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Research Paper

Correlation Structure of Time-Changed Lévy Processes

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Dedicated to Professor Francesco Mainardi
on the occasion of his retirement

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Abstract

Time-changed Lévy processes include the fractional Poisson process, and the scaling limit of a continuous time random walk. They are obtained by replacing the deterministic time variable by a positive non-decreasing random process. The use of time-changed processes in modeling often requires the knowledge of their second order properties such as the correlation function. This paper provides the explicit expression for the correlation function for time-changed Lévy processes. The processes used to model random time include subordinators and inverse subordinators, and the time-changed Lévy processes include limits of continuous time random walks. Several examples useful in applications are discussed.

Keywords: Lévy processes, subordinators, inverse subordinators, correlation function, Mittag-Leffler function.

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1. Introduction and notation.

Time-changed Lévy processes arise in many applications. Gorenflo and Mainardi [15] show that a continuous time random walk (CTRW) with

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power law distributed random waiting times between each random jump converges to a Lévy process time-changed by an inverse stable subordinator, see also [30,32]. The CTRW is used as a model of anomalous diffusion in physics, finance, hydrology, and other fields [6,7,29,36,42]. Mainardi et al. [26,27] study the fractional Poisson process, where the exponential waiting time distribution is replaced by a Mittag-Leffler distribution, see also [5,20, 40]. Meerschaert et al. [34] showed that the same fractional Poisson process can also be obtained via an inverse stable time-change. Recent work in finance questioned the classical geometric Brownian motion (gBM) model, and random activity time models have been developed [16,21]. In these and other applications [12,17,43] it is useful to compute the correlation function of the time-changed process, and this paper develops explicit computational formulae.

The simplest CTRW model assumes that the independent identically distributed (iid) particle jumps J_n with mean $\mu = \mathbb{E}[J_n] = 0$ and finite variance $\sigma^2 = \mathrm{Var}(J_n) = \mathbb{E}[(J_n - \mu)^2]$ are separated by iid waiting times W_n, with $P(W_n > t) \sim t^{-\alpha}/\Gamma(1-\alpha)$ as $t \to \infty$. Then the particle arrives at location $S_n = J_1 + \cdots + J_n$ at time $T_n = W_1 + \cdots + W_n$. The number of jumps by time $t > 0$ is given by the renewal process $N_t = \max\{n \geq 0 : T_n \leq t\}$. The extended central limit theorem [35, Theorem 4.5] yields $n^{-1/\alpha} T_{[nt]} \Rightarrow L(t)$, a standard α-stable subordinator with $\mathbb{E}[e^{-sL(t)}] = e^{-ts^\alpha}$ for all s, $t > 0$. Since $\{N_t \geq n\} = \{T_n \leq t\}$, a continuous mapping argument yields $n^{-\alpha} N_{nt} \Rightarrow Y(t) = \inf\{u > 0 : L(u) > y\}$, an inverse α-stable subordinator. Since the inverse process $Y(t)$ is constant over the jump intervals of $L(t)$ (whose length is, in general, not an exponential random variable), it is not a Markov process. The increments of $Y(t)$ are neither stationary nor independent [30]. The CTRW $S(N_t)$ gives the particle location at time $t > 0$, with long time limit $n^{-\alpha/2} S(N_{nt}) \Rightarrow X(Y(t))$ in Skorokhod's M_1 topology, where $X(t)$ Brownian motion [30, Theorem 4.2]. The proof uses the fact that $X(t)$ and $L(t)$, two independent Lèvy processes, have no common points of discontinuity almost surely, and applies the continuous mapping theorem. The outer process $X(t)$ models the random walk, and the inner process $Y(t)$ accounts for particle waiting times.

A very general class of CTRW models was considered in [32], where the particle jumps $J_n^{(c)}$ form a triangular array in $\mathbb{R}^d$, and the waiting times form another triangular array $W_n^{(c)}$ in $\mathbb{R}_+$. The pair of row sums $(S^{(c)}(cu), T^{(c)}(cu))$ is assumed to converge to $(X(u), L(u))$, $u \geq 0$ as $c \to \infty$ in Skorokhod's J_1 topology on $D([0,\infty), \mathbb{R}^d \times \mathbb{R}_+)$, where $(X(u), L(u))$ is a Lévy process on $\mathbb{R}^d \times \mathbb{R}_+$. In this setting $X(t)$ could be an arbitrary Lévy process, and $L(t)$ is a subordinator (i.e. a one-dimensional Lévy process

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with nonnegative increments). The Laplace transform of $L(t)$ is

$$\mathbb{E}[e^{-sL(t)}] = e^{-t\phi(s)}, \ s \geq 0,$$

where the Laplace exponent is a Bernstein function

$$\phi(s) = \mu s + \int_{(0,\infty)} (1 - e^{-sx})\nu(dx), s \geq 0. \tag{1}$$

If the drift coefficient $\mu = 0$, or if the Lévy measure ν satisfies $\nu(0, \infty) = \infty$, then L is strictly increasing. If, in addition, $X(t)$ and $L(t)$ have no common points of discontinuity almost surely which is, for example, true if jumps are independent of waiting times, then Straka and Henry [44] showed that the CTRW $X^{(c)}(t) = S^{(c)}(N_t^{(c)})$ with $N_t^{(c)} = \max\{n \geq 0 : T^{(c)}(n) \leq t\}$ converges to $X(Y(t))$ in Skorokhod's J_1 topology on $D([0, \infty), \mathbb{R}^d)$. Here, as before, $Y(t) = \inf\{u > 0 : L(u) > y\}$.

For the case where $X(t)$ is Brownian motion and $L(t)$ is a standard stable subordinator, a formula for the correlation function has been obtained by Janczura and Wyłomańska [17] using the result of Magdziarz [24, Theorem 2.1], see also [25, Section 2], who showed that $X(Y(t))$ is a martingale with respect to a suitably defined filtration. Then Janczura and Wyłomańska [17] computed that for $0 \leq s \leq t$

$$\mathrm{corr}(X(Y(t)), X(Y(s))) = \left(\frac{s}{t}\right)^{\alpha/2}.$$

The present paper uses a different method to compute the correlation function, and treats a more general case when the outer process is any Lévy process, and the inner process is any random time-change, both with finite second moment. Then the explicit formula is derived for the correlation function of several other time-changed processes that arise in applications.

