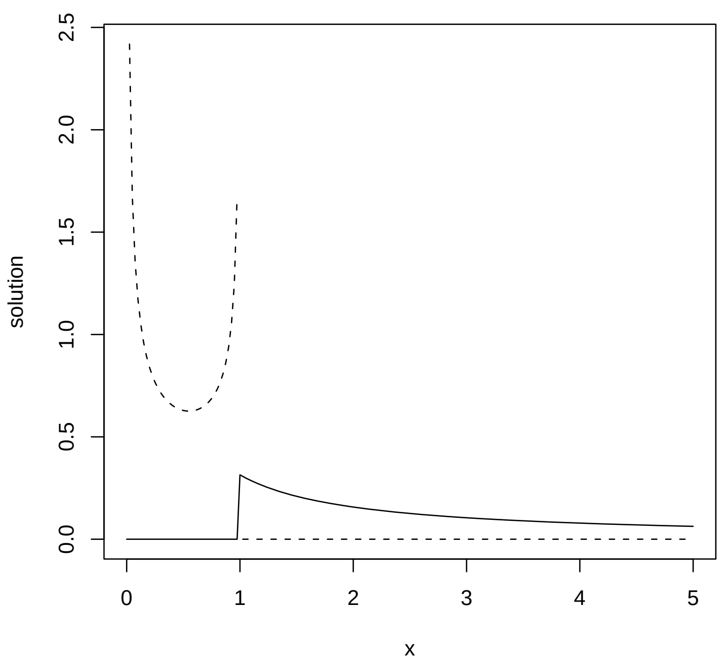

Fig. 5: Solution $a_2(x,t)$ to the coupled OCTRW limit equation (45) at $t = 1.0$ in the case $\beta = 0.45$ (solid line), compared with the solution $c_2(x,t)$ to the coupled CTRW limit equation (42) with $t = 1.0$ and $\beta = 0.45$ (dashed line).

Figure 5 compares the CTRW and OCTRW limit pdf in the case where both the waiting times and the jumps are heavy tailed with $\beta = 0.45$. Note the striking difference between the CTRW and OCTRW limit pdf. The CTRW limit density $c_2(x, t)$ in (41) is supported on $0 < x < t$ and has moments of all orders. The OCTRW limit $a_2(x, t)$ in (44) falls of like $x^{-1-\beta}$ as $x \to \infty$ and hence its moments of order $> \beta$ diverge. Recall that in this model both the jumps and the waiting times are positive random variables.

The coupled CTRW $S(N(t))$ lies between 0 and t because the jumps and the waiting times are equal, but at any time $t > 0$ that is not a jump time $T(n)$, the particle has experienced a portion of the waiting time J_{n+1}, but not the jump Y_{n+1}. The coupled OCTRW $S(N(t)+1)$ lies between t and ∞ because at any time $t > 0$ that is not a jump time $T(n)$, the particle has already experienced the jump Y_{n+1}, but only a portion of the waiting time J_{n+1}. Hence the limit CTRW pdf is concentrated on $0 < x < t$, and the limit OCTRW pdf is supported on $x > t$.

It may seem strange that the difference of a single jump can have such a profound effect on the limit pdf. However, in the case of heavy tails, we explained in Example 2.4 that $c^{-1/\alpha}M([ct]) \Rightarrow Z(t)$, where $M(n) = \max(Y_1, \ldots, Y_n)$. Since we also have $c^{-1/\alpha}S([ct]) \Rightarrow A(t)$, the largest jump is the same order of magnitude as the entire sum. Hence a single jump Y_i can be comparable to the entire sum of jumps $S(n)$, and likewise for the waiting times.

Example 5.3. (Gaussian mixture)
Suppose that D is a β-stable random variable with $\mathbb{E}(e^{-sD}) = e^{-s^\beta}$ and the conditional distribution of Y given $D = t$ is normal with mean zero and variance $2t$, as in Shlesinger, Klafter, and Wong [43]. Then Y is symmetric stable with index $\alpha = 2\beta$, since

$$\mathbb{E}[e^{ikY}] = \mathbb{E}[e^{-Dk^2}] = e^{-|k|^{2\beta}},$$

using the fact that a normal with mean zero and variance $2t$ has FT e^{-tk^2}. If we take (Y_n, J_n) iid with (Y, J) then (18) holds and it follows from (38) that the Lévy measure of (A, D) is given by

