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Distributed-order fractional diffusions on bounded domains

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ABSTRACT

Fractional derivatives can be used to model time delays in a diffusion process. When the order of the fractional derivative is distributed over the unit interval, it is useful for modeling a mixture of delay sources. This paper provides explicit strong solutions and stochastic analogues for distributed-order time-fractional diffusion equations on bounded domains, with Dirichlet boundary conditions.

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1. Introduction

This paper develops explicit strong solutions for distributed-order time-fractional diffusion equations on bounded domains $D \subset \mathbb{R}^d$, with Dirichlet boundary conditions. The abstract partial differential equation $\partial u/\partial t = Lu$ models a diffusion process. The simplest case $L = \Delta = \sum_j \partial^2 u/\partial x_j^2$ governs a Brownian motion $B(t)$ with density $u(t,x)$, for which the square root scaling $u(t,x) = t^{-1/2}u(1, t^{-1/2}x)$ holds [13]. The time-fractional diffusion equation $\partial^\beta u/\partial t^\beta = Lu$ which is based on Caputo fractional derivative of order $0 < \beta < 1$ is used to model anomalous sub-diffusion, in which a cloud of particles spreads slower than the square root of time [18,19,30,33,36]. Baeumer and Meerschaert [2] investigated the solutions to the initial-value problem in the general Banach space setting, where L is the generator of a uniformly bounded, strongly continuous semigroup. Their results can be used to establish weak solutions. See also the pioneering work of Kochubei [18,19]. Baeumer et al. [3] proved strong solutions for the case where L is a uniformly elliptical operator on $\mathbb{R}^d$, and Meerschaert et al. [28] extended this result to bounded domains on $\mathbb{R}^d$, with Dirichlet boundary conditions.

When $L = \Delta$, the solution $u(t,x)$ is the density of a time-changed Brownian motion $B(E_t)$, where the non-Markovian time change $E_t = \inf\{\tau > 0\colon\ W_\tau > t\}$ is the inverse, or first passage time, of a stable subordinator W_t with index β. The scaling $W_{ct} = c^{1/\beta} W_t$ in law implies $E_{ct} = c^\beta E_t$ in law for the inverse process, so that $u(t,x) = t^{-\beta/2}u(1, t^{-\beta/2}x)$. Scaling properties for a related fractional differential equation were developed by Buckwar and Luchko [5], see Remark 2.3. The process $B(E_t)$ is the long-time scaling limit of a random walk [24,25], when the random waiting times between jumps belong to the β-stable domain of attraction. Roughly speaking, a power-law distribution of waiting times leads to a fractional time derivative in the governing equation. Recently, Barlow and Černý [4] obtained $B(E_t)$ as the scaling limit of a random walk in a random environment. More generally, for a uniformly elliptic operator L on a bounded domain $D \subset \mathbb{R}^d$, under

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suitable technical conditions and assuming Dirichlet boundary conditions, the diffusion equation $\partial u/\partial t = Lu$ governs a Markov process $X(t)$ killed at the boundary, and the corresponding fractional diffusion equation $\partial^\beta u/\partial t^\beta = Lu$ governs the time-changed process $X(E_t)$ [28].

In some applications, waiting times between particle jumps evolve according to a more complicated process, which cannot be adequately described by a single power law. A mixture of power laws leads to a distributed-order fractional derivative in time [8,26,29]. An important application of distributed-order diffusions is to model ultraslow diffusion where a plume of particles spreads at a logarithmic rate [34]. This paper considers the distributed-order time-fractional diffusion equations with Dirichlet boundary conditions. Hahn et al. [15] discussed the solutions of such equations on $\mathbb{R}^d$, and the connections with certain subordinated processes. Kochubei [20] proved strong solutions on $\mathbb{R}^d$ for the case $L = \Delta$. Luchko [21] proved the uniqueness and continuous dependence on initial conditions on bounded domains. This paper constructs explicit classical solutions on bounded domains, and identifies the underlying stochastic process, which can be useful for particle tracking [22,37].

2. Distributed order fractional derivatives

The Caputo fractional derivative [6] is defined for $0 < \beta < 1$ as

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(r,x)}{\partial r} \frac{dr}{(t-r)^\beta}. \tag{2.1}$$

Its Laplace transform

$$\int_0^\infty e^{-st} \frac{\partial^\beta u(t,x)}{\partial t^\beta}\, ds = s^\beta \tilde{u}(s,x) - s^{\beta-1} u(0,x) \tag{2.2}$$

incorporates the initial value in the same way as the first derivative. The distributed order fractional derivative is

$$\mathbb{D}^{(\nu)} u(t,x) := \int_0^1 \frac{\partial^\beta u(t,x)}{\partial t^\beta}\, \nu(d\beta), \tag{2.3}$$

where ν is a finite Borel measure with $\nu(0,1) > 0$.

For a function $u(t,x)$ continuous in $t \geqslant 0$, the Riemann–Liouville fractional derivative of order $0 < \beta < 1$ is defined by

$$\left(\frac{\partial}{\partial t}\right)^\beta u(t,x) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \int_0^t \frac{u(r,x)}{(t-r)^\beta}\, dr. \tag{2.4}$$

Its Laplace transform

$$\int_0^\infty e^{-st} \left(\frac{\partial}{\partial t}\right)^\beta u(t,x)\, ds = s^\beta \tilde{u}(s,x). \tag{2.5}$$

If $u(\cdot,x)$ is absolutely continuous on bounded intervals (e.g., if the derivative exists everywhere and is integrable) then the Riemann–Liouville and Caputo derivatives are related by

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = \left(\frac{\partial}{\partial t}\right)^\beta u(t,x) - \frac{t^{-\beta} u(0,x)}{\Gamma(1-\beta)}. \tag{2.6}$$

The Riemann–Liouville fractional derivative is more general, as it does not require the first derivative to exist. It is also possible to adopt the right-hand side of (2.6) as the definition of the Caputo derivative, see for example Kochubei [20]. Then the (extended) distributed order derivative is

$$\mathbb{D}_1^{(\nu)} u(t,x) := \int_0^1 \left[\left(\frac{\partial}{\partial t}\right)^\beta u(t,x) - \frac{t^{-\beta} u(0,x)}{\Gamma(1-\beta)}\right] \nu(d\beta), \tag{2.7}$$

which exists for $u(t,x)$ continuous, and agrees with the usual definition (2.3) when $u(t,x)$ is absolutely continuous.