2. Correlation function.

In this section, we prove a general result that can be used to compute the correlation function of a time-changed Lévy process $Z(t) = X(Y(t))$ where X, Y are independent, and in general Y may be non-Markovian with non-stationary and non-independent increments. For example, it might be an inverse subordinator, as in CTRW limit theory. Then Z may also be also non-Markovian with non-stationary and non-independent increments. The next result gives an explicit expression for the correlation function of this time-changed process.

Theorem 2.1. *Suppose that $X(t)$, $t \geq 0$ is a homogeneous Lévy process with $X(0) = 0$, and $Y(t)$ is a non-decreasing process independent of X.*

If $\psi(\xi) = -\log \mathbb{E}e^{i\xi X(1)}$ *is the characteristic exponent of the Lévy process, then the characteristic function of the process* $Z(t) = X(Y(t))$ *is given by*

$$\mathbb{E}e^{i\xi Z(t)} = \mathbb{E}e^{-\psi(\xi)Y(t)}.$$

Moreover, if $\mathbb{E}X(1)$ *and* $U(t) = \mathbb{E}Y(t)$ *exist, then* $\mathbb{E}Z(t)$ *exists and*

$$\mathbb{E}[Z(t)] = U(t)\mathbb{E}[X(1)]; \tag{2}$$

if X *and* Y *have finite second moments, so does* Z *and*

$$\operatorname{Var}[Z(t)] = [\mathbb{E}X(1)]^2 \operatorname{Var}[Y(t)] + U(t) \operatorname{Var}[X(1)], \tag{3}$$

and the covariance function is given by

$$\operatorname{Cov}[Z(t), Z(s)] = \operatorname{Var}[X(1)]U(\min(t, s)) + [\mathbb{E}X(1)]^2 \operatorname{Cov}[Y(t), Y(s)]. \tag{4}$$

Proof. Using the independence of the processes X and Y we get

$$\begin{aligned}\mathbb{E}e^{i\xi Z(t)} = \mathbb{E}e^{i\xi X(Y(t))} &= \int \mathbb{E}e^{i\xi X(y)}\mathbb{P}(Y(t) \in dy) \\ &= \int e^{-y\psi(\xi)}\mathbb{P}(Y(t) \in dy) = \mathbb{E}e^{-\psi(\xi)Y(t)}.\end{aligned}$$

Differentiating the characteristic function, we can work out the moments of the random variables (provided that they exist; for even moments this is guaranteed by the differentiability of the characteristic function). From $\mathbb{E}e^{i\xi X(t)} = e^{-t\psi(\xi)}$, we see that $\mathbb{E}X(t) = it\psi'(0)$ and $\operatorname{Var} X(t) = t\psi''(0)$. Thus,

$$\mathbb{E}Z(t) = -i \left.\frac{d}{d\xi}\mathbb{E}e^{i\xi Z(t)}\right|_{\xi=0} = -i\mathbb{E}Y(t)\psi'(0) = U(t)\mathbb{E}X(1)$$

and

$$\begin{aligned}\mathbb{E}\big[Z(t)^2\big] = -\left.\frac{d^2}{d\xi^2}\mathbb{E}e^{i\xi Z(t)}\right|_{\xi=0} &= \mathbb{E}\left[Y(t)\psi''(0) - Y^2(t)\{\psi'(\xi)\}^2\right] \\ &= U(t) \operatorname{Var} X(1) - \mathbb{E}\big[Y(t)^2\big]\big[\mathbb{E}X(1)\big]^2 \\ &= U(t) \operatorname{Var} X(1) + \big[\mathbb{E}X(1)\big]^2 \operatorname{Var} Y(t) + \big[\mathbb{E}Y(t)\big]^2 \big[\mathbb{E}X(1)\big]^2.\end{aligned}$$

This proves (3).

Since the outer process $X(t)$ has independent increments, for $0 < s < t$ we have

$$\begin{aligned}\mathbb{E}[X(t)X(s)] &= \mathbb{E}[(X(t) - X(s))X(s)] + \mathbb{E}[X(s)^2] \\ &= \mathbb{E}[X(t) - X(s)]\mathbb{E}[X(s)] + \mathbb{E}[X(s)^2] \\ &= (t - s)s\mathbb{E}[X(1)]^2 + \mathrm{Var}[X(s)] + s^2\mathbb{E}[X(1)]^2 \\ &= ts\mathbb{E}[X(1)]^2 + s\,\mathrm{Var}[X(1)].\end{aligned}$$

Then, since the processes X and Y are independent, a simple conditioning argument yields

$$\mathbb{E}[X(Y(t))\,X(Y(s))] = \mathbb{E}[Y(t)Y(s)]\mathbb{E}[X(1)]^2 + \mathbb{E}[Y(s)]\,\mathrm{Var}[X(1)].$$

Then the covariance function of the time-changed process is

$$\begin{aligned}&\mathrm{Cov}[Z(t), Z(s)] \\ &\quad= \mathbb{E}[Y(t)Y(s)]\mathbb{E}[X(1)]^2 + \mathbb{E}[Y(s)]\,\mathrm{Var}[X(1)] - \mathbb{E}[Z(t)]\mathbb{E}[Z(s)] \\ &\quad= \mathbb{E}[Y(t)Y(s)]\mathbb{E}[X(1)]^2 + \mathbb{E}[Y(s)]\,\mathrm{Var}[X(1)] - U(t)U(s)\mathbb{E}[X(1)]^2 \\ &\quad= U(s)\,\mathrm{Var}[X(1)] + \mathbb{E}[X(1)]^2\,\mathrm{Cov}[Y(t), Y(s)],\end{aligned}$$

by another conditioning argument. □

Remark 2.1. In the special case $EX(1) = 0$, the results of Theorem 2.1 simplify. Now the time-changed process $Z(t) = X(Y(t))$ has mean zero, its variance is

$$\mathrm{Var}[Z(t)] = U(t)\,\mathrm{Var}[X(1)],$$

its covariance function is

$$\mathrm{Cov}[Z(t), Z(s)] = \mathrm{Var}[X(1)]U(\min(t, s)),$$

and its correlation function is

$$\mathrm{corr}[Z(t), Z(s)] = \frac{U(\min(t, s))}{\sqrt{U(t)U(s)}} = \sqrt{\frac{U(\min(t, s))}{U(\max(t, s))}}.$$

This special case is relevant to many applications.

