Tempered Fractional Calculus

2014 International Conference on Fractional Differentiation and Its Applications

University of Catania, Italy
23–25 June 2014

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Abstract

Fractional derivatives and integrals are convolutions with a power law. Including an exponential term leads to tempered fractional derivatives and integrals. Tempered fractional Brownian motion, the tempered fractional integral or derivative of a Brownian motion, is a new stochastic process whose increments can exhibit semi-long range dependence. A tempered Grunwald-Letnikov formula provides the basis for finite difference methods to solve tempered fractional diffusion equations. The tempered finite difference operator is also useful in time series analysis, where it provides a useful new stochastic model for turbulent velocity data. Tempered stable processes are the limits of random walk models, where the power law probability of long jumps is tempered by an exponential factor. These random walks converge to tempered stable stochastic process limits, whose probability densities solve tempered fractional diffusion equations. Tempered power law waiting times lead to tempered fractional time derivatives. Applications include geophysics and finance.
Acknowledgments

Farzad Sabzikar, Statistics and Probability, Michigan State

Inmaculada Aban, Biostatistics, Univ Alabama Birmingham
Boris Baeumer, Maths & Stats, University of Otago, New Zealand
Jinghua Chen, School of Sciences, Jimei University, China
Peter Franz Kern, Math, Universität Düsseldorf, Germany
Anna K. Panorska, Mathematics and Statistics, U of Nevada
M.S. Phanikumar, Civil & Environ. Eng., Michigan State
Parthanil Roy, Indian Statistical Institute, Kolkata
Hans-Peter Scheffler, Math, Universität Siegen, Germany
Qin Shao, Mathematics, University of Toledo, Ohio
Alla Sikorskii, Statistics and Probability, Michigan State
Aklilu Zeleke, Lyman Briggs School, Michigan State
Yong Zhang, Department of Geological Sciences, U Alabama
Recent Books

*Stochastic Models for Fractional Calculus*
Mark M. Meerschaert and Alla Sikorskii
De Gruyter Studies in Mathematics *43*, 2012
Advanced graduate textbook
ISBN 978-3-11-025869-1

*Mathematical Modeling, 4th Edition*
Mark M. Meerschaert
Advanced undergraduate / beginning graduate textbook
(new sections on particle tracking and anomalous diffusion)
ISBN 978-0-12-386912-8
Fourier transform definition

Define the Fourier transform \( \hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx \).

Positive fractional derivatives and integrals of order \( \alpha > 0 \) satisfy

\[
\mathbb{D}_x^{\alpha} f(x) \iff (ik)^\alpha \hat{f}(k) \quad \text{and} \quad \mathbb{I}_x^{\alpha} f(x) \iff (ik)^{-\alpha} \hat{f}(k)
\]

Negative fractional derivatives and integrals of order \( \alpha > 0 \) satisfy

\[
\mathbb{D}_{-x}^{\alpha} f(x) \iff (-ik)^\alpha \hat{f}(k) \quad \text{and} \quad \mathbb{I}_{-x}^{\alpha} f(x) \iff (-ik)^{-\alpha} \hat{f}(k)
\]

Tempered fractional derivatives with \( \alpha, \lambda > 0 \) satisfy

\[
\mathbb{D}_{x,\lambda}^{\alpha} f(x) \iff (\lambda + ik)^\alpha \hat{f}(k) \quad \text{and} \quad \mathbb{I}_{x,\lambda}^{\alpha} f(x) \iff (\lambda + ik)^{-\alpha} \hat{f}(k)
\]

\[
\mathbb{D}_{-x,\lambda}^{\alpha} f(x) \iff (\lambda - ik)^\alpha \hat{f}(k) \quad \text{and} \quad \mathbb{I}_{-x,\lambda}^{\alpha} f(x) \iff (\lambda - ik)^{-\alpha} \hat{f}(k)
\]

Note the Fourier symbol \((\lambda \pm ik)^{-\alpha}\) no longer blows up at \(k = 0\).
Riemann-Liouville definition

Tempered fractional integrals can also be defined by

\[
\mathbb{I}^\alpha,\lambda_x f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(u)(x-u)^{\alpha-1} e^{-\lambda(x-u)} du
\]

\[
\mathbb{I}^\alpha,\lambda_{-x} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} f(u)(u-x)^{\alpha-1} e^{-\lambda(u-x)} du
\]

for any \(\alpha, \lambda > 0\). For \(\lambda = 0\), these are Riemann-Liouville fractional integrals.