Distributed order fractional derivatives are connected with random walk limits. For each $c > 0$, take a sequence of i.i.d. waiting times (J_n^c) and i.i.d. jumps (Y_n^c). Let $X^c(n) = Y_1^c + \cdots + Y_n^c$ be the particle location after n jumps, and $T^c(n) = J_1^c + \cdots + J_n^c$ the time of the nth jump. Suppose that $X^c(cu) \Rightarrow A(t)$ and $T^c(cu) \Rightarrow W_t$ as $c \to \infty$, where the limits $A(t)$

and W_t are independent Lévy processes. The number of jumps by time $t \geqslant 0$ is $N_t^c = \max\{n \geqslant 0\colon\ T^c(n) \leqslant t\}$, and [27, Theorem 2.1] shows that $X^c(N_t^c) \Rightarrow A(E_t)$, where

$$E_t = \inf\{\tau\colon\ W_\tau > t\}. \tag{2.8}$$

A specific mixture model from [26] gives rise to distributed order fractional derivatives: Let (B_i), $0 < B_i < 1$, be i.i.d. random variables such that $P\{J_i^c > u \mid B_i = \beta\} = c^{-1}u^{-\beta}$, for $u \geqslant c^{-1/\beta}$. Then $T^c(cu) \Rightarrow W_t$, a subordinator with $\mathbb{E}[e^{-sW_t}] = e^{-t\psi_W(s)}$, where

$$\psi_W(s) = \int_0^\infty \left(e^{-sx} - 1\right) \phi_W(dx). \tag{2.9}$$

Then the associated Lévy measure is

$$\phi_W(t,\infty) = \int_0^1 t^{-\beta}\,\mu(d\beta), \tag{2.10}$$

where μ is the distribution of B_i. An easy computation gives

$$\psi_W(s) = \int_0^1 s^\beta\,\Gamma(1-\beta)\,\mu(d\beta). \tag{2.11}$$

Then, Theorem 3.10 in [26] shows that $c^{-1}N_t^c \Rightarrow E_t$, where E_t is given by (2.8). The Lévy process $A(t)$ defines a strongly continuous convolution semigroup with generator L, and $A(E_t)$ is the stochastic solution to the distributed order-fractional diffusion equation

$$\mathbb{D}^{(\nu)}u(t,x) = Lu(t,x), \tag{2.12}$$

where $\mathbb{D}^{(\nu)}$ is given by (2.3) with $\nu(d\beta) = \Gamma(1-\beta)\,\mu(d\beta)$. The condition

$$\int_0^1 \frac{1}{1-\beta}\,\mu(d\beta) < \infty \tag{2.13}$$

is imposed to ensure that $\nu(0,1) < \infty$. Since $\phi_W(0,\infty) = \infty$ in (2.9), Theorem 3.1 in [27] implies that E_t has a Lebesgue density

$$g(t,x) = \int_0^t \phi_W(t-y,\infty)\,P_{W_x}(dy).$$

Note that E_t is almost surely continuous and nondecreasing.

We say that a function h is a mild solution of a fractional differential equation if its Laplace (or Fourier) transform solves the corresponding algebraic equation. The following lemma shows that $h(t,\lambda) = \mathbb{E}[e^{-\lambda E_t}]$ is an eigenfunction of the distributed-order fractional derivative $\mathbb{D}^{(\nu)}$ in the mild sense. It also shows that $h(t,\lambda)$ is continuous in $t > 0$, and hence is also an eigenfunction of the extended distributed order derivative (2.7).

Lemma 2.1. *For any $\lambda > 0$, $h(t,\lambda) = \int_0^\infty e^{-\lambda x} g(t,x)\,dx = \mathbb{E}[e^{-\lambda E_t}]$ is a mild solution of the distributed-order fractional differential equation*

$$\mathbb{D}^{(\nu)}h(t,\lambda) = -\lambda h(t,\lambda); \quad h(0,\lambda) = 1. \tag{2.14}$$

Proof. First note that $h(0,\lambda) = \mathbb{E}(1) = 1$. Using (2.2), (2.11) and (2.3), compute the Laplace transform of $\mathbb{D}^{(\nu)}h(t,\lambda)$ as

$$\begin{aligned}\int_0^\infty e^{-st}\mathbb{D}^{(\nu)}h(t,\lambda)\,dt &= \int_0^\infty e^{-st}\int_0^1 \frac{\partial^\beta h(t,\lambda)}{\partial t^\beta}\,\nu(d\beta)\,dt\\ &= \int_0^1\int_0^\infty e^{-st}\frac{\partial^\beta h(t,\lambda)}{\partial t^\beta}\,dt\,\nu(d\beta)\end{aligned}$$

$$= \int_0^1 \left(s^\beta \tilde{h}(s,\lambda) - s^{\beta-1}\right) \nu(d\beta)$$

$$= \left(\tilde{h}(s,\lambda) - \frac{1}{s}\right)\psi_W(s), \tag{2.15}$$

by applying a Fubini argument which holds because $\psi_W(s) < \infty$.

The Laplace transform of $g(t,x)$, the density of E_t, is given by [27, Eq. (3.13)]:

$$\tilde{g}(s,x) = \int_0^\infty e^{-st} g(t,x)\,dt = \frac{1}{s}\psi_W(s) e^{-x\psi_W(s)}. \tag{2.16}$$

Then the double Laplace transform

$$\tilde{h}(s,\lambda) := \int_0^\infty e^{-st} h(t,\lambda)\,dt = \int_0^\infty e^{-st}\left(\int_0^\infty e^{-\lambda x} g(t,x)\,dx\right)dt$$

$$= \int_0^\infty e^{-\lambda x}\left(\int_0^\infty e^{-st} g(t,x)\,dt\right)dx$$

$$= \frac{\psi_W(s)}{s}\int_0^\infty e^{-(\lambda+\psi_W(s))x}\,dx \tag{2.17}$$

$$= \frac{\psi_W(s)}{s(\lambda+\psi_W(s))}. \tag{2.18}$$

That is, $\tilde{h}(s,\lambda)$ satisfies

$$\lambda\tilde{h}(s,\lambda) = \left(\frac{1}{s} - \tilde{h}(s,\lambda)\right)\psi_W(s). \tag{2.19}$$

Since E_t has continuous paths, the dominated convergence theorem implies that $t \to \mathbb{E}[e^{-\lambda E_t}] = h(t,\lambda)$ is a continuous function. Then (2.14) follows from (2.15), (2.19) and the uniqueness of the Laplace transform. □

The next theorem extends a deep result of Kochubei [20] to show that the functions $h(t,\lambda)$ from Lemma 2.1 are eigenfunctions of the distributed-order fractional derivative (2.3) in the strong sense. A function h is a classical (strong) solution of the distributed-order fractional differential equation (2.14) if the equality holds pointwise, and the distributed-order fractional derivative exists in the classical sense. The proof also shows that $\partial_t h(t,\lambda)$ exists, and gives an explicit upper bound (2.21) on its absolute value, which will be useful in the next section.