3. Applications.

In this section, we compute the correlation function for several examples that are important in applications. In view of Theorem 2.1, the main technical issue is the computation of the renewal function $U(t)$ for the time-change process Y. The first example deals with a time-change process that is a subordinator, and time-changes in the gBM model for a risky asset. The time-change allows one to obtain distributions of log-returns (increments of the logarithm of a price) that are heavier-tailed and higher-peaked than Gaussian. This distributional property is one of the 'stylized facts' that are typical of financial data [14].

Example 3.1 (Inverse Gaussian subordinator). The inverse Gaussian subordinator $Y(t)$ is obtained as a hitting time process:

$$Y(t) = \inf\{u \geq 0 : \gamma u + W(u) = \delta t\}, \, t \geq 0,$$

where W is the standard Brownian motion, and γ, $\delta > 0$, see [1,45]. The process $Y(t), t \geq 0$, is a Lévy process with $U(t) = \mathbb{E}[Y(t)] = \delta t/\gamma$, and

$$\mathrm{Cov}[Y(t), Y(s)] = \mathrm{Var}[Y(1)] \min(t, s) = \frac{\delta}{\gamma^3} \min(t, s).$$

If the outer process $X(t)$ is a Brownian motion with drift, with mean μt and variance $\sigma^2 t$, then the time-changed process Z is used to specify the normal inverse Gaussian (NIG) model for a risky asset [4]. This process is also the limit of a CTRW with finite variance jumps and finite mean waiting times [19]. Theorem 2.1 implies that the NIG process has mean $\mathbb{E}[Z(t)] = \mu\delta t/\gamma$, and covariance function

$$\mathrm{Cov}[Z(t), Z(s)] = \frac{\delta(\mu^2 + \sigma^2\gamma^2)}{\gamma^3} \min(t, s),$$

for any $s, t \geq 0$. A variance gamma model [23] similar to NIG model is obtained when the inverse Gaussian subordinator is replaced by a Gamma subordinator.

The remaining examples deal with the random time-changes that are the inverse or hitting time processes of a Lévy subordinator L with the Laplace exponent ϕ defined in equation (1). The inverse or first passage time process of L

$$Y(t) = \inf\{u \geq 0 : L(u) > t\}, t \geq 0 \tag{5}$$

is nondecreasing, and its sample paths are almost surely continuous if L is strictly increasing. For any Lévy subordinator L, Veillette and Taqqu [45] show that the renewal function $U(t) = \mathbb{E}[Y(t)]$ of its inverse (5) has Laplace transform $\tilde{U}$ given by:

$$\tilde{U}(s) = \int_0^\infty U(t)e^{-st}dt = \frac{1}{s\phi(s)}, \tag{6}$$

where ϕ is Laplace exponent of L. Thus, U characterizes the inverse process Y, since ϕ characterizes L. For example, it follows easily from [46, Theorem 4.2] that the second moment of Y is

$$\mathbb{E}Y^2(t) = \int_0^t 2U(t-\tau)dU(\tau).$$

The covariance function of Y is given by [46, Corollary 4.3]:

$$\mathrm{Cov}[Y(t_1), Y(t_2)] = \int_0^{\min(t_1,t_2)} (U(t_1-\tau)+U(t_2-\tau))dU(\tau) - U(t_1)U(t_2). \tag{7}$$

For many inverse subordinators, the Laplace exponent ϕ can be written explicitly using (6), but the inversion to obtain the renewal function may be difficult. Numerical methods for the inversion were proposed in [45,46]. Below we give examples from applications where the Laplace transform can be inverted analytically and where its asymptotic behavior can be found in order to characterize the behavior of the correlation function of the time-changed process.

Example 3.2 (Inverse stable subordinator). Suppose $L(t)$ is standard α-stable subordinator with index $0 < \alpha < 1$, so that the Laplace exponent $\phi(s) = s^\alpha$ for all $s > 0$. Bingham [9] and Bondesson et al. [10] showed that the inverse stable subordinator (5) has a Mittag-Leffler distribution:

$$\mathbb{E}\left[e^{-sY(t)}\right] = \sum_{n=0}^{\infty} \frac{(-st^\alpha)^n}{\Gamma(\alpha n + 1)} = E_\alpha(-st^\alpha),$$

where E_α is Mittag-Leffler function:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}.$$

When the outer process $X(t)$ is a homogeneous Poisson process, the time-changed process $X(Y(t))$ is fractional Poisson process [5,20,26,27,34,39,40]. More generally, for any Lévy process $X(t)$, the time-changed process

$X(Y(t))$ is a CTRW limit where the waiting times between particle jumps belong to the domain of attraction of the stable subordinator $L(t)$, see [32].

Since $\tilde{U}(s) = 1/s^{\alpha+1}$, the renewal function

$$U(t) = \mathbb{E}[Y(t)] = \frac{t^\alpha}{\Gamma(1+\alpha)}. \tag{8}$$

For $0 < s \le t$, substitute (8) into (7) to see that the covariance function of the inverse stable subordinator is

(9)
$$\begin{aligned}
\mathrm{Cov}[Y(t), Y(s)] &= \frac{\alpha}{\Gamma(1+\alpha)^2} \int_0^s \left((t-\tau)^\alpha + (s-\tau)^\alpha\right) \tau^{\alpha-1} d\tau - \frac{(ts)^\alpha}{\Gamma(1+\alpha)^2} \\
&= \frac{\alpha t^{2\alpha}}{\Gamma(1+\alpha)^2} \int_0^{s/t} (1-u)^\alpha u^{\alpha-1} du \\
&\quad + \frac{\alpha s^{2\alpha}}{\Gamma(1+\alpha)^2} B(\alpha, \alpha+1) - \frac{(ts)^\alpha}{\Gamma(1+\alpha)^2} \\
&= \frac{1}{\Gamma(1+\alpha)^2} \Big[\alpha t^{2\alpha} B(\alpha, \alpha+1; s/t) \\
&\qquad + \alpha s^{2\alpha} B(\alpha, \alpha+1) - (ts)^\alpha\Big],
\end{aligned}$$

using a substitution $u = \tau/t$, where $B(a, b; x) := \int_0^x u^{a-1}(1-u)^{b-1} du$ is the incomplete Beta function, and $B(a, b) := \Gamma(a)\Gamma(b)/\Gamma(a+b) = B(a, b; 1)$ is the Beta function. An equivalent form of the covariance function in terms of the hypergeometric function was obtained in [45, Equation (74)]. Apply the Taylor series expansion $(1-u)^{b-1} = 1 + (1-b)u + O(u^2)$ as $u \to 0$ to see that

$$B(a, b; x) = \frac{x^a}{a} + (1-b)\frac{x^{a+1}}{a+1} + O(x^{a+2}) \quad \text{as } x \to 0.$$

