JOURNAL OF MULTIVARIATE ANALYSIS 19, 342–347 (1986)

Domains of Attraction of Nonnormal Operator-Stable Laws

MARK M. MEERSCHAERT

Vector Research, Inc.

Communicated by P. Révész

A sequence of independent, identically distributed random vectors $X_1, X_2, X_3,...$ is said to belong to the domain of attraction of a random vector Y is there exist linear operators A_n and constant vectors b_n such that $A_n(X_1,..., X_n)+b_n$ converges in distribution to Y. We present a simple, necessary, and sufficient condition for the existence of such A_n, B_n in the case where Y has no normal component. © 1986 Academic Press, Inc.

1. INTRODUCTION

Suppose that $X, X_1, X_2,...$ are independent random vectors on $\mathbb{R}^k$ with common distribution μ. Under suitable conditions on μ we can find linear operators A_n and constants b_n such that $A_n(X_1 + \cdots + X_n) + b_n$ converges in distribution to a nontrivial limit. For example if $E\|X\|^2 < \infty$ we can take $A_n = n^{-1/2}I$ and $b_n = -nEX$ and the limiting distribution is normal with mean zero. The class of all nontrivial limit distributions obtained in this way is called the operator-stable distributions. We say that X is in the domain of attraction of Y operator-stable if $A_n(X_1 + \cdots + X_n) + b_n$ converges in distribution to Y for some A_n, b_n. The limit law Y is said to be full, or nondegenerate, if it is not almost surely contained in some $(k-1)$ dimensional hyperplane. In this case A_n must be invertible for all large n, and the distribution of $A_n^{-1}(Y-b_n)$ approximates that of $(X_1 + \cdots + X_n)$. We are interested therefore in obtaining necessary and sufficient conditions for X to belong to the domain of attraction of a full operator-stable law.

Operator-stable laws have been investigated by Sharpe [9], Kucharczak [5], and several others. Since an operator-stable law is the weak limit of the triangular array $A_nX_1 + \cdots + A_nX_n + b_n$ it is infinitely divisible.

Received April 24, 1984; revised January 7, 1985.

AMS 1980 subject classification: 60F05.

Key words and phrases: domains of attraction, regular variation, stable laws, infinitely divisible laws.

0047-259X/86 $3.00

Copyright © 1986 by Academic Press, Inc.
All rights of reproduction in any form reserved.

Infinitely divisible laws on $\mathbb{R}^k$ were characterized by P. Lévy [6] who gave the following result. Y is infinitely divisible if and only if there exists a triple (a, Q, ϕ), where $a \in \mathbb{R}^k$, Q is a nonnegative quadratic form on $\mathbb{R}^k$, and ϕ is a Borel measure on $\mathbb{R}^k - \{0\}$ which is finite on sets bounded away from the origin and which satisfies

$$\int_{0<\|x\|<1} \|x\|^2 \phi\{dx\} < \infty, \tag{1.1}$$

such that the characteristic function of Y can be written in the form e^{ψ}, where

$$\psi(t) = i(a, t) - \frac{1}{2} Q(t) + \int_{x \neq 0} \left[e^{i(t,x)} - 1 - \frac{i(t, x)}{1 + (x, x)} \right] \phi\{dx\}. \tag{1.2}$$

Necessary and sufficient conditions for the convergence of a triangular array of random vectors to a weak limit were given by Rvačeva [8]. An application of her Theorem 2.3 yields immediately that

$$A_n(X_1 + \cdots + X_n) + b_n \Rightarrow Y \tag{1.3}$$

holds for some full Y with Lévy representation $(a, 0, \phi)$ if and only if

$$n\mu\{A_n^{-1}\, dx\} \to \phi\{dx\}; \tag{1.4a}$$

$$\lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}}\, n \left[\int_{0<\|x\|<\varepsilon} (x, t)^2 \mu\{A_n^{-1}\, dx\} - \left(\int_{0<\|x\|<\varepsilon} (x, t)\, \mu\{A_n^{-1}\, dx\} \right)^2 \right] = 0. \tag{1.4b}$$

In this case Y is operator-stable and its Lévy measure ϕ satisfies

$$\lambda\phi\{dx\} = \phi\{\lambda^{-E}\, dx\}, \qquad \forall \lambda > 0, \tag{1.5}$$

where λ^A denotes the operator $\exp[(\log \lambda)\, A]$ and E is a nonsingular linear operator on $\mathbb{R}^k$ whose eigenvalues all have real part greater than $\frac{1}{2}$ (cf. [9]).