For \(0 < \alpha < 1\) and \(\lambda > 0\) we also have

\[
\mathbb{D}^\alpha,\lambda_x f(x) = \lambda^\alpha f(x) + \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{f(x) - f(u)}{(x-u)^{\alpha+1}} e^{-\lambda(x-u)} du
\]

\[
\mathbb{D}^\alpha,\lambda_{-x} f(x) = \lambda^\alpha f(x) + \frac{\alpha}{\Gamma(1-\alpha)} \int_{x}^{\infty} \frac{f(x) - f(u)}{(u-x)^{\alpha+1}} e^{-\lambda(u-x)} du
\]

For \(\lambda = 0\), these are Marchaud fractional derivatives.
Grünwald-Letnikov definition

For $\lambda > 0$ and $0 < \alpha < 1$ we have

$$\lim_{h \to 0} h^{-\alpha} \Delta_h^{\alpha,\lambda} f(x) = D_x^{\alpha,\lambda} f(x)$$

where the tempered fractional difference

$$\Delta_h^{\alpha,\lambda} f(x) = \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j e^{-\lambda j h} f(x - jh) - (1 - e^{-\lambda h})^\alpha f(x).$$

using the fractional binomial coefficients

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha + 1)}{j!\Gamma(\alpha - j + 1)}$$

Can use to construct finite difference codes [special session].

These codes are mass-preserving since (by the Binomial formula)

$$\sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j e^{-\lambda j h} = (1 - e^{-\lambda h})^\alpha$$
Tempered fractional Brownian motion

Yaglom noise is defined by

\[ Y_{\alpha,\lambda}(t) = \int_{-\infty}^{t} (t - u)^{-\alpha} e^{-\lambda(t-u)} \, W(u) \, du = \mathbb{I}^{1-\alpha,\lambda}_{t} W(t) \]

where \( W(u) \) is a white noise.

Tempered fractional Brownian motion (TFBM) is defined by

\[ B_{\alpha,\lambda}(t) = Y_{\alpha,\lambda}(t) - Y_{\alpha,\lambda}(0) = \int_{-\infty}^{\infty} \left[ e^{-\lambda(t-u)} + (t - u)^{+\alpha} - e^{-\lambda(0-u)} + (0 - u)^{+\alpha} \right] \, W(u) \, du \]

where \( (x)^{+} = x \) for \( x > 0 \) and \( (x)^{+} = 0 \) for \( x \leq 0 \).

When \( \lambda = 0 \), Yaglom noise is undefined, and TFBM = FBM.
FBM and tempered FBM

FBM (thin line) and TFBM (thick line) with same $W(u)$ for $H = 0.3$, $\lambda = 0.03$ (left) and $H = 0.7$, $\lambda = 0.01$ (right).

Sample paths are Hölder continuous of order $H = 1/2 - \alpha$.

Scaling: $B_{\alpha,\lambda}(ct) \overset{d}{=} c^H B_{\alpha,c\lambda}(t)$. 
Tempered fractional time series

The ARMA\((p, q)\) time series model is

\[
Y_t - \sum_{j=1}^{p} \phi_j Y_{t-j} = Z_t + \sum_{i=1}^{q} \theta_i Z_{t-i}
\]

where \(Z_t \sim \text{IID}(0, \sigma^2)\).

We say that \(X_t\) follows an ARTFIMA\((p, \alpha, \lambda, q)\) model if

\[
Y_t = \Delta_{1, \lambda} \alpha X_t = \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j e^{-\lambda j} X_{t-j}
\]

follows an ARMA\((p, q)\) model.

Then \(X_t = \Delta_{1, \lambda}^{-\alpha} Y_t\), a kind of tempered fractional integration.
Spectral approach

Kolmogorov argued that the spectral density of turbulence data follows $h(k) \approx k^{-5/3}$ in the inertial range of frequency $k$.

Tempered fractional Gaussian noise (TFGN) is defined by

$$X_j = B_{\alpha, \lambda}(j) - B_{\alpha, \lambda}(j - 1) = Y_{\alpha, \lambda}(j) - Y_{\alpha, \lambda}(j - 1).$$

Its spectral density (FT of the autocovariance function) is

$$h(k) = C \frac{|e^{-ik} - 1|^2}{(\lambda^2 + k^2)^{H+1/2}} \approx C k^{1-2H}$$

for $k$ small and $\lambda \approx 0$. Set $H = 4/3$ for Kolmogorov scaling.