Then it follows that for $s > 0$ fixed and $t \to \infty$ we have

$$\begin{aligned}
F(\alpha; s, t) &:= \alpha t^{2\alpha} B(\alpha, \alpha+1; s/t) - (ts)^\alpha \\
&= \alpha t^{2\alpha} \frac{(s/t)^\alpha}{\alpha} - \alpha \frac{(s/t)^{\alpha+1}}{\alpha+1} + O((s/t)^{\alpha+2}) - (ts)^\alpha \\
&= -\alpha \frac{(s/t)^{\alpha+1}}{\alpha+1} + O((s/t)^{\alpha+2}),
\end{aligned}$$

so that

$$\mathrm{Cov}[Y(t), Y(s)] = \frac{1}{\Gamma(1+\alpha)^2} \Big[\alpha s^{2\alpha} B(\alpha, \alpha+1) + F(\alpha; s, t)\Big], \tag{10}$$

where $F(\alpha; s, t) \to 0$ as $t \to \infty$. Hence

$$\mathrm{Cov}[Y(t), Y(s)] \xrightarrow[t\to\infty]{} \frac{\alpha s^{2\alpha} B(\alpha, \alpha+1)}{\Gamma(1+\alpha)^2} = \frac{s^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Letting $s = t$ it follows from (9) that

$$\begin{aligned}\mathrm{Var}[Y(t)] &= \frac{1}{\Gamma(1+\alpha)^2}\left[2t^{2\alpha}\frac{\alpha\Gamma(\alpha)\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} - t^{2\alpha}\right] \\ &= t^{2\alpha}\left[\frac{2}{\Gamma(2\alpha+1)} - \frac{1}{\Gamma(1+\alpha)^2}\right],\end{aligned} \tag{11}$$

which agrees with the computation in [2, Section 5.1]. From (10) and (11) it follows that for $0 < s \le t$

$$\mathrm{corr}[Y(s), Y(t)] = \frac{\left[\alpha s^{2\alpha} B(\alpha, \alpha+1) + F(\alpha; s, t)\right]}{(st)^\alpha \left[\frac{2\Gamma(1+\alpha)^2}{\Gamma(2\alpha+1)} - 1\right]}$$

where $F(\alpha; s, t) \to 0$ as $t \to \infty$, and hence

$$\mathrm{corr}[Y(s), Y(t)] \sim \left(\frac{s}{t}\right)^\alpha \left[2 - \frac{\Gamma(2\alpha+1)}{\Gamma(1+\alpha)^2}\right]^{-1} \quad \text{as } t \to \infty.$$

This power law decay of the correlation function can be viewed as a long range dependence for the inverse stable subordinator $Y(t)$, since the correlation function is not integrable at infinity.

From (2) and (8) we can see that the time-changed process $Z(t) = X(Y(t))$ has mean

$$\mathbb{E}[Z(t)] = \frac{t^\alpha \mathbb{E}[X(1)]}{\Gamma(1+\alpha)}. \tag{12}$$

Substituting (11) into (3) yields the variance of the time-changed process:

$$\mathrm{Var}[Z(t)] = \frac{t^\alpha\, \mathrm{Var}[X(1)]}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}[\mathbb{E}X(1)]^2}{\alpha}\left(\frac{1}{\Gamma(2\alpha)} - \frac{1}{\alpha\Gamma(\alpha)^2}\right). \tag{13}$$

This formula was derived previously by Beghin and Orsingher [5] in the special case where outer process $X(t)$ is a Poisson process, so that $Z(t)$ is a fractional Poisson process. It follows from (4), (8), and (10) that for $0 < s \le t$ the covariance function of $Z(t) = X(Y(t))$ is

$$\begin{aligned}&\mathrm{Cov}[Z(t), Z(s)] \\ &\quad = \frac{s^\alpha\, \mathrm{Var}[X(1)]}{\Gamma(1+\alpha)} + \frac{[\mathbb{E}X(1)]^2}{\Gamma(1+\alpha)^2}\left[\alpha s^{2\alpha} B(\alpha, \alpha+1) + F(\alpha; s, t)\right],\end{aligned} \tag{14}$$

where $F(\alpha; s,t) \to 0$ as $t \to \infty$, hence for a fixed $s > 0$

$$\mathrm{Cov}[Z(t), Z(s)] \xrightarrow[t\to\infty]{} \frac{s^\alpha \,\mathrm{Var}[X(1)]}{\Gamma(1+\alpha)} + \frac{s^{2\alpha}[\mathbb{E}X(1)]^2}{\Gamma(1+2\alpha)}.$$

In particular, for the fractional Poisson process the covariance function is

$$\begin{aligned}
&\mathrm{Cov}[Z(t), Z(s)] \\
&\quad = \frac{s^\alpha \lambda}{\Gamma(1+\alpha)} + \frac{\lambda^2}{\Gamma(1+\alpha)^2}\Big[\alpha s^{2\alpha} B(\alpha, \alpha+1) + F(\alpha; s, t)\Big],
\end{aligned}$$

where $\lambda > 0$ is the intensity of the outer Poisson process.

