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RESEARCH PAPER

FRACTIONAL WAVE EQUATIONS WITH ATTENUATION

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Abstract

Fractional wave equations with attenuation have been proposed by Caputo [5], Szabo [28], Chen and Holm [7], and Kelly et al. [11]. These equations capture the power-law attenuation with frequency observed in many experimental settings when sound waves travel through inhomogeneous media. In particular, these models are useful for medical ultrasound. This paper develops stochastic solutions and weak solutions to the power law wave equation of Kelly et al. [11].

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Key Words and Phrases: fractional derivative, wave equation, stable law, continuous time random walk, subordination, attenuation, dispersion

1. Introduction

In medical ultrasound, high frequency sound waves are transmitted through human tissue. The sound waves are attenuated with distance traveled through the inhomogeneous medium (human tissue), so that the wave amplitude at a distance r from the source falls off like $e^{-\alpha(\omega)r}$ where the attenuation coefficient $\alpha(\omega)$ follows a power law: $\alpha(\omega) = \alpha_0|\omega|^y$ depending on the frequency ω. Experimental evidence indicates that the power law exponent y lies in the interval $1 \le y \le 1.5$ for sound wave attenuation in human tissue, see for example Duck [8].

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Physicists and engineers have proposed several different models for sound wave propagation with power law attenuation. The wave equation of Szabo [28] modifies the traditional wave equation by adding a time-fractional derivative term of higher order. The power law wave equation proposed in Kelly et al. [11] modifies the Szabo wave equation by adding another time-fractional derivative term. The power law wave equation has an exact analytical solution in terms of stable probability densities, see Section **2** for a brief review.

The stable density is connected to a fractional diffusion equation in a continuous time random walk framework, see Meerschaert and Sikorskii [21]. Power law waiting times between particle jumps lead to a fractional time derivative, while power law jump lengths lead to a fractional derivative in space. A stochastic model for the power law wave equation has been developed, based on a continuous time random walk. The stochastic model explains the appearance of a stable density in the analytical solution to the power law wave equation of Kelly et al. [11] and provides a simple explanation for frequency dependent attenuation in terms of statistical physics. We review the basic ideas here, and then we proceed to develop explicit distributional solutions to the power law wave equation.

Fractional wave equations have a long history, see for example [1, 9, 12, 13, 23, 27]. The recent book of Mainardi [17] gives a nice review. Attenuated wave equations offer new challenges, and many interesting problems remain open. It is our hope that this paper will inspire further research in this important area.

2. Fractional wave equations

The traditional wave equation

$$\partial_t^2 p(\boldsymbol{x},t) = c_0^2 \Delta_{\boldsymbol{x}} p(\boldsymbol{x},t) \tag{2.1}$$

models sound wave propagation in an ideal conducting medium, where c_0 is the speed of sound in that medium, and $p(\boldsymbol{x},t)$ is the pressure. Szabo [28] proposed the following model (see also [6, 15]):

$$\Delta_{\boldsymbol{x}} p(\boldsymbol{x},t) = \frac{1}{c_0^2}\partial_t^2 p(\boldsymbol{x},t) + \frac{2\alpha_0}{c_0\cos(\pi y/2)}\partial_t^{y+1} p(\boldsymbol{x},t) \tag{2.2}$$

to account for attenuated wave conduction in a heterogenous medium. Here ∂_t^y is a (distributional) Riemann-Liouville fractional derivative in the time variable, so that $\partial_t^y f(t)$ has Fourier transform (FT) $(-i\omega)^y \hat{f}(\omega)$, where

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t)\,dt$$

is the FT of $f(t)$. Note that, in general, $f(t)$ here is a tempered distribution or generalized function, which can be associated with a pointwise defined function under suitable conditions. Equation (2.2) interpolates between the Blackstock equation [4] for sound wave propagation in a viscous fluid ($y = 2$) and the telegrapher's equation for wave propagation in conductive media ($y = 0$).