Theorem 2.2. *Let $\nu(d\beta) = p(\beta)\,d\beta$ for some $p \in C^1(0,1)$, and $0 < \beta_0 < \beta_1 < 1$ be such that*

$$C(\beta_0,\beta_1,p) = \int_{\beta_0}^{\beta_1} \sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta > 0. \tag{2.20}$$

Then $h(t,\lambda)$, the mild solution to (2.14), satisfies $|\partial_t h(t,\lambda)| \leqslant \lambda k(t)$, where

$$k(t) = \left[C(\beta_0,\beta_1,p)\pi\right]^{-1}\left[\Gamma(1-\beta_1)t^{\beta_1-1} + \Gamma(1-\beta_0)t^{\beta_0-1}\right], \tag{2.21}$$

and hence is a classical solution.

Proof. Using (2.19) in Kochubei [20], which follows from inverting the Laplace transform in (2.17) of $h(t,\lambda)$, we have

$$h(t,\lambda) = \frac{-\lambda}{\pi}\int_0^\infty r^{-1} e^{-tr}\Phi(r,1)\,dr \tag{2.22}$$

where

$$\Phi(r,1)=\frac{\int_0^1 r^\beta\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta}{[\int_0^1 r^\beta\cos(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta+\lambda]^2+[\int_0^1 r^\beta\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta]^2}.$$

First we show that $|\partial_t h(t,\lambda)|<\lambda k(t)$. Note that

$$\begin{aligned}\left|\partial_t h(\lambda,t)\right|&=\left|\frac{-\lambda}{\pi}\int_0^\infty r^{-1}\left[\partial_t e^{-tr}\right]\Phi(r,1)\,dr\right|\\&=\frac{\lambda}{\pi}\int_0^\infty e^{-tr}\Phi(r,1)\,dr\\&=\frac{\lambda}{\pi}\int_0^\infty\frac{e^{-tr}\int_0^1 r^\beta\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta}{[\int_0^1 r^\beta\cos(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta+\lambda]^2+[\int_0^1 r^\beta\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta]^2}\,dr\\&\leqslant\lambda\pi^{-1}\int_0^\infty\frac{e^{-tr}\,dr}{\int_0^1 r^\beta\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta}\\&=\lambda l(t)\quad\text{(say)},\end{aligned}$$

where $l(t)$ is a function of t only. For example, $l(t)=Ct^{\beta-1}$ in the case of a simple fractional derivative. Now,

$$\int_0^1\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta\geqslant\int_{\beta_0}^{\beta_1}\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta=C(\beta_0,\beta_1,p)>0,\tag{2.23}$$

by (2.20). For $r>1$, and $\beta_0\leqslant\beta\leqslant\beta_1\leqslant1$, we have $r^{\beta_0}\leqslant r^\beta\leqslant r^{\beta_1}$ and so

$$\begin{aligned}\int_{\beta_0}^{\beta_1}r^\beta\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta&\geqslant\int_{\beta_0}^{\beta_1}r^{\beta_0}\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta\\&=r^{\beta_0}C(\beta_0,\beta_1,p).\end{aligned}\tag{2.24}$$

For $0<r\leqslant1$, and $\beta_0\leqslant\beta\leqslant\beta_1\leqslant1$, we have $r^{\beta_0}\geqslant r^\beta\geqslant r^{\beta_1}$ and so

$$\begin{aligned}\int_{\beta_0}^{\beta_1}r^\beta\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta&\geqslant\int_{\beta_0}^{\beta_1}r^{\beta_1}\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta\\&=r^{\beta_1}C(\beta_0,\beta_1,p).\end{aligned}\tag{2.25}$$

Using the above facts, we obtain

$$\begin{aligned}l(t)&=\pi^{-1}\int_0^\infty\frac{e^{-tr}\,dr}{\int_0^1 r^\beta\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta}\\&=\pi^{-1}\left[\int_0^1\frac{e^{-tr}\,dr}{\int_0^1 r^\beta\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta}+\int_1^\infty\frac{e^{-tr}\,dr}{\int_0^1 r^\beta\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta}\right]\\&\leqslant\pi^{-1}\left[\int_0^1\frac{e^{-tr}\,dr}{\int_{\beta_0}^{\beta_1} r^\beta\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta}+\int_1^\infty\frac{e^{-tr}\,dr}{\int_{\beta_0}^{\beta_1} r^\beta\sin(\beta\pi)\Gamma(1-\beta)p(\beta)\,d\beta}\right]\\&\leqslant\left[C(\beta_0,\beta_1,p)\pi\right]^{-1}\left[\int_0^1 r^{-\beta_1}e^{-tr}\,dr+\int_1^\infty r^{-\beta_0}e^{-tr}\,dr\right]\\&\leqslant\left[C(\beta_0,\beta_1,p)\pi\right]^{-1}\left[\Gamma(1-\beta_1)t^{\beta_1-1}+\Gamma(1-\beta_0)t^{\beta_0-1}\right]=k(t)\end{aligned}$$

and so $|\partial_t h(t,\lambda)| \leqslant \lambda k(t)$. Hence, it follows from (2.3) that

$$\begin{aligned}\left|\mathbb{D}^{(\nu)} h(t,\lambda)\right| &\leqslant \left|\int_0^1 \frac{\partial^\beta}{\partial t^\beta} h(t,\lambda)\Gamma(1-\beta)p(\beta)\,d\beta\right| \\ &\leqslant \int_0^1 \frac{1}{\Gamma(1-\beta)}\int_0^t \left|\frac{\partial h(s,\lambda)}{\partial s}\right| \frac{ds}{(t-s)^\beta}\Gamma(1-\beta)p(\beta)\,d\beta \\ &\leqslant \lambda \int_0^1 \frac{1}{\Gamma(1-\beta)}\int_0^t k(s)\frac{ds}{(t-s)^\beta}\Gamma(1-\beta)p(\beta)\,d\beta \\ &< \infty, \end{aligned} \tag{2.26}$$

using (2.13) and the beta density formula. Thus, the distributed-order derivative $\mathbb{D}^{(\nu)} h(t,\lambda)$ exists. Also, it follows now from Lemma 2.1 that $h(t,\lambda)$ is an eigenfunction in the strong sense. □

Remark 2.3. Since Theorem 2.2 shows that $h(t,\lambda)$ is absolutely continuous and the derivative bound (2.26) holds, the two distributed-order derivatives of $h(t,\lambda)$ defined in (2.3) and in (2.7) agree. Hence, $h(t,\lambda)$ is also a classical solution of Eq. (2.14) with $\mathbb{D}^{(\nu)}$ replaced by $\mathbb{D}_1^{(\nu)}$. The time-fractional diffusion equations using the Riemann–Liouville fractional derivative (2.4) have also been considered. For example, scaling relations for solutions of these equations are developed in Buckwar and Luchko [5]. Also, the distributed-order Riemann–Liouville fractional derivative can be defined similar to (2.3), and the corresponding diffusion equations in the Riemann–Liouville sense have been considered in [1,12].