For $0 < s \le t$, the time-changed process $Z(t) = X(Y(t))$ has correlation

$$\mathrm{corr}[Z(t), Z(s)] = \frac{\mathrm{Cov}[Z(s), Z(t)]}{\sqrt{\mathrm{Var}[Z(s)]\ \ \mathrm{Var}[Z(t)]}}$$

where $\mathrm{Cov}[Z(s), Z(t)]$ is given by (14) and the remaining terms are specified in (13). The asymptotic behavior of the correlation depends on whether the outer process has zero mean. If $\mathbb{E}[X(1)] \neq 0$, then for any $s > 0$ fixed we have

$$\mathrm{Var}[Z(t)] \sim \frac{t^{2\alpha}[\mathbb{E}X(1)]^2}{\alpha}\left(\frac{1}{\Gamma(2\alpha)} - \frac{1}{\alpha\Gamma(\alpha)^2}\right) \quad \text{as } t \to \infty,$$

and so we have

$$\mathrm{corr}[Z(t), Z(s)] \sim t^{-\alpha} C(\alpha, s) \quad \text{as } t \to \infty,$$

where

$$C(\alpha, s) = \left(\frac{1}{\Gamma(2\alpha)} - \frac{1}{\alpha\Gamma(\alpha)^2}\right)^{-1}\left[\frac{\alpha\,\mathrm{Var}[X(1)]}{\Gamma(1+\alpha)[\mathbb{E}X(1)]^2} + \frac{\alpha s^\alpha}{\Gamma(1+2\alpha)}\right].$$

On the other hand, if $\mathbb{E}[X(1)] = 0$, then the covariance function of the time-changed process for $0 < s \le t$ simplifies to

$$\mathrm{Cov}[Z(t), Z(s)] = \mathrm{Var}[X(1)]\frac{s^\alpha}{\Gamma(1+\alpha)}.$$

and $\mathrm{corr}[Z(t), Z(s)] = (s/t)^{\alpha/2}$, a formula obtained by Janczura and Wyłomańska [17] for the special case when the outer process $X(t)$ is a Brownian motion.

In summary, the correlation function of $Z(t)$ decays like a power law $t^{-\alpha}$ when $\mathbb{E}[X(1)] \neq 0$, and even more slowly, like the power law $t^{-\alpha/2}$ when $\mathbb{E}[X(1)] = 0$. In either case, the non-stationary time-changed process $Z(t)$ exhibits long range dependence. If $\mathbb{E}[X(1)] = 0$, the time-changed process $Z(t) = X(Y(t))$ also has uncorrelated increments: Since $\mathrm{Cov}[Z(t), Z(s)]$ does not depend on t, we have $\mathrm{Var}[Z(s)] = \mathrm{Cov}[Z(s), Z(s)] = \mathrm{Cov}[Z(s), Z(t)]$ and hence, since the covariance is additive, $\mathrm{Cov}[Z(s), Z(t) - Z(s)] = 0$ for $0 < s < t$. Uncorrelated increments together with long range dependence is a hallmark of financial data [42], and hence this process can be useful to model such data. Since the outer process $X(t)$ can be any Lévy process with a finite second moment, the distribution of the time-changed process $Z(t) = X(Y(t))$ can take many forms. Similar long range dependent behavior has been obtained for a fractional Pearson diffusion, the time-change of a stationary diffusion process using the inverse stable subordinator [22].

Example 3.3 (Inverse tempered stable subordinator). The standard tempered stable subordinator $L(t)$ with $0 < \alpha < 1$ is a Lévy process with tempered stable increments [3,41]. The Lévy measure of the unit increment is

$$\nu(dx) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-\alpha-1} e^{-\lambda x}, x > 0,$$

and then (e.g., see [35, Section 7.2]):

$$\mathbb{E}[e^{-sL(t)}] = e^{-t\phi(s)} = \exp\{t((\lambda+s)^\alpha - \lambda^\alpha\}.$$

When $s \to 0$, the Laplace exponent $\phi(s) = (\lambda + s)^\alpha - \lambda^\alpha \sim s\alpha\lambda^{\alpha-1}$ as $s \to 0$, and hence

$$\tilde{U}(s) = \frac{1}{s\phi(s)} = \frac{1}{s((\lambda+s)^\alpha - \lambda^\alpha)} \sim s^{-2}\frac{\lambda^{1-\alpha}}{\alpha} \quad \text{as } s \to 0.$$

The Karamata Tauberian theorem (e.g., see [8, p.10]) implies that $U(t) \sim t^p$ as $t \to \infty$ is equivalent to $\tilde{U}(s) \sim s^{-1-p}\Gamma(1+p)$ as $s \to 0$. Hence the renewal function behaves as follows:

$$U(t) \sim t\frac{\lambda^{1-\alpha}}{\alpha} \quad \text{as } t \to \infty.$$

The same Karamata Tauberian theorem also implies that $U(t) \sim t^p$ as $t \to 0$ is equivalent to $\tilde{U}(s) \sim s^{-1-p}\Gamma(1+p)$ as $s \to \infty$. Since $\phi(s) \sim s^\alpha$ as $s \to \infty$, $\tilde{U}(s) \sim s^{-\alpha-1}$ as $s \to \infty$, and hence

$$U(t) \sim \frac{t^\alpha}{\Gamma(1+\alpha)} \quad \text{as } t \to 0.$$

When the outer process X has zero mean, then from Remark 2.1, the variance of the time-changed process

$$\mathrm{Var}[Z(t)] = U(t)\,\mathrm{Var}[X(1)]$$

grows as t when $t \to \infty$. For a fixed $s > 0$ and $t \to \infty$, the correlation function of the process Z decays as $1/\sqrt{t}$:

$$\mathrm{corr}[Z(t), Z(s)] = \sqrt{\frac{U(s)}{U(t)}} \sim \frac{\sqrt{\alpha U(s)}}{\lambda^{(1-\alpha)/2}\sqrt{t}}.$$

When t is fixed and $s \to 0$, then

$$\mathrm{corr}[Z(t), Z(s)] = \sqrt{\frac{U(s)}{U(t)}} \sim \frac{s^{\alpha/2}}{\sqrt{\Gamma(1+\alpha)U(t)}}.$$

The inverse tempered stable subordinator models transient anomalous diffusion, since it smoothly transitions between the inverse stable subordinator at early time, to a linear clock at late time. This has proven useful in applications to geophysics [33,47,48] and finance [11].