Kelly et al. [11] modify the Szabo wave equation, including an additional term:

$$\frac{1}{c_0^2}\partial_t^2 p + \frac{2\alpha_0}{c_0 b}\partial_t^{y+1} p + \frac{\alpha_0^2}{b^2}\partial_t^{2y} p = \Delta_{\boldsymbol{x}} p. \tag{2.3}$$

Here $p = p(\boldsymbol{x}, t)$ and $b = \cos(\pi y/2)$. For small $\alpha_0 > 0$, the additional term is negligible, so that (2.3) can be used to approximate (2.2). This is useful because equation (2.3) admits an exact analytic solution in terms of stable densities: Take FT in the time variable, and rewrite as a Helmholtz equation: $-k(\omega)^2 \hat{p} = \Delta_{\boldsymbol{x}} \hat{p}$ where $k(\omega) = c_0^{-1}\omega - b^{-1}\alpha_0(-i)^{y+1}\omega^y$. The Green's function solution in three dimensions is

$$\hat{p}(\boldsymbol{x}, \omega) = \frac{1}{4\pi r} e^{ik(\omega) r} = \frac{e^{i\omega r/c_0}}{4\pi r} \cdot e^{-\alpha_0 (r/b)(-i\omega)^y}, \tag{2.4}$$

where $r = \|\boldsymbol{x}\|$ is the radial distance from the source. The first term in (2.4) models wave motion, and the second term is the FT of a stable density $g_y(t, r)$ with index $0 < y \le 2$ [21]. Invert the FT in (2.4) to get a time convoluton $p(\boldsymbol{x}, t) = p_0(\boldsymbol{x}, t) * g_y(t, r)$, where $p_0(\boldsymbol{x}, t) = \delta(t - r/c_0)/(4\pi r)$ solves the lossless wave equation (2.1).

Numerical solutions are developed in [16].

3. Stochastic model

The continuous time random walk (CTRW) model in Meerschaert et al. [22] assumes that a random travel time

$$W_n = \Delta r^{1/y} Z_n + \frac{\Delta r}{c_0}$$

is required to traverse the nth spherical shell of thickness Δr, where Z_n has a Pareto distribution with

$$\mathrm{P}(Z_n > t) = \frac{\alpha_0 t^{-y}}{b\,\Gamma(1-y)} \tag{3.1}$$

for $t > 0$ sufficiently large (if $1 < y < 2$, substitute $Z_n - \mathrm{E}[Z_n]$ for Z_n). The time for a randomly selected packet of wave energy to reach the radial distance $r = n\Delta r$ from the source is

$$T_n := W_1 + \cdots + W_n \Rightarrow D(r) := D_0(r) + \frac{r}{c_0}$$

by [21, Theorem 3.37] as $\Delta r \to 0$, where $\Rightarrow$ denotes convergence in distribution. Here $D_0(r)$ is a stable Lévy motion with density function $g_y(t, r)$.

The number of spherical shells traversed by any given time $t > 0$ is $N_t = \max\{n \geq 0 : T_n \leq t\}$, and [3, Theorem 3.1] shows that N_t converges to E_t as $\Delta r \to 0$, where

$$E_t = \inf\{r : D(r) > t\}$$

is an *inverse stable subordinator*. The random variable $D(r)$ has probability density $p_1(t, r) = g_y(t - r/c_0, r)$, with Laplace transform $\tilde{p}_1(s, r) = e^{-r\psi(s)}$, where $\psi(s) = s/c_0 + (\alpha_0/b)s^y$. Then $\partial_r \tilde{p}_1 = -\psi(s)\tilde{p}_1$. Inverting the (distributional) Laplace transform yields $\partial_r p_1(t, r) = -\mathbb{D}_t p_1(t, r)$, where

$$\mathbb{D}_t = \psi(\partial_t) = \frac{1}{c_0}\partial_t + \frac{\alpha_0}{b}\partial_t^y \tag{3.2}$$