3. Distributed order time-fractional diffusion equations

In this section, we prove classical (strong) solutions to distributed-order time-fractional diffusion equations $\mathbb{D}^{(\nu)}u = Lu$ on bounded domains $D \subset \mathbb{R}^d$. Let $C^k(D), C^{k,\alpha}(D)$ and $C^k(\bar{D})$ respectively be the space of k-times differentiable functions in D, the space of k-times differential functions with k-th derivative Hölder continuous of index α, and the space of functions that have all the derivatives up to order k extendable continuously up to the boundary ∂D of D. Let $D_\infty = (0,\infty) \times D$ and write $u \in C^k(\bar{D})$ if for each fixed $t > 0$, $u(t,\cdot) \in C^k(\bar{D})$. Write $u \in C_b^k(\bar{D}_\infty)$ when $u \in C^k(\bar{D}_\infty)$ is bounded.

A uniformly elliptic operator L in divergence form is a linear operator on $L^2(\mathbb{R}^d)$ defined for $u \in C^2(\mathbb{R}^d)$ by

$$Lu = \sum_{i,j=1}^d \frac{\partial(a_{ij}(x)(\partial u/\partial x_i))}{\partial x_j} \tag{3.1}$$

with $a_{ij} = a_{ji}$ and

$$\lambda \sum_{i=1}^n y_i^2 \leqslant \sum_{i,j=1}^n a_{ij}(x) y_i y_j \leqslant \lambda^{-1}\sum_{i=1}^n y_i^2, \quad \forall y \in \mathbb{R}^d \tag{3.2}$$

for some $\lambda > 0$. Let X_t solve $dX_t = \sigma(X_t)\,dW_t + b(X_t)\,dt$ with $X_0 = x_0$, where σ is a $d \times d$ matrix, and W_t is a Brownian motion. Let $\tau_D(X) = \inf\{t \geqslant 0\colon\ X(t) \notin D\}$ be the first exit time. Then the semigroup

$$T(t)f(x) = E_x\left[f(X_t)I\left(\tau_D(X) > t\right)\right] = \int_D p_D(t,x,y)f(y)\,dy$$

has generator L_D of the form (3.1) with $a = \sigma\sigma^T$ by an application of the Itô formula. The operator L_D has eigenvalues $0 < \mu_1 < \mu_2 \leqslant \mu_3 \cdots$ with $\mu_n \to \infty$, and eigenfunctions

$$L_D\psi_n(x) = -\mu_n\psi_n(x), \quad x \in D\colon \psi_n|_{\partial D} = 0. \tag{3.3}$$

The heat kernel $p_D(t,x,y) = \sum_{n=1}^\infty e^{-\mu_n t}\psi_n(x)\psi_n(y)$, where the series converges absolutely and uniformly on $[t_0,\infty) \times D \times D$ for all $t_0 > 0$. Since the eigenfunctions (ψ_n) form a complete orthonormal basis, we can write $f(x) = \sum_n \bar{f}(n)\psi_n(x)$ for any $f \in L^2(\mathbb{R}^d)$. We will call $\bar{f}(n) = \int_D \psi_n(x)f(x)\,dx$ the ψ_n-transform of f.

In the special case $L_D = \Delta_D$, the corresponding Markov process is a killed Brownian motion. We denote the eigenvalues and the eigenfunctions of Δ_D by $\{\lambda_n, \phi_n\}_{n=1}^\infty$, where $\phi_n \in C^\infty(D)$. Let

$$\mathcal{H}_\Delta(D_\infty) \equiv \left\{u : D_\infty \to \mathbb{R}\colon\ \Delta u \in C(D_\infty),\ \left|\partial_t u(t,x)\right| \leqslant k(t)g(x),\ g \in L^\infty(D),\ t > 0\right\},$$

where $k(t)$ is given by (2.21).

The next result provides strong solutions, in particular in the space $\mathcal{H}_\Delta(D_\infty) \cap C_b(\bar{D}_\infty) \cap C^1(\bar{D})$, to distributed-order time-fractional diffusion equations on bounded domains.

Theorem 3.1. *Let D be a bounded domain with $\partial D \in C^{1,\alpha}$ for some $0 < \alpha < 1$, and $T_D(t)$ be the killed semigroup of Brownian motion $\{X(t)\}$ on D. Let E_t be the inverse (2.8) of the subordinator W_t, independent of $\{X(t)\}$, with Lévy measure (2.10). Suppose that $\mu(d\beta) = p(\beta)\,d\beta$, as in Theorem 2.2, and $\mathbb{D}^{(\nu)}$ is the distributed-order fractional derivative defined by (2.3). Then, for any $f \in D(\Delta_D) \cap C^1(\bar{D}) \cap C^2(D)$ for which the eigenfunction expansion of Δf with respect to the complete orthonormal basis $\{\phi_n\colon n \in \mathbb{N}\}$ converges uniformly and absolutely, the unique classical solution of the distributed order time-fractional diffusion equation*

$$\begin{aligned}
&\mathbb{D}^{(\nu)} u(t,x) = \Delta u(t,x), \quad x \in D,\ t > 0,\\
&u(t,x) = 0, \quad x \in \partial D,\ t > 0,\\
&u(0,x) = f(x), \quad x \in D,
\end{aligned} \tag{3.4}$$

for $u \in \mathcal{H}_\Delta(D_\infty) \cap C_b(\bar{D}_\infty) \cap C^1(\bar{D})$, is given by

$$\begin{aligned}
u(t,x) &= \mathbb{E}_x\big[f\big(X(E_t)\big) I\big(\tau_D(X) > E_t\big)\big]\\
&= \int_0^\infty T_D(l) f(x) g(t,l)\,dl\\
&= \sum_{n=1}^\infty \bar{f}(n)\phi_n(x) h(t,\lambda_n),
\end{aligned} \tag{3.5}$$

where $h(t,\lambda) = \mathbb{E}(e^{-\lambda E_t}) = \int_0^\infty e^{-\lambda x} g(t,x)\,dx$ is the Laplace transform of E_t.