The ARTFIMA(0, $\alpha$, $\lambda$, 0) model has spectral density

$$h(k) = C |1 - e^{-(\lambda + ik)}|^{-2\alpha} \approx C k^{-2\alpha}$$

for $k$ small and $\lambda \approx 0$. Set $\alpha = 5/6$ for Kolmogorov scaling.
Spectral density of TFGN and ARTFIMA

Log-log plot of spectral density for TFGN (solid), ARTFIMA (dotted), and Kolmogorov scaling (dashed). Here $\lambda = 0.03$. 
Lake Huron water velocity spectrum

Spectrum of Saginaw Bay velocity data (symbols), fitted ARTFIMA model (thick line) with $\alpha = 5/6$ and $\lambda = 0.006$, and Kolmogorov model (thin line).
Davenport spectrum for wind gusts

Spectral density of wind gustiness from Davenport (1961). The Davenport spectrum is the same as TFGN with $H = 5/6$. 
Fractional diffusion and probability

The space-time fractional diffusion equation

\[ \mathbb{D}_t^\beta p(x, t) = a\mathbb{D}_x^\alpha p(x, t) + b\mathbb{D}_{-x}^\alpha p(x, t) + \delta(x) \frac{t^{-\beta}}{\Gamma(1 - \beta)} \]

governs a stochastic process \( X_t \) with power law jumps in space

\[ \mathbb{P}(|\Delta Y_u| > x) \approx Ax^{-\alpha} \]

and power law waiting times

\[ \mathbb{P}(\Delta D_u > t) \approx Bt^{-\beta} \]

for \( 0 < \alpha < 2 \) and \( 0 < \beta < 1 \).

Graph of sample paths (particle traces): \( (t, X_t) = (D_u, Y_u) \).

Hence \( \mathbb{D}_t^\beta \) codes power law waiting times, \( \mathbb{D}_\pm^\alpha \) power law jumps.
Continuous time random walks

For particle jumps \( \mathbb{P}(J > x) \approx Ax^{-\alpha} \), \( J_1 + \cdots + J_{[nu]} \approx n^{1/\alpha}Y_u \).

For waiting times \( \mathbb{P}(W > t) \approx Bx^{-\beta} \), \( W_1 + \cdots + W_{[nu]} \approx n^{1/\beta}D_u \).
Fractional diffusion particle tracking

Here $\alpha = 2.0, 1.7$ (left/right) and $\beta = 1.0, 0.9$ (top/bottom).
Tempered fractional diffusion

The tempered space-fractional diffusion equation

\[ \mathbb{D}_t p(x, t) = a\partial_{x,\lambda}^\alpha p(x, t) + b\partial_{-x}^{\alpha,\lambda} p(x, t) \]

governs a stochastic process \( X_t \) with tempered power law jumps

\[ \mathbb{P}(|\Delta X_u| > x) \approx Ax^{-\alpha}e^{-\lambda x} \]

for \( 0 < \alpha < 2 \) and \( \lambda > 0 \).

This *normalized* tempered fractional derivative is defined by

\[
\partial_{\pm x}^{\alpha,\lambda} f(x) \iff [(\lambda \pm ik)^\alpha - \lambda^\alpha] \hat{f}(k) \quad \text{for} \quad 0 < \alpha < 1, \text{ and}
\]

\[
\partial_{\pm x}^{\alpha,\lambda} f(x) \iff [(\lambda \pm ik)^\alpha - \lambda^\alpha \mp ik\alpha\lambda^{\alpha-1}] \hat{f}(k) \quad \text{for} \quad 1 < \alpha < 2.
\]

The \( \lambda^\alpha \) term makes \( p(x, t) \) a PDF.

The \( ik\alpha\lambda^{\alpha-1} \) term makes \( p(x, t) \) have zero mean.
Tempered fractional diffusion particle tracking

Tempered stable process with $\alpha = 1.2$ transitions from Brownian motion to stable Lévy motion as $\lambda$ decreases.
Tempered power laws in finance

AMZN stock price changes fit a tempered power law model

\[ P(X > x) \approx x^{-0.6} e^{-0.3x} \quad \text{for } x \text{ large} \]
Tempered power laws in hydrology

Tempered power law model \( P(X > x) \approx x^{-0.6}e^{-5.2x} \) for increments in hydraulic conductivity at the MADE site.
Tempered power laws in atmospheric science

Tempered power law model $P(X > x) \approx x^{-0.2}e^{-0.01x}$ for daily precipitation data at Tombstone AZ.
Tempered stable pdf in macroeconomics

One-step BARMA forecast errors for annual inflation rates fit a symmetric tempered stable with $\alpha = 1.1$ and $\lambda = 12$. 
Tempered time-fractional diffusion model

Fitted concentration data from a 3-D supercomputer simulation. ADE fit uses $\alpha = 2, \beta = 1$. Without cutoff uses $\lambda = 0$. 
Summary

- Tempered fractional derivatives

- Tempered fractional Brownian motion

- Applications to turbulence

- Tempered fractional diffusion

- Numerical methods [special session]

- Applications to finance and geophysics
References


Simulating tempered stable laws (JCAP 2010)

Simulation codes for stable random variates are widely available.

If $X > 0$ has stable density density $f(x)$, TS density is

$$f_\lambda(x) = \frac{e^{-\lambda x} f(x)}{\int_0^\infty e^{-\lambda u} f(u) \, du}$$

Take $Y \sim \exp(\lambda)$ independent of $X$, $(X_i, Y_i)$ IID with $(X, Y)$.