$$\phi(dx, dt) = \mathcal{N}_{0,2t}(dx)\phi_D(dt)$$

where $\mathcal{N}_{0,2t}$ is a normal distribution with mean zero and variance $2t$. Then the Lévy symbol of (A, D) equals

$$\psi(k, s) = (s + k^2)^\beta. \tag{46}$$

By (32) the CTRW limit $A(E(t)-)$ has FLT

$$\bar{c}_3(k, s) = \frac{s^{\beta-1}}{(s + k^2)^\beta}. \tag{47}$$

Inverting the FLT [3, Example 5.2] shows that the CTRW limit has the density

$$c_3(x,t) = \int_0^t \frac{1}{\sqrt{4\pi u}} \exp\left(-\frac{x^2}{4u}\right) c_2(u,t)\, du \tag{48}$$

with $c_2(u, t)$ as in (41). This density solves the governing equation

$$(\partial_t - \partial_x^2)^\beta c_3(x,t) = \delta_0(x) \frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{49}$$

By (33) the OCTRW limit $A(E(t))$ has FLT

$$\bar{a}_3(k,s) = \frac{(s+k^2)^\beta - |k|^{2\beta}}{(s+k^2)^\beta} \tag{50}$$

and has the density [13, Example 5.3]

$$a_3(x,t) = \int_t^\infty \frac{1}{\sqrt{4\pi u}} \exp\left(-\frac{x^2}{4u}\right) a_2(u,t)\, du \tag{51}$$

with $a_2(u, t)$ as in (44). In view of (30) it solves the governing equation

$$(\partial_t - \partial_x^2)^\beta a_3(x,t) = \frac{1}{\Gamma(1-\beta)} \int_t^\infty \frac{1}{\sqrt{4\pi u}} \exp\left(-\frac{x^2}{4u}\right) \beta u^{-\beta-1}\, du \tag{52}$$

Figure 6 plots the pdf of the OCTRW limit and the CTRW limit in the case $\beta = 0.8$ at time $t = 1$. The difference is striking. The CTRW limit pdf has a sharp peak, and a much lighter tail than the OCTRW limit pdf. See Meerschaert and Scalas [25] for an application to finance.

Acknowledgment: Mark M. Meerschaert was partially supported by ARO MURI grant W911NF-15-1-0562 and USA National Science Foundation grants DMS-1462156 and EAR-1344280.

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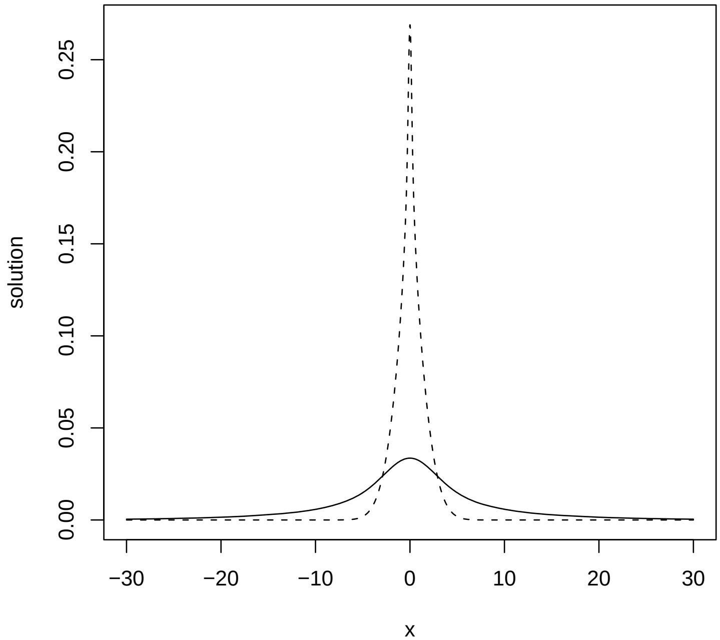

Fig. 6: Solution $a_3(x, t)$ to the coupled OCTRW limit equation (52) with $t = 1.0$ and $\beta = 0.8$ (solid line), and solution $c_3(x, t)$ to the corresponding CTRW limit equation (49) with $t = 1.0$ and $\beta = 0.8$ (dashed line).

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