Proof. The proof is similar to Theorem 3.1 in [28]. Assume that $u(t,x)$ solves (3.4). Use Green's second identity to get

$$\int_D \phi_n(x)\Delta u(t,x)\,dx = \int_D u(t,x)\Delta\phi_n(x)\,dx = -\lambda_n \int_D u(t,x)\phi_n(x)\,dx = -\lambda_n \bar{u}(t,n).$$

Using (2.1) and (2.3), we get

$$\begin{aligned}
\int_D \phi_n(x)\mathbb{D}^{(\nu)} u(t,x)\,dx &= \int_D \phi_n(x) \int_0^1 \frac{\partial^\beta}{\partial t^\beta} u(t,x)\Gamma(1-\beta)p(\beta)\,d\beta\,dx\\
&= \int_D \phi_n(x) \int_0^1 \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(s,x)}{\partial s}\frac{ds}{(t-s)^\beta}\Gamma(1-\beta)p(\beta)\,d\beta\,dx\\
&= \int_D \phi_n(x) \int_0^1 \int_0^t \frac{\partial u(s,x)}{\partial s}\frac{ds}{(t-s)^\beta} p(\beta)\,d\beta\,dx\\
&= \int_0^1 \int_0^t \left(\int_D \phi_n(x)\frac{\partial}{\partial s} u(s,x)\,dx\right)\frac{ds}{(t-s)^\beta} p(\beta)\,d\beta \quad \text{(by Fubini, see below)}\\
&= \int_0^1 \int_0^t \frac{\partial}{\partial s}\left(\int_D \phi_n(x) u(s,x)\,dx\right)\frac{ds}{(t-s)^\beta} p(\beta)\,d\beta\\
&= \int_0^1 \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial}{\partial s}\bar{u}(s,n)\frac{ds}{(t-s)^\beta}\Gamma(1-\beta)p(\beta)\,d\beta\\
&= \mathbb{D}^{(\nu)}\bar{u}(s,n).
\end{aligned} \tag{3.6}$$

The Fubini–Tonelli argument for the interchange of order of integration in (3.6) can be justified as follows

$$\begin{aligned}\left|\int_D \phi_n(x)\mathbb{D}^{(\nu)}u(t,x)\,dx\right| &= \left|\int_D \phi_n(x)\int_0^1\int_0^t \frac{\partial u(s,x)}{\partial s}\frac{ds}{(t-s)^\beta}p(\beta)\,d\beta\,dx\right|\\ &\leqslant \int_D \left|\phi_n(x)\right|\int_0^1\int_0^t \left|\frac{\partial u(s,x)}{\partial s}\right|\frac{ds}{(t-s)^\beta}p(\beta)\,d\beta\,dx\\ &\leqslant \int_D \left|\phi_n(x)\right|\left|g(x)\right|dx\int_0^1\int_0^t k(s)\frac{ds}{(t-s)^\beta}p(\beta)\,d\beta\\ &\leqslant \sqrt{|D|}\|\phi_n\|_{L^2(D)}\|g\|_{L^\infty}\int_0^1\int_0^t \left[C(\beta_0,\beta_1,p)\pi\right]^{-1}\left[\Gamma(1-\beta_1)s^{\beta_1-1}+\Gamma(1-\beta_0)s^{\beta_0-1}\right]\\ &\quad\times\frac{ds}{(t-s)^\beta}p(\beta)\,d\beta,\end{aligned}$$

using (2.21). Further, using the property of beta density, for $0<\gamma,\eta<1$,

$$\int_0^t \frac{1}{(t-s)^\gamma}s^{\eta-1}\,ds = t^{\eta-\gamma}\int_0^1 (1-u)^{(1-\gamma)-1}u^{\eta-1}\,du = B(1-\gamma,\eta)t^{\eta-\gamma},$$

where $B(a,b)$ denotes the usual beta function. Thus,

$$\begin{aligned}&\left|\int_D \phi_n(x)\mathbb{D}^{(\nu)}u(t,x)\,dx\right|\\ &\quad\leqslant \sqrt{|D|}\|\phi_n\|_{L^2(D)}\|g\|_{L^\infty}\left[C(\beta_0,\beta_1,p)\pi\right]^{-1}\\ &\qquad\times\left[\int_0^1\int_0^t \Gamma(1-\beta_1)s^{\beta_1-1}\frac{ds}{(t-s)^\beta}p(\beta)\,d\beta+\int_0^1\int_0^t \Gamma(1-\beta_0)s^{\beta_0-1}\frac{ds}{(t-s)^\beta}p(\beta)\,d\beta\right]\\ &\quad= \sqrt{|D|}\|\phi_n\|_{L^2(D)}\|g\|_{L^\infty}\left[C(\beta_0,\beta_1,p)\pi\right]^{-1}\\ &\qquad\times\left[\Gamma(1-\beta_1)\int_0^1 t^{\beta_1-\beta}B(1-\beta,\beta_1)p(\beta)\,d\beta+\Gamma(1-\beta_0)\int_0^1 t^{\beta_0-\beta}B(1-\beta,\beta_0)p(\beta)\,d\beta\right]\\ &\quad<\infty,\end{aligned}$$

which justifies the use of Fubini–Tonelli theorem in (3.6).

Now apply the ϕ_n-transforms to both sides of (3.4) to get

$$\mathbb{D}^{(\nu)}\bar{u}(t,n) = -\lambda_n\bar{u}(t,n). \tag{3.7}$$

Since u is uniformly continuous on $C([0,\epsilon]\times\bar{D})$, it is uniformly bounded on $[0,\epsilon]\times\bar{D}$. Thus, by the dominated convergence theorem, we have $\lim_{t\to 0}\int_D u(t,x)\phi_n(x)\,dx = \bar{f}(n)$. Hence, $\bar{u}(0,n)=\bar{f}(n)$. A similar argument shows that $t\mapsto\bar{u}(t,n)$ is a continuous function of $t\in[0,\infty)$ for every n.