Let $N = \min\{n : X_n \leq Y_n\}$. Then $X_N \sim f_\lambda(x)$.

Proof: Compute $P(X_N \leq x) = P(X \leq x | X \leq Y)$ by conditioning, then take $d/dx$ to verify.
Triangular array scheme (SPL 2011)

Take $P(J > x) \approx Cx^{-\alpha}$ with $1 < \alpha < 2$. Triangular array limit

$$\sum_{k=1}^{[nt]} n^{-1/\alpha} J_k - b_t^{(n)} \Rightarrow Y_t$$

is stable. Define tempering variables:

$$P(Z > u) = u^\alpha \int_u^\infty r^{-\alpha-1} e^{-\lambda r} dr$$

Replace $n^{-1/\alpha} X_k$ by $Z_k$ if $n^{-1/\alpha} X_k > Z_k$.

Triangular array limit is tempered stable.

Exponential tempering: sum of $\alpha$ and $\alpha - 1$ tempered stables.
Tail estimation (CIS 2012)

Hill-type estimator: Assume $P(X > x) \approx Cx^{-\alpha}e^{-\lambda x}$ for $x$ large, use order statistics $X(1) \leq X(2) \leq \cdots \leq X(n)$.

Conditional MLE given $X(n-k+1) > L \geq X(n-k)$:

$$T_1 := \sum_{i=1}^{k} (\log X(n-i+1) - \log L)$$

$$T_2 := \sum_{i=1}^{k} (X(n-i+1) - L)$$

$$1 = \sum_{i=1}^{k} \frac{x(n-i+1)}{kx(n-i+1) + \hat{\alpha}(T_2 - T_1x(n-i+1))}$$

$$\hat{\lambda} = (k - \hat{\alpha}T_1)/T_2$$

$$\hat{C} = \frac{k}{n} L \hat{\alpha} e^{\hat{\lambda}L}$$

R code available at [www.stt.msu.edu/users/mcubed/TempParetoR.zip](http://www.stt.msu.edu/users/mcubed/TempParetoR.zip)
Testing for pure power law tail (JASA 06)

Null hypothesis $H_0 : P(X > x) = Cx^{-\alpha}$ Pareto for $x > L$.

Test based on extreme value theory rejects $H_0$ if

$$X_{(1)} < \left( \frac{nC}{-\ln q} \right)^{1/\alpha}$$

where $\alpha, C$ can be estimated using Hill’s estimator

$$\hat{\alpha}_H = \left[ k^{-1} \sum_{i=1}^{k} \{\ln X_{(n-i+1)} - \ln X_{(n-k)}\} \right]^{-1}$$

$$\hat{C}_H = \left( \frac{k}{n} \right) \left( X_{(n-k)} \right)^{\hat{\alpha}_H}$$

Simple $p$-value formula $p = \exp\{ -n C X_{(n)}^{-\alpha} \}$. 
Tempered fractional calculus operators (SPA 2014)

\[ I_{\pm x}^\alpha,\lambda f(x) \iff (\lambda \pm ik)^{-\alpha} \hat{f}(k) \quad \text{for} \quad f \in L^2(\mathbb{R}) = \{ f : \int |f(x)|^2 \, dx < \infty \}. \]

Define the fractional Sobolev space

\[ W^{\alpha,2}(\mathbb{R}) := \{ f \in L^2(\mathbb{R}) : \int (\lambda^2 + k^2)^{\alpha/2} |\hat{f}(k)|^2 \, dk < \infty \}. \]

Then \( D_{\pm x}^\alpha,\lambda f(x) \iff (\lambda \pm ik)^\alpha \hat{f}(k) \quad \text{for} \quad f \in W^{\alpha,2}(\mathbb{R}). \)

Semigroup property: \( I_{\pm}^\alpha,I_{\pm}^\beta f = I_{\pm}^{\alpha+\beta} f \quad \text{for} \quad f \in L^2(\mathbb{R}), \) and \( D_{\pm}^\alpha,D_{\pm}^\beta f = D_{\pm}^{\alpha+\beta} f \quad \text{for} \quad f \in W^{\alpha+\beta,2}(\mathbb{R}). \)

Adjoint property: \( \langle f, I_{+}^\alpha g \rangle_2 = \langle I_{-}^\alpha f, g \rangle_2 \quad \text{for} \quad f, g \in L^2(\mathbb{R}), \) and \( \langle f, D_{+}^\alpha g \rangle_2 = \langle D_{-}^\alpha f, g \rangle_2 \quad \text{for} \quad f, g \in W^{\alpha,2}(\mathbb{R}), \) using the \( L^2 \) inner product \( \langle f, g \rangle_2 = \int f(x)g(x) \, dx \)