is a pseudo-differential operator (e.g., see Jacob [10]). Now suppose that $0 < y < 1$. Since $t = D(r)$ and $r = E_t$ are inverse processes, and $D(r)$ is strictly increasing, we have $\mathrm{P}(E_t \leq r) = \mathrm{P}(D(r) \geq t)$. Then the probability density $h(r, t)$ of E_t is given by

$$\begin{aligned} \mathbb{P}\{E_t \leq r\} &= \mathbb{P}\{D(r) \geq t\} \\ &= \mathbb{P}\{D_0(r) + r/c_0 \geq t\} \\ &= \int_{t-(r/c_0)}^{\infty} p_1(t, u)\, du \end{aligned} \tag{3.3}$$

for $t > 0$ and $r > 0$. Take Laplace transforms in (3.3) to see that

$$\tilde{h}(r, s) = -\frac{d}{dr}\left[s^{-1}e^{-r\psi(s)}\right] = s^{-1}\psi(s)e^{-r\psi(s)}$$

and then take Laplace transforms in the other variable to see that

$$\bar{h}(\lambda, s) = \int_0^{\infty} e^{-\lambda r}\tilde{h}(r, s)\, dr = \frac{s^{-1}\psi(s)}{\lambda + \psi(s)}\,.$$

Rewrite in the form

$$\psi(s)\bar{h}(\lambda, s) = -\lambda\bar{h}(\lambda, s) + s^{-1}\psi(s)$$

and invert the (distributional) Laplace transforms, using the fact that $t^{-y}/\Gamma(1 - y)$ has Laplace transform s^{y-1} by Example 2.9 in [21], to see that the densities $h(r, t)$ of the inverse process E_t solve

$$\mathbb{D}_t h(r, t) = -\partial_r h(r, t) + \delta(r)\left[\frac{\delta(t)}{c_0} + \frac{\alpha_0}{b}\frac{t^{-y}}{\Gamma(1 - y)}\right]. \tag{3.4}$$

See Kolokoltsov [14] for an alternative derivation.

Let $f_y(\cdot)$ be the standard stable density with index y, center 0, skewness 1 and scale 1 in the parameterization of Samorodnitzky and Taqqu [25].

Then $D_0(r)$ has pdf $p_1(t,r) = (\alpha_0 r)^{-1/y} f_y(t(\alpha_0 r)^{-1/y})$. Using the relation (3.3) and taking the derivative with respect to r, the density of E_t is:

$$h(r,t) = \left(\frac{t - r/c_0}{yr} + \frac{1}{c_0}\right)(\alpha_0 r)^{-1/y} f_y\left(\frac{t - r/c_0}{(\alpha_0 r)^{1/y}}\right) I(0 < r < c_0 t), \quad (3.5)$$

see also [20, Eq. (40)].

Now we will show that the function (3.5) solves the power law wave equation in one dimension. Take Laplace transforms in the remaining variable r to see that

$$\bar{h}(\lambda, s) = \frac{s^{-1}\psi(s)}{\psi(s)+\lambda} \cdot \frac{\psi(s)-\lambda}{\psi(s)-\lambda} = \frac{s^{-1}\psi(s)[\psi(s)-\lambda]}{\psi(s)^2 - \lambda^2},$$

a special case of Corollary 3.5 in [20]. Rearrange to get

$$\psi(s)^2 \bar{h}(\lambda, s) = \lambda^2 \bar{h}(\lambda, s) + \bar{R}(\lambda, s),$$

where $\bar{R}(\lambda, s) = s^{-1}\psi(s)[\psi(s) - \lambda]$. Invert both (distributional) Laplace transforms to see that

$$\mathbb{D}_t^2 h(r,t) = \partial_r^2 h(r,t) + R(r,t)$$

which is equivalent to the power law wave equation with initial/boundary term

$$\begin{aligned} R(r,t) = \frac{1}{c_0^2}\delta(r)\delta'(t) + \frac{2\alpha_0}{c_0 b}\delta(r)\delta^{(y)}(t) - \frac{1}{c_0}\delta'(r)\delta(t) \\ + \frac{\alpha_0^2}{b^2}\delta(r)\frac{t^{-2y}}{\Gamma(1-2y)} - \frac{\alpha_0}{b}\delta'(r)\frac{t^{-y}}{\Gamma(1-y)}, \end{aligned}$$