Denote the Laplace transform $t\to s$ of $u(t,x)$ by $\tilde{u}(s,x)=\int_0^\infty e^{-st}u(t,x)\,dt$ and call $\hat{u}(s,n)=\int_D \psi_n(x)\tilde{u}(s,x)\,dx$ the ψ_n-Laplace transform of u. Taking Laplace transforms on both sides of (3.7) and using (2.15), we get

$$\int_0^1 \left(s^\beta\hat{u}(s,n)-s^{\beta-1}\bar{u}(0,n)\right)\Gamma(1-\beta)p(\beta)\,d\beta = -\lambda_n\hat{u}(s,n) \tag{3.8}$$

which leads to

$$\hat{u}(s,n) = \frac{\bar{f}(n)\int_0^1 s^{\beta-1}\Gamma(1-\beta)p(\beta)\,d\beta}{\int_0^1 s^\beta\Gamma(1-\beta)p(\beta)\,d\beta+\lambda_n}. \tag{3.9}$$

Use (2.11) to get

$$\begin{aligned}\hat{u}(s,n) &= \frac{\bar{f}(n)\psi_W(s)}{s(\psi_W(s)+\lambda_n)}\\ &= \frac{1}{s}\bar{f}(n)\psi_W(s)\int_0^\infty e^{-(\psi_W(s)+\lambda_n)l}\,dl\\ &= \int_0^\infty e^{-\lambda_n l}\bar{f}(n)\frac{1}{s}\psi_W(s)e^{-l\psi_W(s)}\,dl, \end{aligned}\tag{3.10}$$

using the property of the exponential density. The ϕ_n-transform of the killed semigroup $T_D(l)f(x) = \sum_{m=1}^\infty e^{-\lambda_m l}\phi_m(x)\bar{f}(m)$ is found as follows. Since $\{\phi_n,\ n\in\mathbb{N}\}$ is a complete orthonormal basis of $L^2(D)$, we get

$$\begin{aligned}\overline{\big[T_D(l)f\big]}(n) &= \int_D \phi_n(x)T_D(l)f(x)\,dx\\ &= \int_D \phi_n(x)\int_D p_D(l,x,y)f(y)\,dy\,dx\\ &= \int_D \phi_n(x)\int_D \sum_{m=1}^\infty e^{-\lambda_m l}\phi_m(x)\phi_m(y)f(y)\,dy\,dx\\ &= \int_D \phi_n(x)\sum_{m=1}^\infty e^{-\lambda_m l}\phi_m(x)\int_D \phi_m(y)f(y)\,dy\,dx\\ &= \int_D \phi_n(x)\sum_{m=1}^\infty e^{-\lambda_m l}\phi_m(x)\bar{f}(m)\,dx\\ &= \sum_{m=1}^\infty e^{-\lambda_m l}\bar{f}(m)\int_D \phi_n(x)\phi_m(x)\,dx\\ &= e^{-l\lambda_n}\bar{f}(n). \end{aligned}\tag{3.11}$$

Since $T_D(t)$ is a contraction semigroup on $L^2(D)$, $T_D(t)f\in L^2(D)$ and hence Fubini–Tonelli applies. By (3.20) in [26], we have

$$\frac{1}{s}\psi_W(s)e^{-\psi_W(s)l} = \int_0^\infty e^{-st}g(t,l)\,dt,\tag{3.12}$$

where $g(t,l)$ is the smooth density of E_t. Using the results (3.11), (3.12) and (3.10), we get

$$\begin{aligned}\int_0^\infty e^{-st}\bar{u}(t,n)\,dt = \hat{u}(s,n) &= \int_0^\infty \overline{\big[T_D(l)f\big]}(n)\left[\int_0^\infty e^{-st}g(t,l)\,dt\right]dl\\ &= \int_0^\infty e^{-st}\left[\int_0^\infty \overline{\big[T_D(l)f\big]}(n)g(t,l)\,dl\right]dt. \end{aligned}$$

By the uniqueness of the Laplace transform,

$$\begin{aligned}\bar{u}(t,n) &= \int_0^\infty \overline{\big[T_D(l)f\big]}(n)g(t,l)\,dl\\ &= \bar{f}(n)\int_0^\infty e^{-\lambda_n l}g(t,l)\,dl \quad \big(\text{using (3.11)}\big)\\ &= \bar{f}(n)h(t,\lambda_n), \end{aligned}\tag{3.13}$$

where $h(t,\lambda)=\int_0^\infty e^{-l\lambda}g(t,l)\,dl$ is the Laplace transform of E_t. Inverting the ϕ_n-transform $\bar{u}(t,n)$ in (3.13), we get an L^2-convergent solution of (3.4) as

$$u(t,x)=\sum_{n=1}^{\infty}\bar{u}(t,n)\phi_n(x)=\sum_{n=1}^{\infty}\bar{f}(n)\phi_n(x)h(t,\lambda_n) \tag{3.14}$$

for each $t\geqslant 0$. In order to complete the proof, it will suffice to show that the series (3.14) converges pointwise, and satisfies all the conditions in (3.4).

Step 1. Since $h(t,\lambda)$ is the Laplace transform of E_t, it is completely monotone and non-increasing in $\lambda\geqslant 0$ with $0<h(t,\lambda_n)\leqslant 1$. Then an elementary estimate shows that (3.14) convergence uniformly in $t\in[0,\infty)$ in the L^2 sense.

Step 2. Using $\{E_t\leqslant x\}=\{W_x\geqslant t\}$ [26, Eq. (3.16)], it is easy to check that $E_t\Rightarrow E_0\equiv 0$ in distribution as $t\to 0+$ and hence the Laplace transforms converge: $h(t,\lambda_n)\to 1$. Then another elementary estimate shows that $t\to u(t,\cdot)\in C((0,\infty);L^2(D))$ and $\|u(t,\cdot)-f\|_{2,D}\to 0$, as $t\to 0$. The continuity of $t\mapsto u(t,\cdot)$ in $L^2(D)$ at every point $t\in(0,\infty)$ follows by a similar argument.

Step 3. As λ_n is increasing in n, and $h(t,\lambda_n)$ is non-increasing for $n\geqslant 1$, $\|u(t,\cdot)\|_{2,D}\leqslant h(t,\lambda_1)\|f\|_{2,D}$ by Parseval's identity.

Step 4. Note $h(t,\lambda)=\mathbb{E}(e^{-\lambda E_t})\leqslant 1$ for all $t\geqslant 0$ and $\lambda\geqslant 0$. Then a straightforward argument shows that

$$u_N(t,x)=\sum_{n=1}^{N}\bar{f}(n)\phi_n(x)h(t,\lambda_n)$$

is a Cauchy sequence in $L^\infty(D)$ uniformly in $t\geqslant 0$. Then the series (3.14) defining $u(t,x)$ converges uniformly and absolutely.