2. Results

The main result of this paper extends a theorem of Feller [1] which states that a random variable with distribution function $F(x) = \mu(-\infty, x]$ is in the domain of attraction of a nonnormal stable law in $\mathbb{R}^1$ if and only if the tailsum $F(-x)+1-F(x)$ varies regularly at infinity with index $\rho\in(-2,0)$, and for some $0\leqslant c\leqslant 1$

$$\lim_{x\to\infty}\frac{F(-x)}{F(-x)+1-F(x)}=c.$$

An equivalent condition is that for some $a_n\to\infty$, $nF(-a_nx)\to C_1x$ and $n[1-F(a_nx)]\to C_2x$. In this case (1.3) holds with $A_n=a_n^{-1}$, and the Lévy measure of Y is given by $\phi(-\infty,-x)=C_1x^{\rho}$ and $\phi(x,\infty)=C_2x^{\rho}$. That is, X is in the domain of attraction of Y if and only if there exists $a_n\to\infty$ such that $n\mu\{a_n\,dx\}\to\phi\{dx\}$.

THEOREM. *X is in the domain of attraction of a full nonnormal operator-stable law Y with Levy measure ϕ if and only if there exists a sequence of linear operators $\{A_n\}$ such that $\|A_n\|\to 0$ and $n\mu\{A_n^{-1}\,dx\}\to\phi\{dx\}$. In this case* (1.3) *holds for some sequence of constant vectors $\{b_n\}$.*

The proof of this theorem requires a few preliminary results.

LEMMA 1. *Suppose B is a linear operator on $\mathbb{R}^k$ and all eigenvalues of B have real part greater than some $\alpha>0$. For any $\varepsilon>0$ there exists $\lambda_0>0$ such that $\|\lambda^Bx\|>\lambda^{\alpha-\varepsilon}\|x\|$ for all $\lambda\geqslant\lambda_0$ and $x\neq 0$.*

Proof. Transformation groups of the form $\{e^{tB}: t\in\mathbb{R}\}$ have been extensively studied in the literature on linear differential equations on $\mathbb{R}^k$. The above result is an easy computation using, for example, Hirsch and Smale [3, Chap. 6]. ∎

Define a real-valued function f on $\mathbb{R}^k-\{0\}$ by setting $f(t)=\phi(B_t)$ where

$$B_t=\{x\in\mathbb{R}^k: |(x,t)|>1\}. \tag{2.1}$$

The measure ϕ can be represented as a mixture of Lévy measures which satisfy (1.5) and are concentrated on a single orbit of the transformation group $\{\lambda^E: \lambda>0\}$ (cf. [5]). Since ∂B_t is bounded away from the origin and $\|\lambda^E\|\to 0$ as $\lambda\to 0$, the set $\{\lambda>0: \lambda^Ex\in\partial B_t\}$ has Lebesgue measure zero for any $x\in\mathbb{R}^k$. Thus $\phi(\partial B_t)=0$ for all t, and it follows that f is continuous.