Example 3.4 (Inverse stable mixture). Now consider a mixture of standard α-stable subordinators with Laplace exponent

$$\phi(s) = \int_0^1 q(w) s^w dw = \int_0^\infty (1 - e^{-sx}) l_q(x) dx,$$

where $q(w)$ is a probability density on $(0, 1)$, and the density $l_q(x)$ of the Lévy measure is given by

$$l_q(x) = \int_0^1 \frac{w x^{-w-1}}{\Gamma(1-w)} q(w) dw. \tag{15}$$

Such mixtures are used in time-fractional models of accelerating subdiffusion, see Mainardi et al. [28] and Chechkin et al. [12]. They can also be used to model ultraslow diffusion, see Sokolov et al. [43], Meerschaert and Scheffler [31], and Kovács and Meerschaert [18].

The α-stable subordinator corresponds to the choice $q(w) = \delta(w - \alpha)$, where $\delta(\cdot)$ is the delta function. The model

$$q(w) = C_1\delta(w - \alpha_1) + C_2\delta(w - \alpha_2), \quad C_1 + C_2 = 1,$$

with $\alpha_1 > \alpha_2$ was considered in Chechkin et al. [12]. The subordinator L in this case is the linear combination of two independent stable subordinators with $\phi(s) = C_1 s^{\alpha_1} + C_2 s^{\alpha_2}$, so that

$$\tilde{U}(s) = \frac{1}{s(C_1 s^{\alpha_1} + C_2 s^{\alpha_2})}.$$

This Laplace transform can be explicitly inverted using the following property [37]:

$$\int_0^\infty e^{-st} t^{\gamma k+\delta-1} E^{(k)}_{\gamma,\delta}(\pm b t^\gamma)dt = \frac{k! s^{\gamma-\delta}}{(s^\gamma \mp b)^{k+1}}, \quad \mathrm{Re}(s) > |b|^{1/\gamma}, \tag{16}$$

where

$$E^{(k)}_{\alpha,\beta}(y) = \frac{d^k}{dy^k} E_{\alpha,\beta}(y)$$

is the k-th order derivative ($k = 0, 1, 2, \dots$) of two-parameter Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0,\ \beta > 0. \tag{17}$$

Setting $k = 0$, $\delta = \alpha_1 + 1$, $\gamma = \alpha_1 - \alpha_2$, and $b = C_2/C_1$ we get

$$U(t) = \mathbb{E}[Y(t)] = \frac{t^{\alpha_1}}{C_1} E_{\alpha_1-\alpha_2,\alpha_1+1}(-C_2 t^{\alpha_1-\alpha_2}/C_1).$$

Then (2) implies that the time-changed process $Z(t) = X(Y(t))$ has mean

$$\mathbb{E}[Z(t)] = \frac{t^{\alpha_1}\mathbb{E}[X(1)]}{C_1} E_{\alpha_1-\alpha_2,\alpha_1+1}(-C_2 t^{\alpha_1-\alpha_2}/C_1).$$

When $\mathbb{E}[X(1)] = 0$, Remark 2.1 shows that the time-changed process has zero mean and variance

$$\mathrm{Var}[Z(t)] = \mathrm{Var}[Z(1)]U(t) = \frac{\mathrm{Var}[X(1)]}{C_1} t^{\alpha_1} E_{\alpha_1-\alpha_2,\alpha_1+1}(-C_2 t^{\alpha_1-\alpha_2}/C_1).$$

In the case when the outer process is Brownian motion, this expression for the mean square displacement of the time-changed process was obtained in [12] using a different method. Veillette and Taqqu [45] derive the asymptotic behavior of the variance using a Tauberian theorem for the Laplace transform. We use the properties of two-parameter Mittag-Leffler function to obtain more a precise asymptotic expansion of the variance. From [38, Theorem 1.4] for real $z < 0$, $0 < \alpha < 2$ and any positive integer p

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-1-p}) \quad \text{as } z \to \infty.$$

With $p = 1$ we obtain

$$\mathrm{Var}[Z(t)] = \mathrm{Var}[Z(1)] \frac{t^{\alpha_2}}{C_2 \Gamma(\alpha_2 + 1)} + O(t^{2\alpha_2-\alpha_1}) \quad \text{as } t \to \infty.$$

When $t \to 0$, equation (17) yields

$$\mathrm{Var}[Z(t)] = \mathrm{Var}[Z(1)] \frac{t^{\alpha_1}}{C_1 \Gamma(\alpha_1 + 1)} + O(t^{2\alpha_1 - \alpha_2}) \quad \text{as } t \to 0.$$

When $\mathbb{E}[X(1)] = 0$, the behavior of the covariance function for a fixed $s > 0$ and $t \to \infty$ is obtained from Remark 2.1. We have

$$\mathrm{corr}[Z(t), Z(s)] \sim C_3(s, \alpha_1, \alpha_2) t^{-\alpha_2/2},$$

where

$$C_3(s, \alpha_1, \alpha_2) = \left(\frac{C_2 \Gamma(\alpha_2 + 1) s^{\alpha_1} E_{\alpha_1 - \alpha_2, \alpha_1 + 1}(-C_2 s^{\alpha_1 - \alpha_2}/C_1)}{C_1} \right)^{1/2}.$$

When t is fixed and $s \to 0$

$$\mathrm{corr}[Z(t), Z(s)] \sim C_4(t, \alpha_1, \alpha_2) s^{\alpha_1/2},$$

where

$$C_4(t, \alpha_1, \alpha_2) = \left(\Gamma(\alpha_1 + 1) t^{\alpha_1} E_{\alpha_1 - \alpha_2, \alpha_1 + 1}(-C_2 t^{\alpha_1 - \alpha_2}/C_1) \right)^{-1/2}.$$

For the case when $\mathbb{E}[X(1)] \neq 0$, we can also explicitly compute the variance of the time-changed process using equation (3). From [46, Theorem 3.1], the Laplace transform of $\mathbb{E}[Y^2(t)] := U(t; 2)$ equals

$$\tilde{U}(s; 2) = \frac{2}{s \phi^2(s)} = \frac{2 s^{-1-2\alpha_2}}{C_1^2 (s^{\alpha_1 - \alpha_2} + C_2/C_1)^2}.$$

Invert the Laplace transform using (16) with $k = 1$, $\gamma = \alpha_1 - \alpha_2$, $\delta = \alpha_1 + \alpha_2 + 1$, and $b = C_2/C_1$ to get

$$U(t; 2) = \frac{2}{C_1^2} t^{2\alpha_1} E^{(1)}_{\alpha_1 - \alpha_2, \alpha_1 + \alpha_2 + 1} \left(-C_2 t^{\alpha_1 - \alpha_2}/C_1 \right).$$