Step 5. Since the eigenfunction expansion of Δf converges absolutely and uniformly, the series $\sum_{n=1}^\infty \bar{f}(n)h(t,\lambda_n)\Delta\phi_n(x)$ is absolutely convergent in $L^\infty(D)$ uniformly in $(0,\infty)$. Since $\mathbb{D}^{(\nu)}h(t,\lambda)=-\lambda h(t,\lambda)$, we have

$$\sum_{n=1}^{\infty}\bar{f}(n)\phi_n(x)\mathbb{D}^{(\nu)}h(t,\lambda_n)=\sum_{n=1}^{\infty}\bar{f}(n)h(t,\lambda_n)\Delta\phi_n(x),$$

where both series converge absolutely and uniformly. Then $\mathbb{D}^{(\nu)}$ and Δ can be applied term by term in (3.14) to show that (3.4) is a classical (strong) solution to (3.4). Since Δf has an absolutely and uniformly convergent series expansion with respect to (ϕ_n), it follows using Theorem 2.2 that $u\in\mathcal{H}_\Delta(D_\infty)\cap C_b(\bar{D}_\infty)$.

Step 6. Using the bounds given in Theorem 8.33 of [14] and the absolute and uniform convergence of the series defining f, a simple estimate shows that $u\in C^1(\bar{D})$.

Step 7. Using (3.14) and (3.11), we get

$$\begin{aligned}
u(t,x)&=\sum_{n=1}^{\infty}\phi_n(x)\int_0^\infty \overline{\big[T_D(l)f\big]}(n)g(t,l)\,dl\\
&=\int_0^\infty\Bigg[\sum_{n=1}^{\infty}\phi_n(x)\bar{f}(n)e^{-l\lambda_n}\Bigg]g(t,l)\,dl\\
&=\int_0^\infty T_D(l)f(x)g(t,l)\,dl\\
&=\mathbb{E}_x\big[f\big(X(E_t)\big)I\big(\tau_D(X)>E_t\big)\big].
\end{aligned} \tag{3.15}$$

Thus, (3.5) is proved.

Step 8. Given two solutions u_i, $i=1,2$, of (3.4) with the same initial data $u_i(0,x)=f(x)$, $U=u_1-u_2$ is also a solution with the corresponding $f\equiv 0$. Then it is easy to check that $U(t,x)=0$ for all $(t,x)\in[0,\infty)\times D$ which proves uniqueness. □

The next result provides sufficient conditions for Theorem 3.1 to hold, which can easily be verified in practical applications.

Corollary 3.2. *Let $f \in C_c^{2k}(D)$ be a 2k-times continuously differentiable function with compact support in D. If $k > 1 + 3d/4$, then* (3.4) *has a classical (strong) solution. In particular, if $f \in C_c^\infty(D)$, then the solution of* (3.4) *is in $C^\infty(D)$.*

Proof. The proof is essentially identical to Corollary 3.4 in [28]. □

Next we extend Theorem 3.1 to a more general setting. Define

$$\mathcal{H}_L(D_\infty) = \{u : D_\infty \to \mathbb{R}\colon\ Lu(t,x) \in C(D_\infty)\};$$
$$\mathcal{H}_L^b(D_\infty) = \mathcal{H}_L(D_\infty) \cap \{u\colon\ |\partial_t u(t,x)| \leqslant k(t)g(x),\ g \in L^\infty(D),\ t > 0\},$$

where $k(t)$ is defined in (2.21), and L is a uniformly elliptic operator in divergence form (3.1) such that $a_{ij} = a_{ji}$ and (3.2) holds for some $\lambda > 0$.

Theorem 3.3. *Let $\{X(t)\}$ be a continuous Markov process with generator L defined in* (3.1)*. Then, under the conditions of Theorem* 3.1*, for any $f \in D(L_D) \cap C^1(\bar{D}) \cap C^2(D)$ the (classical) solution of*

$$\begin{aligned}
&\mathbb{D}^{(\nu)}u(t,x) = Lu(t,x), \quad x \in D,\ t \geqslant 0;\\
&u(t,x) = 0, \quad x \in \partial D,\ t \geqslant 0;\\
&u(0,x) = f(x), \quad x \in D,
\end{aligned} \tag{3.16}$$

for $u \in \mathcal{H}_L^b(D_\infty) \cap C_b(\bar{D}_\infty) \cap C^1(\bar{D})$, is given by

$$\begin{aligned}
u(t,x) &= \mathbb{E}_x\big[f\big(X(E_t)\big)I\big(\tau_D(X) > E_t\big)\big]\\
&= \int_0^\infty T_D(l)f(x)g(t,l)\,dl = \sum_0^\infty \bar{f}(n)\psi_n(x)h(t,\mu_n).
\end{aligned} \tag{3.17}$$

Proof. The proof is similar to Theorem 3.6 in [28]. Suppose $u(t,x) = G(t)F(x)$ solves (3.16) so that $F(x)\mathbb{D}^{(\nu)}G(t) = G(t)LF(x)$. Divide both sides by $G(t)F(x)$ to get

$$\frac{\mathbb{D}^{(\nu)}G(t)}{G(t)} = \frac{LF(x)}{F(x)} = -\mu.$$

The eigenvalue problem $LF(x) = -\mu F(x),\ x \in D,\ F|_{\partial D} = 0$ is solved by an infinite sequence of pairs (μ_n, ψ_n), where (ψ_n) forms a complete orthonormal set in $L^2(D)$. Also, $\mathbb{D}^{(\nu)}G(t) = -\mu G(t)$ is solved by $G(t) = G_0(n)h(t,\mu_n)$, where $G_0(n) = \bar{f}(n)$. The sequence $u_N(t,x) = \sum_{n=1}^N \bar{f}(n)h(t,\mu_n)\psi_n(x)$ is Cauchy in $L^2(D) \cap L^\infty(D)$, uniformly in $t \in [0,\infty)$. The series defining u and Lu converge absolutely and uniformly so that $\mathbb{D}^{(\nu)}$ and L can be applied term by term. Then

$$u(t,x) = \sum_{n=1}^\infty \bar{f}(n)h(t,\mu_n)\psi_n(x) \tag{3.18}$$

is a classical solution. The stochastic solution and the uniqueness also follow as before. □

Remark 3.4. The stochastic solution in Theorems 3.1 and 3.3 can also be written as

$$u(t,x) = \mathbb{E}_x\big[f\big(X(E_t)\big)I\big(\tau_D\big(X(E)\big) > t\big)\big].$$

The argument is similar to Corollary 3.2 in [28].

