

Preface

This book is based on a series of lecture notes for a graduate course in the Department of Statistics and Probability at Michigan State University. The goal is to prepare graduate students for research in the area of fractional calculus, anomalous diffusion, and heavy tails. The book covers basic limit theorems for random variables and random vectors with heavy tails. This includes regular variation, triangular arrays, infinitely divisible laws, random walks, and stochastic process convergence in the Skorokhod topology. The basic ideas of fractional calculus and anomalous diffusion are introduced in the context of probability theory. Each section of the book provides material roughly equivalent to one lecture. Most sections conclude with some additional details, intended for individual reading, and to make the book relatively self-contained.

Heavy tails are applied in finance, insurance, physics, geophysics, cell biology, ecology, medicine, and computer engineering. A random variable has heavy tails if $P(|X| > x) \approx Cx^{-\alpha}$ for some $\alpha > 0$. If $\alpha < 2$, then the second moment of X is undefined, so the usual central limit theorem does not apply. A heavy-tailed version of the central limit theorem leads to a stable distribution. Random walks with heavy tails converge to a stable Lévy motion, similar to Brownian motion. The densities of Brownian motion solve the diffusion equation, which provides a powerful link between differential equations and stochastic processes. Densities of a stable Lévy motion solve a fractional diffusion equation like

$$\frac{\partial}{\partial t}p(x, t) = c \frac{\partial^\alpha}{\partial x^\alpha}p(x, t)$$

using a fractional derivative of order α . Fractional derivatives are limits of fractional difference quotients, using the same fractional difference operator that appears in time series models of long range dependence.

Vector models with heavy tails are useful in many applications, since physical space is not one dimensional. A heavy tailed version of the central limit theorem for random vectors leads to an operator stable limit distribution, which can have a combination of normal and stable components. Vector random walks with heavy tails converge to an operator stable Lévy motion. Probability densities of this random walk limit solve a vector fractional diffusion equation like

$$\frac{\partial}{\partial t}p(x, y, t) = c_1 \frac{\partial^\alpha}{\partial x^\alpha}p(x, y, t) + c_2 \frac{\partial^\beta}{\partial y^\beta}p(x, y, t)$$

with a different fractional derivative in each coordinate.

Many interesting research problems in this area remain open. This book will guide the motivated reader to gain the essential background, insights, and techniques needed to read and understand current research papers and begin making their own contributions to this rapidly growing field.

In this second edition of the text, we have corrected all known errors and added three sections on applications: The fractional Poisson process; distributed order frac-

tional derivatives; and the correlation structure of fractional processes. We would like to thank all the people who wrote to us about the first edition with helpful ideas and corrections. It is our hope that the second edition continues to be a useful contribution to the literature on this interesting area of theoretical and applied research.

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