Therefore the variance of the inner process is

$$\mathrm{Var}[Y(t)] = \frac{t^{2\alpha_1}}{C_1^2} \Big[2 E^{(1)}_{\alpha_1 - \alpha_2, \alpha_1 + \alpha_2 + 1} \left(-C_2 t^{\alpha_1 - \alpha_2}/C_1 \right) - E_{\alpha_1 - \alpha_2, \alpha_1 + 1} \left(-C_2 t^{\alpha_1 - \alpha_2}/C_1 \right)^2 \Big],$$

and the variance of the time-changed process $Z(t) = X(Y(t))$ is

$$\mathrm{Var}[Z(t)] = \mathrm{Var}[X(1)] \frac{t^{\alpha_1}}{C_1} E_{\alpha_1 - \alpha_2, \alpha_1 + 1}(-C_2 t^{\alpha_1 - \alpha_2}/C_1) + \mathbb{E}[X(1)]^2 \frac{t^{2\alpha_1}}{C_1^2} \big[2 E^{(1)}_{\alpha_1 - \alpha_2, \alpha_1 + \alpha_2 + 1} \left(-C_2 t^{\alpha_1 - \alpha_2}/C_1 \right) - E_{\alpha_1 - \alpha_2, \alpha_1 + 1} \left(-C_2 t^{\alpha_1 - \alpha_2}/C_1 \right)^2 \big].$$

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This extends the results of Chechkin et al. [12] for the mean square displacement to the case when the outer process has a non-zero mean.

When the mixture is uniform, i.e., $q(w) = 1$ on $[0,1]$, an explicit expression for the renewal function was given by Veillette and Taqqu [45]:

$$U(t) = \gamma_e + \log(t) + \int_t^\infty e^{t-z} z^{-1} dz,$$

where $\gamma_e \doteq 0.57721$ is the Euler constant. Since the integral term remains bounded for t large, the mean of the time-changed process $Z(t) = X(Y(t))$ grows very slowly, like $\log(t)$, as $t \to \infty$. When the mean of the outer process is zero, the variance of the time-changed process also grows like $\log(t)$, and for fixed s and $t \to \infty$ the correlation function decays as $1/\sqrt{\log(t)}$:

$$\mathrm{corr}[Z(s), Z(t)] \sim \sqrt{U(s)/\log(t)}.$$

The next example of an inverse stable mixture models ultraslow diffusion [31]. Take any $\alpha > 0$ and let

$$p(\beta) = C\frac{\beta^{\alpha-1}}{\Gamma(1-\beta)} I(0 < \beta < 1),$$

where

$$C^{-1} = \int_0^1 \frac{\beta^{\alpha-1}}{\Gamma(1-\beta)} d\beta < \infty$$

since $\Gamma(1-\beta) \to 1$ as $\beta \to 0$. In this case the subordinator L has the Lévy measure

$$\nu(u,\infty) = \int_0^\infty u^{-\beta} p(\beta)\, d\beta,$$

and its density is

$$l_q(u) = \int_0^\infty \beta u^{-\beta-1} p(\beta) d\beta = \int_0^1 \frac{\beta u^{-\beta-1}}{\Gamma(1-\beta)} C\beta^{\alpha-1} d\beta,$$

so that in $q(\beta) = C\beta^{\alpha-1}$ in (15).

The subordinator L can be obtained as a limit of a triangular array constructed as follows. Take $\{B_i\}$ iid with pdf $p(\beta)$. Define a triangular array by taking J_i^c iid for each $c > 0$ such that $\mathbb{P}[J_i^c > t \mid B_i = \beta] = c^{-1}t^{-\beta}$ for $t \geq 1$. Note that if we take J_i iid such that $\mathbb{P}[J_i > t \mid B_i = \beta] = t^{-\beta}$ for $t \geq 1$ then we can let $J_i^c = c^{-1/\beta} J_i$ when $B_i = \beta$. Then, conditional on $B_i = \beta$, J_i has a distribution in the domain of attraction of a β-stable subordinator. Then Theorem 3.4 in [31] implies that

$$J_1^c + \cdots + J_{[ct]}^c \Rightarrow L(t) \quad \text{as } c \to \infty,$$

where $\mathbb{E}[e^{-sL(t)}] = e^{-t\phi(s)}$ for all $t \geq 0$, and

$$\begin{aligned}\phi(s) &= \int_0^\infty \left(1 - e^{-su}\right) \nu(du) \\ &= \int_0^1 \int_0^\infty \left(1 - e^{-su}\right) \beta u^{-\beta-1} du \, p(\beta) \, d\beta \\ &= \int_0^1 \Gamma(1-\beta) s^\beta p(\beta) \, d\beta \\ &= C \int_0^1 s^\beta \beta^{\alpha-1} \, d\beta \\ &= C \int_0^1 e^{-\beta \log(1/s)} \beta^{\alpha-1} \, d\beta,\end{aligned}$$

where we have used a formula from [35, p. 114] to arrive at the third line. Let $z = \log(1/s)$ and make a substitution $y = \beta z$ to see that

$$\int_0^1 e^{-z\beta} \beta^{\alpha-1} \, d\beta = \int_0^z e^{-y} (y/z)^{\alpha-1} z^{-1} \, dy = z^{-\alpha} \Gamma(\alpha, z)$$

in terms of the incomplete gamma function $\Gamma(\alpha, z) = \int_0^z e^{-y} y^{\alpha-1} \, dy$. Then we have

$$\phi(s) = C(\log(1/s))^{-\alpha} \Gamma(\alpha, \log(1/s)),$$

and the Laplace transform of the renewal function $U(t) = \mathbb{E}[Y(t)]$ is

$$\tilde{U}(s) = s^{-1} \frac{(\log(1/s))^\alpha}{C\Gamma(\alpha, \log(1/s))}.$$

Hence we have $\tilde{U}(s) = s^{-1} L(1/s)$ where

$$L(x) = \frac{(\log x)^\alpha}{C\Gamma(\alpha, \log x)} \sim \frac{(\log x)^\alpha}{C\Gamma(\alpha)} \quad \text{as } x \to \infty$$

is slowly varying as $x \to \infty$. Now we apply the Karamata Tauberian theorem (e.g., see [13, Theorem 4, p. 446]) with $\rho = 1$ to conclude that

$$U(t) \sim \frac{(\log t)^\alpha}{C\Gamma(\alpha)} \quad \text{as } t \to \infty.$$

As in the previous example of a uniform mixture (the special case $\alpha = 1$), the mean of the time-changed process $Z(t) = X(Y(t))$ grows very slowly, like $(\log(t))^\alpha$, as $t \to \infty$. When the mean of the outer process is zero, Remark 2.1 shows that the variance of the time-changed process also grows like $(\log(t))^\alpha$. For a fixed $s > 0$ the correlation function decays slowly:

$$\mathrm{corr}[Z(s), Z(t)] \sim \sqrt{C\Gamma(\alpha) U(s)} (\log(t))^{-\alpha/2}$$

as $t \to \infty$.