which we obtained by inverting the (distributional) Laplace-Laplace transform $\bar{R}(\lambda, s)$, where $\delta^{(y)}$ is the yth order fractional derivative of the Dirac delta function. We have shown that the probability density function (3.5) of the inverse subordinator E_t solves the one dimensional power law wave equation. This is a point source solution, since $E_0 = 0$, so that $h(r, 0) = \delta(r)$. The random variable E_t represents the (random) distance r traveled by time t, so $h(r,t)$ represents the pressure at time t. Thus the power law wave equation governs a CTRW limit with deterministic jumps and random waiting times. The fractional time derivatives come from the power law waiting times in the CTRW framework.

For the case $y > 1$, the arguments are a bit different, because now $D(r)$ can decrease, so that $\mathrm{P}(E_t \le r) \ne \mathrm{P}(D(r) \ge t)$. The governing equation for E_t in this case was derived by Baeumer et al. [2]. Using that formula, the argument that the hitting time density $h(r,t)$ solves the power law wave equation is similar, with slightly different boundary conditions to account for the possibility that $D(r)$ is decreasing. Three dimensional solutions are also available, obtained by replacing the time variable t in a solution

$p_0(\boldsymbol{x}, t)$ to the traditional wave equation by the random time E_t required for a packet of sound energy to reach the radial distance r from the source. See [22] for more details.

4. A convolution of two solutions in one dimension

Let $H(t) := I(t \geq 0)$ denote the Heaviside function, and $\mu_\alpha(t) := t^{\alpha-1}H(t)/\Gamma(\alpha)$. It is not hard to check that the functions $\{\mu_\alpha : \alpha > 0\}$ form a convolution family $\mu_\alpha * \mu_\beta = \mu_{\alpha+\beta}$ for all $\alpha, \beta > 0$, with $\mu_1(t) = H(t)$ and $\mu_2(t) = tH(t)$. The Riemann-Liouville fractional integral can then be defined as a convolution: $\mathbb{I}_t^\alpha f(t) = [f * \mu_\alpha](t)$. Since the Riemann-Liouville fractional derivative $\partial_t^y = \partial_t^1 \mathbb{I}_t^{1-y}$ for $0 < y < 1$, it follows from (3.4) that

$$(\mathbb{D}_t + \partial_r)h(r,t) = \delta(r)\mathbb{D}_t H(t) = \delta(r)\left[\frac{1}{c_0}\delta(t) + \frac{\alpha_0}{b}\mu_{1-y}(t)\right] \tag{4.1}$$

in the sense of distributions, see also Kolokoltsov [14, eq.(25)]. Then the function $\check{h}(r,t) = h(-r,t)$ solves

$$(\mathbb{D}_t - \partial_r)\check{h}(r,t) = \delta(r)\mathbb{D}_t H(t) = \delta(r)\left[\frac{1}{c_0}\delta(t) + \frac{\alpha_0}{b}\mu_{1-y}(t)\right], \tag{4.2}$$

and a quick application of distributional calculus shows that

$$\begin{aligned}(\mathbb{D}_t^2 - \partial_r^2)[h * \check{h}] &= [(\mathbb{D}_t - \partial_r)\check{h}] * [(\mathbb{D}_t + \partial_r)h] \\ &= [\delta(r)\mathbb{D}_t H(t)] * [\delta(r)\mathbb{D}_t H(t)] \\ &= \delta(r)\mathbb{D}_t^2(tH(t)) \\ &= \delta(r)\left\{\frac{1}{c_0^2}\delta(t) + \frac{2\alpha_0}{c_0 b}\mu_{1-y}(t) + \frac{\alpha_0^2}{b^2}\mu_{2-2y}(t)\right\},\end{aligned}$$

i.e., the function $h * \check{h}$ solves the power law wave equation in one dimension with source term $\delta(r)\mathbb{D}_t^2(tH(t))$.