LEMMA 2. *Suppose K is a compact subset of $\mathbb{R}^k-\{0\}$. For all $\varepsilon>0$ sufficiently small there exists $\mu_0>1$ such that $f(\mu x)\leqslant\mu_0^{2-\varepsilon}f(x)$ whenever $1\leqslant\mu\leqslant\mu_0$ and $x\in K$.*

Proof. It suffices to prove the theorem in the case $K=\{x\in\mathbb{R}^k: a\leqslant\|x\|\leqslant b\}$, where $0<a<b$. By (1.5) and (2.1) we have $\lambda f(t)=f(\lambda^{E^*}t)$ for all $\lambda>0$ and $t\neq 0$. If $x\in K$ and $\mu>0$ there exists $x'\in K$ and $\lambda>0$ such that

$\mu x = \lambda^{E^*}x'$ and then $f(\mu x) = \lambda f(x')$. The desired result follows from Lemma 1 by a straightforward computation. ▮

Let μ_θ denote the distribution of the random variable $|(X, \theta)|$ and define for $r > 0$ and $\|\theta\| = 1$

$$U(r, \theta) = \int_0^r s^2 \mu_\theta\{ds\}$$
$$V(r, \theta) = \int_r^\infty \mu_\theta\{ds\} \tag{2.2}$$

(compare with Feller [1, p. 282]). The key to the proof of the above theorem is the following lemma, which states that $V(r, \theta)$ is $R-0$ varying (cf. Seneta [10]) as a function of r, uniformly in θ.

LEMMA 3. *Suppose* $\|A_n\| \to 0$ *and* $n\mu\{A_n^{-1}\,dx\} \to \phi\{dx\}$. *Then for all* $\delta > 0$ *sufficiently small there exist positive reals* r_0 *and* $\lambda_0 \geq 1$ *such that*

$$V(r\lambda, \theta)/V(r, \theta) \geq \lambda_0^{\delta-2} \tag{2.3}$$

whenever $1 \leq \lambda \leq \lambda_0$ *and* $r \geq r_0$.

Proof. Define $g(t) = \mu(B_t)$, where B_t is defined by (2.1). For all $t \in \mathbb{R}^k - \{0\}$ we have $ng(A_n^* t) = n\mu(A_n^{-1} B_t) \to \phi(B_t) = f(t)$, and furthermore this convergence is uniform on compact subset of $\mathbb{R}^k - \{0\}$. Let $n(r, \theta) = \max\{n\colon \|A_n^{*-1}(\theta/r)\| \leq 1\}$. From the fact that $\|A_n\| \to 0$ it follows that $n(r, \theta) \to \infty$ as $r \to \infty$ uniformly in θ. Writing n for $n(r, \theta)$ and $y_n = A_n^{*-1}(\theta/r)$ we have

$$\frac{V(r\lambda, \theta)}{V(r, \theta)} = \frac{g(\theta/r\lambda)}{g(\theta/r)} = \frac{ng(A_n^* y_n/\lambda)}{ng(A_n^* y_n)}.$$

From $n\mu\{A_{n+1}^{-1}\,dx\} \to \phi\{dx\}$ it follows that $\|A_{n+1} A_n^{-1}\|$ remains bounded away from zero and infinity as $n \to \infty$, and hence for some $r_0 > 0$ the set $\{y_n\colon r \geq r_0, \|\theta\| = 1\}$ is compactly contained in $\mathbb{R}^k - \{0\}$. Now the desired result follows from Lemma 2 and the fact that $ng(A_n^* t) \to f(t)$ uniformly on compacta. ▮

Proof of Theorem. The weak convergence of the left-hand side of (1.3) requires $\|A_n\| \to 0$. Suppose $\|A_n\| \to 0$ and (1.4a) holds. We will be done if we can show that (1.4b) follows. By the Schwartz inequality, it is enough to show that for all $\|t\| = 1$

$$\lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}}\, n \int_{0 < \|x\| < \varepsilon} (x, t)^2 \mu\{A_n^{-1}\,dx\} = 0. \tag{2.4}$$

The expression under the limit in (2.4) is nonnegative and bounded above by $n\rho_n^2 \, U(\varepsilon/\rho_n, \theta_n)$ where $\rho_n = \|A_n^* t\| \to 0$ as $n \to \infty$ and $\theta_n = A_n^* t/\rho_n$ is a unit vector. Integrating by parts in (2.2) we obtain

$$U(r, \theta) = -r^2 V(r, \theta) + 2 \int_0^r s V(s, \theta) \, ds. \tag{2.5}$$

