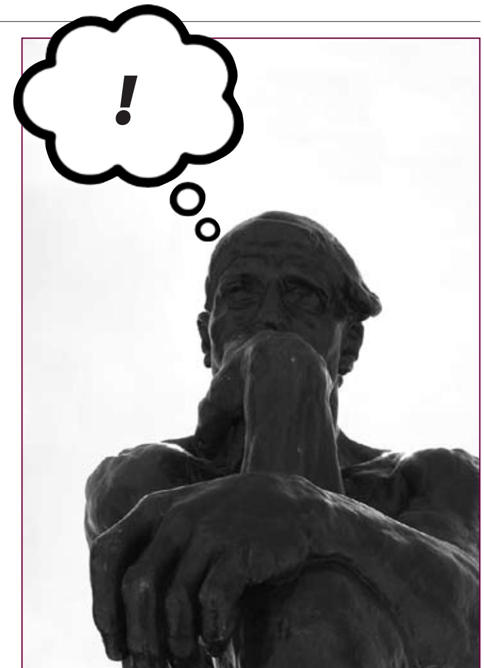
Members' Discovery: Fractional Cauchy Problems

Fractional derivatives were invented by Leibniz in 1695, as a natural extension of integer-order derivatives. They are limits of fractional difference quotients, using the fractional difference operator common in time series analysis. This paper develops stochastic methods for solving fractional Cauchy problems on bounded domains. A fractional Cauchy problem replaces the first order time derivative in the usual Cauchy problem by a fractional derivative of order less than one. The simplest Cauchy problem is the diffusion equation. Its point source solution is a family of mean zero Gaussian densities that govern the underlying Brownian motion. This connection between stochastic process and partial differential equation is an important theoretical technique, and a useful toolbox in practical applications.

Cauchy problems are abstract partial differential equations that govern the transition densities of Markov processes. They are useful for modeling complex particle motions, including the dispersion of pollutants in underground aquifers. The underlying Markov process forms the basis for particle tracking algorithms, where a Cauchy problem solution is approximated by a histogram of independent particle traces. This paper identifies the stochastic process that underlies a fractional Cauchy problem as the Markov process for the original Cauchy problem, subordinated to an inverse stable, and killed when it exits the domain. This facilitates particle tracking solutions for a broad class of fractional evolution equations on bounded domains. The proof involves eigenfunction expansions to reduce the fractional Cauchy problem to an infinite system of linear time-fractional ordinary differential equations in the eigenfunction coordinates. These are solved using Mittag-Leffler functions, the fractional analogue of the exponential. This is related to a result of Bingham, that stable subordinator hitting times have Mittag-Leffler distributions.

Iterated Brownian motion is another interesting class of processes, in which the time index of a Brownian motion is replaced by the absolute value of another independent Brownian motion. The densities of iterated Brownian motion solve a Cauchy problem that involves a fourth order derivative in space. Iterated Brownian motion has the same one dimensional distributions as Brownian motion subordinated to the inverse of an independent stable subordinator with index one half. Solutions to the one half order fractional diffusion equation also solve the fourth order Cauchy problem behind iterated Brownian motion. The paper identifies the boundary conditions that make the two differential equations equivalent on a bounded domain.

This line of research began by accident. A graduate student in Hydrology named David Benson approached Meerschaert in 1996, asking how to invert a certain Laplace transform. Meerschaert recognized the transform of a stable density, and Benson explained the connection to fractional diffusion. At this time, Benson knew nothing about heavy tails, infinite variance, or stable limits. Meerschaert was equally ignorant of fractional derivatives, partial differential equations, and ground water hydrology. Their first joint paper in 1999, in a physics journal, explained the fractional diffusion equation behind multivariable stable laws, and the connection to vector random walks with heavy tailed jumps. Applications since then have included models of ground water pollution, contaminant transport in rivers, and tick-by-tick financial data.



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We welcome IMS members' short news items regarding their discoveries, through publication or invention. The *Bulletin* will publish selected items to share the discoveries. These items can be as short as just one or two paragraphs, and are subject to editing.

These results are published in the paper "Fractional Cauchy Problems on Bounded Domains" by Mark M. Meerschaert, Erkan Nane and Palaniappan Vellaisamy, in a forthcoming issue of the *Annals of Probability*: see http://www.e-publications.org/ims/submission/index.php/AOP/user/submissionFile/2786?confirm=2f04f086. Communicated by the Editor, with contributions from Mark Meerschaert (Michigan State University).