5. Distributional solution

Write $\mathbf{D}(Y)$ for the space of test functions on an open subset $Y \subseteq \mathbb{R}^n$ and $\mathbf{D}'(Y)$ for the corresponding space of distributions.

LEMMA **5.1**. *Let $Q(x,t) \in \mathbf{D}'(\mathbb{R}^{d+1})$ solve the standard wave equation*

$$[\partial_t^2 - \Delta_x]Q(x,t) = f_2(x)\delta(t) + f_1(x)\delta'(t) \tag{5.1}$$

in $\mathbb{R}^{d+1}$ for $d = 1$ or $d = 3$ with initial displacement f_1 and initial velocity f_2, where $Q(x,t)$ is supported on $\mathbb{R}^d \times [0,\infty)$. Then $Q(x,t)$ has the following properties:

• *$Q(x,t)$ is a function in t and a distribution in x. More precisely, there exists a mapping $(0,\infty) \ni t \mapsto q(x,t) \in \mathbb{D}'_x(\mathbb{R}^d)$ such that $\int q(\phi,t)\psi(t)dt = \langle Q(x,t), \phi(x)\psi(t)\rangle$ for all $\phi(x) \in \mathbf{D}(\mathbb{R}^d)$ and $\psi(t) \in \mathbf{D}(\mathbb{R})$.*

• *For every $\phi(x) \in \mathbf{D}(\mathbb{R}^d)$ the mapping $t \mapsto q(\phi,t)$ is in $C^2((0,\infty))$ and bounded on $[0,\infty)$.*

P r o o f. If $d = 1$, we have

$$2q(x,t) = [H(x+t) - H(x-t)] * f_2(x) + [\delta(x+t) + \delta(x-t)] * f_1(x)$$

and for $d = 3$ we have

$$4\pi q(x,t) = t\int_{\|\omega\|=1} f_2(x+\omega t)d\omega + \partial_t\left(t\int_{\|\omega\|=1} f_1(x+\omega t)d\omega\right)$$

and for these q, the mapping $t \mapsto q(\phi,t)$ is smooth. □

Lemma **5.2**. *Let $Q(x,t)$ and $q(x,t)$ be as above. Then*

$$\frac{\partial^2}{\partial t^2} q(\phi,t) = q(\Delta_x\phi, t). \tag{5.2}$$

P r o o f. Localizing the distributions on both sides of (5.1) to the open set $\mathbb{R}^d \times (0,\infty)$, we have

$$[\Delta_x - \partial_t^2]Q = 0. \tag{5.3}$$

Now let $\phi(x) \in \mathbf{D}(\mathbb{R}^d)$ and $\psi(t) \in \mathbf{D}((0,\infty))$ be arbitrary. Then

$$0 = \langle Q, \psi(t)\Delta_x\phi(x) - \phi(x)\psi''(t)\rangle \tag{5.4}$$

and hence by definition of $q(x,t)$ we have

$$\int_0^\infty q(\Delta_x\phi,t)\psi(t)dt = \int_0^\infty q(\phi,t)\psi''(t)dt = \int_0^\infty \frac{\partial^2}{\partial t^2}q(\phi,t)\psi(t)dt \tag{5.5}$$

Since $\psi(t)$ is arbitrary and $t \mapsto q(\phi,t)$ is continuous, this shows the statement. □

Now define the mapping $(0,\infty) \ni t \mapsto p(x,t) \in \mathbf{D}'_x(\mathbb{R}^d)$ via

$$p(\phi,t) = \int_0^\infty q(\phi,\tau)h(\tau,t)d\tau, \qquad \phi \in \mathbf{D}(\mathbb{R}^d), \tag{5.6}$$

and define $P(x,t) \in \mathbf{D}'(\mathbb{R}^{d+1})$ via

$$P(\phi\otimes\psi) = \int p(\phi,t)\psi(t)dt, \qquad \phi \in \mathbf{D}(\mathbb{R}^d), \psi \in \mathbf{D}(\mathbb{R}). \tag{5.7}$$

Recall that tensor products of test functions are dense in the space of test functions on the product space.