By Lemma 3 and Seneta [10, Theorem A.2, part (b)] there exist positive reals c, r_0 such that for all $\|\theta\| = 1$ and $r \geqslant r_0$

$$\int_0^r s V(s, \theta) \, ds \leqslant c r^2 V(r, \theta).$$

Hence for all large n

$$\begin{aligned} n\rho_n^2 U(\varepsilon/\rho_n, \theta_n) &\leqslant \varepsilon^2 (2c-1) \, n V(\varepsilon/\rho_n, \theta_n) \\ &\leqslant \varepsilon^2 (2c-1) \, n \varepsilon^{\delta-2} V(1/\rho_n, \theta_n) \\ &= (2c-1) \, \varepsilon^\delta n g(A_n^* t) \to (2c-1) \, \varepsilon^\delta f(t) \end{aligned}$$

by Lemma 3, Seneta [10, Theorem A.2, part (a)], and the fact that $ng(A_n^* t) \to f(t)$. Equation (2.4) follows. ∎

3. Concluding Remarks

The theory of regular variation has been used to prove new limit theorems in probability and to improve the presentation of known results. The work of Feller on stable laws and domains of attraction in $\mathbb{R}^1$ gives a striking example of the kind of clarity and unification of method which the theory of regular variation can provide. In a multivariable setting, Hahn and Klass [2] have shown that slow variation of the truncated second moment function (the function $U(r, \theta)$, defined in Section 2 above) uniformly in θ is necessary and sufficient for a random vector X to belong to the domain of attraction of a normal law. The arguments of the above section make use of the theory of $R-0$ variation, but more central is the fact that the condition $n\mu\{A_n^{-1} \, dx\} \to \phi\{dx\}$ entails a kind of regular variation of the measure μ at infinity. We mentioned in the proof of the theorem that $ng(A_n^* t) \to f(t)$ is necessary for attraction of X to a full, non-normal, operator-stable limit Y. This is a regular variation condition on $g(t)$ at $t = 0$. We are currently investigating the subject of such limit conditions and attempting to classify the kinds of functions and measures on $\mathbb{R}^k$ which are subject to them, as well as the kinds of limits which can occur.

The theorem presented in this paper reduces to a result obtained by Resnick and Greenwood [7, Theorem 4] in the case where $k=2$ and A_n has a matrix representation which is diagonal with respect to the standard basis for $\mathbb{R}^k$. A similar result was obtained by Jurek [4] in the case where $A_n = n^{-E}$ for all n and E is as in (1.5) above.

REFERENCES

1. FELLER, W. (1971). *An Introduction to Probability Theory and its Applications*, Vol. II, 2nd ed. Wiley, New York.
2. HAHN, M., AND KLASS, M. (1980). Matrix normalization of sums of random vectors in the domain of attraction of the multivariate normal. *Ann. Probab.* **8**, No. 2, 262–280.
3. HIRSCH, M., AND SMALE, S. (1971). *Differential Equations, Dynamical Systems, and Linear Algebra.* Academic Press, New York.
4. JUREK, Z. (1980). Domains of normal attraction of operator-stable measures on euclidean spaces. *Bull. Acad. Polon. Sci., Sér. Sci. Math.* **27** 7–8, 397–409.
5. KUCHARCZAK, J. (1975). Remarks on operator-stable measures. *Colloq. Math.* **34**, No. 1, 109–119.
6. LÉVY, P. (1954). *Théorie de l'Addition des Variables Aléatoires*, 2nd ed. Gauthier-Villars, Paris.
7. RESNICK, S., AND GREENWOOD, P. (1979). A bivariate stable characterization and domains of attraction. *J. Multivariate Anal.* **9** 206–221.
8. RVAČEVA, E. (1962). On domains of attraction of multidimensional distributions. *Select. Transl. Math. Statist. Probab.* **2** 183–205.
9. SHARPE, M. (1969). Operator-stable probability distributions on vector groups. *Trans. Amer. Math. Soc.* **136** 51–65.
10. SENETA, E. (1976). *Regularly Varying Functions.* Lecture Notes in Math. Vol. 508. Springer-Verlag, New York/Berlin.