Remark 3.5. It is not difficult to extend Theorem 3.3 to the case where the mixing measure ν in (2.3) contains atoms. Suppose μ is a finite measure with $\operatorname{supp}(\mu) \subset (0,1)$ and satisfies (2.13). Assume also that $|\partial_t h(t,\lambda)| \leqslant b(\lambda)k_e(t)$ for some functions b and k_e satisfying the condition

$$b(\lambda)\int_0^1\int_0^t \frac{k_e(s)\,ds}{(t-s)^\beta}\,d\mu(\beta) < \infty, \tag{3.19}$$

for $t, \lambda > 0$. Then $h(t,\lambda) = \mathbb{E}(e^{-\lambda E_t})$ is a classical solution of the eigenvalue problem

$$\mathbb{D}^{(\nu)}h(t,\lambda) = -\lambda h(t,\lambda); \quad h(0,\lambda) = 1. \tag{3.20}$$

The proof follows from Lemma 2.1, and the fact that (3.19) is a sufficient condition for $\mathbb{D}^{(\nu)}h(t,\lambda)$ to be defined as a classical function. Suppose that

$$k_e(t)\sum_{n=1}^{\infty} b(\lambda_n)\bar{f}(n)\left|\phi_n(x)\right| < \infty, \tag{3.21}$$

where the series converges absolutely and uniformly for $t > 0$. Define

$$\mathcal{H}_L(D_\infty) = \left\{u : D_\infty \to \mathbb{R}\colon\ Lu(t,x) \in C(D_\infty)\right\};$$
$$\mathcal{H}_L^{b,e}(D_\infty) = \mathcal{H}_L(D_\infty) \cap \left\{u\colon\ \left|\partial_t u(t,x)\right| \leqslant k_e(t)g(x),\ g \in L^\infty(D),\ t > 0\right\},$$

where k_e and b satisfy (3.19) and (3.21). Then under the remaining conditions for Theorem 3.3, the classical solution of (3.16) for $u \in \mathcal{H}_L^{b,e}(D_\infty) \cap C_b(\bar{D}_\infty) \cap C^1(\bar{D})$ is given by (3.17). The proof is similar to Theorem 3.3, using (3.19) and (3.21).

Example 3.6. Let

$$\mu(d\beta) = \sum_{j=1}^{n} c_j^{\beta_j}\left(\Gamma(1-\beta_j)\right)^{-1}\delta(\beta-\beta_j)\,d\beta,$$

where $0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1$, c_j's are positive constants and δ is the Dirac measure. Consider the subordinator defined by

$$W_t = \sum_{j=1}^{n} c_j W_t^{\beta_j},$$

where $W_t^{\beta_j}$'s are the independent stable subordinators. The functions $k_e(t)$ and $b(\lambda)$ that satisfy (3.21), (3.19) and

$$\left|\partial_t h(t,\lambda)\right| \leqslant b(\lambda)k_e(t)$$

are $k_e(t) = (c_j^{\beta_j}\sin(\beta_j\pi))^{-1}(t^{\beta_j-1})$ for all $j = 1,\ldots,n$ and $b(\lambda) = \lambda$, respectively. The proof follows the same steps for Eq. (2.19) in [20] and using the properties of $\mu(\beta)$. Then it follows from Remark 3.5 that (3.17) is the classical solution to (3.16).

4. Discussion

Here we describe some possible extensions and open problems. First we consider the problem of strong solutions to fractional diffusion equations with jumps. The generator L defined by

$$Lu(t,x) = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u(t,x)}{\partial x_i} + \int_{y\neq 0}\left(u(t,x-y) - u(t,x) + \frac{\sum_{i=1}^{d}\frac{\partial u(t,x)}{\partial x_i}y_i}{1+\sum_{i=1}^{d} y_i^2}\right)\phi(x,dy) \tag{4.1}$$

appears in the backward equation $\partial u/\partial t = Lu$ of a Markov process $X(t)$ [17,32]. The probability distribution of the Markov process $X(t)$ solves the forward equation $\partial v/\partial t = L^* v$, where L^* is the L^2 adjoint of the generator L. The integral term in (4.1) generates a jump diffusion (e.g., a stable process). For stable generators, the explicit connection with stochastic differential equations driven by a stable Lévy process was established by Zhang et al. [37] and Chakraborty [7]. In that case, the integral term in (4.1) can be written in terms of fractional derivatives in the space variable. Hahn et al. [15] studied the time-fractional equation $\partial^\beta u/\partial t^\beta = Lu$ in this case, as well as the distributed-order extension. They established a connection with stochastic differential equations driven by a time-changed Lévy process $X(E_t)$, so that their result includes jump diffusions on $\mathbb{R}^d$. It would be interesting to develop strong solutions to fractional and distributed-order jump diffusion equations on $\mathbb{R}^d$.

Note that the general results of [2] remain valid for bounded domains. Hence, the solution formula (3.5) still holds in the appropriate Banach space, and can be used to prove distributional solutions, e.g., in $L^2(\mathbb{R}^d)$. Eigenvalue expansions can be found explicitly in some special cases. The main technical difficulty is to obtain regularity of the eigenfunctions, or at least sharp bounds, for the generator (4.1) in the case of jump diffusions on bounded domains. See Chen et al. [11] for a recent study on this problem. One explicit example is to take $L = -(-\Delta)^{\alpha/2}$ for $0 < \alpha < 2$, the classical fractional power of the Laplacian [16], which generates a spherically symmetric stable Lévy process. This results from (4.1) with $a = b = 0$ and $\phi(x,dy) = C_{d,\alpha}\|y\|^{-\alpha-1}\,dy$, where $C_{d,\alpha}$ is a constant that depends on the stable index α and the dimension d of the space, see for example [23]. This is a type of fractional derivative in space, called the Riesz fractional derivative of order α [31]. Some results for this case are available in Chen and Song [9], Chen et al. [10] and Song and Vondraček [35]. The construction of strong solutions for general time-fractional jump diffusions on bounded domains remains a challenging open problem.

Meerschaert and Scheffler [27] discuss generalized diffusion equations of the form $\psi_W(\partial/\partial t)u(t,x) = Lu(t,x) + \delta(x)\psi_W(t,\infty)$, where $\psi_W(s)$ is the Laplace exponent of a non-decreasing Lévy process (subordinator) whose Lévy measure ϕ_W has infinite total mass, and L is the generator of another Lévy process. This is a distributed order time-fractional diffusion equation (2.12) in the special case when (2.9) holds. As in Section 2, the paper [27] shows that the density $u(t,x)$ of the CTRW scaling limit $A(E_t)$ solves the generalized diffusion equation, when E_t is the inverse of the subordinator W_t with $\mathbb{E}[e^{-sW_t}] = e^{-t\psi_W(s)}$. The problem of finding the strong solutions of generalized diffusion equations remains open even on $\mathbb{R}^d$.

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