THEOREM **5.1**. *The distribution $P(x,t)$ defined above satisfies the power-law wave equation*

$$[\mathbb{D}_t^2 - \Delta_x]P(x,t) = f_2(x)\mathbb{D}_t H(t) + f_1(x)\mathbb{D}_t^2 H(t). \tag{5.8}$$

P r o o f. On the domain $(\tau, t) \in (0,\infty)\times(0,\infty)$ the function $h(\tau,t)$ is smooth and satisfies the differential equation

$$\mathbb{D}_t h(\tau,t) = -\partial_\tau h(\tau,t) \tag{5.9}$$

in the sense of actual real functions (not only distributions), where ∂_τ denotes the pointwise derivative, and we use the pointwise Riemann-Liouville fractional derivative in (3.2). Multiply the above equation by $q(\phi,\tau)$ and integrate over $\tau \in (0,\infty)$. For the left-hand side, the temporal operator $\mathbb{D}_t$ can be carried outside the integral. For the right-hand side, $\tau \mapsto q(\phi,\tau)$ is bounded. Since $\tau \mapsto h(\tau,t)$ is a probability density, it vanishes at ∞. Hence we find via integration by parts

$$-\int_0^\infty q(\phi,\tau)\partial_\tau h(\tau,t)d\tau = \lim_{\tau\downarrow 0} q(\phi,\tau)h(\tau,t) + \int_0^\infty \partial_\tau q(\phi,\tau)h(\tau,t)d\tau.$$

It follows from Kolokoltsov [14, Theorem 4.1] that $h(0+,t) = \mathbb{D}_t H(t)$. Moreover, it follows from Stakgold [26, Section 8.2] that $q(\phi,0+) = f_1(\phi)$ and $\partial_t q(\phi,0+) = f_2(\phi)$. Then we have shown

$$\mathbb{D}_t \int_0^\infty q(\phi,\tau)h(\tau,t)d\tau = f_1(\phi)\mathbb{D}_t H(t) + \int_0^\infty \partial_\tau q(\phi,\tau)h(\tau,t)d\tau.$$

Another integration by parts yields

$$\begin{aligned}
&\mathbb{D}_t^2 \int_0^\infty q(\phi,\tau)h(\tau,t)d\tau \\
&= \mathbb{D}_t\left[f_1(\phi)\mathbb{D}_t H(t) + \int_0^\infty \partial_\tau q(\phi,\tau)h(\tau,t)d\tau\right] \\
&= f_1(\phi)\mathbb{D}_t^2 H(t) - \int_0^\infty \partial_\tau q(\phi,\tau)\partial_\tau h(\tau,t)d\tau \\
&= f_1(\phi)\mathbb{D}_t^2 H(t) + f_2(\phi)\mathbb{D}_t H(t) + \int_0^\infty \partial_\tau^2 q(\phi,\tau)h(\tau,t)d\tau,
\end{aligned}$$

for every $t > 0$ and every $\phi \in \mathbf{D}(\mathbb{R}^d)$, where $\partial_\tau^2 q(\phi,\tau) = q(\Delta_x\phi,\tau)$ by Lemma **5.2**. We now pass over to distibutions on $\mathbb{R}^{d+1}$: Extend both sides of the above equation by 0 for $t \le 0$, multiply them by $\psi(t) \in \mathbf{D}(\mathbb{R})$ and integrate over $\mathbb{R}$ to get

$$\begin{aligned}\langle \mathbb{D}_t^2 P(x,t), \psi(t)\phi(x)\rangle = &\langle f_1(x)\mathbb{D}_t^2 H(t), \phi(x)\psi(t)\rangle \\ &+ \langle f_2(x)\mathbb{D}_t H(t), \phi(x)\psi(t)\rangle \\ &+ \langle P(x,t), \psi(t)\Delta_x\phi(x)\rangle\end{aligned}$$

and the theorem follows easily. □

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