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Example 3.5 (Inverse Poisson subordinator). Consider

$$L(t) = \mu t + N(t), \quad t \geq 0, \mu \geq 0,$$

where $N(t) = \max\{n \geq 1 : T_n \leq t\}$, and $T_n = E_1 + \cdots + E_n$ and E_1, E_2, $E_3, \ldots$ are iid exponential random variables with mean $1/\lambda$, so that $N(t)$ is a homogeneous Poisson process. The Laplace exponent of $L(t)$ is given by

$$\phi(s) = \mu s + \lambda(1 - e^{-s}).$$

For $\mu = 0$, using the definition (5) together with the fact that $\{T_n \leq t\} = \{N_t \geq n\}$, it is not hard to check that the inverse subordinator $Y(t) = T_{[t+1]}$, and therefore $Y(t)$ is distributed as $\Gamma([t+1], 1/\lambda)$. Using the standard formulae for the gamma distribution, we then have

$$U(t) = \mathbb{E}[Y(t)] = \frac{[t+1]}{\lambda}, \quad \operatorname{Var} Y(t) = \frac{[t+1]}{\lambda^2}.$$

For $0 \leq s < t$ the covariance function of the inner process is

$$\operatorname{Cov}[Y(t), Y(s)] = \operatorname{Cov}[(E_1 + \cdots + E_{[t+1]})[(E_1 + \cdots + E_{[s+1]})] = \frac{[s+1]}{\lambda^2}.$$

Therefore for the time-changed process $Z(t) = X(Y(t))$ we have

$$\mathbb{E}[Z(t)] = \frac{\mathbb{E}[X(1)][t+1]}{\lambda}, \quad \operatorname{Var}[Z(t)] = \frac{[t+1]}{\lambda^2}\left([\mathbb{E}X(1)]^2 + \lambda \operatorname{Var}[X(1)]\right),$$

and for $0 \leq s < t$

$$\operatorname{Cov}[Z(t), Z(s)] = \operatorname{Var}[X(1)]\frac{[s+1]}{\lambda} + \mathbb{E}[X(1)]^2\frac{[s+1]}{\lambda^2}.$$

When $\mu \neq 0$, $\tilde{U}(s) \sim s^{-2}/(\lambda + \mu)$ as $s \to 0$, and then the Karamata Tauberian theorem implies that $U(t) \sim t/(\mu + \lambda)$ as $t \to \infty$. When the outer process has zero mean, Remark 2.1 implies that the time-changed process has zero mean, a variance

$$\operatorname{Var}[Z(t)] \sim \operatorname{Var}[X(1)]\frac{t}{\mu + \lambda}$$

that grows linearly, and a covariance function

$$\operatorname{Cov}[Z(t), Z(s)] \sim \operatorname{Var}[X(1)]\frac{\min(t, s)}{\mu + \lambda}$$

that grows linearly as the smaller of s and t. For fixed s and $t \to \infty$, the correlation function of the time-changed process decays like $1/\sqrt{t}$.

Example 3.6 (Fractional Poisson subordinator). If $Y(t)$ is a fractional Poisson process introduced in Example 3.2, then its renewal function is given by (12) with $\mathbb{E}[X(1)] = \lambda$, its variance is given by (13) with $\mathrm{Var}[X(1)] = \lambda$, and its covariance is given by (14). Then the correlation structure of the time-changed $Z(t) = X(Y(t))$ can be obtained from Theorem 2.1 and Remark 2.1. Hence the time-changed process $Z(t) = X(Y(t))$ has mean

$$\mathbb{E}[Z(t)] = \mathbb{E}[X(1)]\frac{t^\alpha \lambda}{\Gamma(1+\alpha)},$$

variance

$$\mathrm{Var}[Z(t)] = \mathbb{E}[X(1)]^2 \left\{ \frac{t^\alpha \lambda}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}\lambda^2}{\alpha}\left(\frac{1}{\Gamma(2\alpha)} - \frac{1}{\alpha\Gamma(\alpha)^2}\right)\right\} + \mathrm{Var}[X(1)]\frac{t^\alpha \lambda}{\Gamma(1+\alpha)},$$

and for $0 \le s < t$ its covariance

$$\mathrm{Cov}[Z(t)] = \mathrm{Var}[X(1)]\frac{s^\alpha \lambda}{\Gamma(1+\alpha)} + \mathbb{E}[X(1)]^2 \left\{ \frac{s^\alpha \lambda}{\Gamma(1+\alpha)} + \frac{\lambda^2}{\Gamma(1+\alpha)^2}\left[\alpha s^{2\alpha} B(\alpha, \alpha+1) + F(\alpha; s, t)\right]\right\}.$$

If the outer process has zero mean, then Remark 2.1 shows that the time-changed process has zero mean, and its variance

$$\mathrm{Var}[Z(t)] = \mathrm{Var}[X(1)]\frac{t^\alpha \lambda}{\Gamma(1+\alpha)}$$

grows like t^α as $t \to \infty$. For $0 \le s < t$ its covariance function is

$$\mathrm{Cov}[Z(t), Z(s)] = \mathrm{Var}[X(1)]\frac{s^\alpha \lambda}{\Gamma(1+\alpha)}$$

and hence its correlation function

$$\mathrm{corr}[Z(t), Z(s)] = \left(\frac{s}{t}\right)^{\alpha/2}$$

decays like $t^{-\alpha/2}$ as $t \to \infty$.

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