Chapter for Handbook of Fractional Calculus with Applications

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Inverse subordinators and time fractional equations

DOI ..., Received ...; accepted ...

Abstract: The inverse stable subordinator is the first passage time of a standard stable subordinator with index $0 < \beta < 1$. The probability density of the inverse stable subordinator can be used to solve time-fractional Cauchy problems, where the usual first derivative in time is replaced by a Caputo fractional derivative of order β. If the Cauchy problem governs a Markov process, then the fractional Cauchy problem governs a time-changed process, where the time parameter is replaced by the inverse stable subordinator. Applications include delayed Brownian motion, and the fractional Poisson process.

Keywords: Inverse subordinator, Cauchy problem, delayed Brownian motion, fractional Poisson process

2010 Mathematics Subject Classification: Primary 60K05; 33E12; 26A33.

1 Introduction

Zaslavsky [52] introduced the time-fractional differential equation

$$\partial_t^\beta m(x, t) = D\partial_x^2 m(x, t) \tag{1}$$

with $0 < \beta < 1$ as a model for Hamiltonian chaos, see also Nigmatullin [40]. Zaslavsky called the stochastic process governed by this equation a “fractal Brownian motion.” Meerschaert and Scheffler [33] showed that equation (1) governs a time-changed Brownian motion $B(E_t)$ where $B(t)$ is a Brownian motion, and E_t is an independent inverse stable subordinator of index β. A typical plot of this “de-

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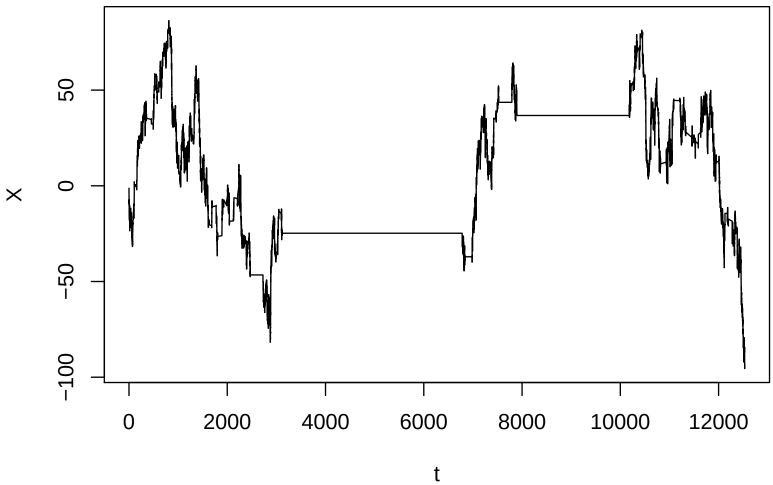

Fig. 1: Typical sample path of delayed Brownian motion $X_t = B(E_t)$ with $D = 0.5$ and $\beta = 0.8$, from [31].

layed Brownian motion" is shown in Figure 1. The effect of the time change is to introduce delays in the particle motion.

Equation (1) is an example of a *fractional Cauchy problem*, where the usual first derivative in time is replaced by a Caputo fractional derivative

$$\partial_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(s)\,ds}{(t-s)^\beta} \tag{2}$$

for some $0 < \beta < 1$. Baeumer and Meerschaert [5] showed that the solution to a fractional Cauchy problem can be written in terms of the probability density function (pdf) of the inverse stable subordinator E_t. First note that

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/(4Dt)}$$

solves the traditional diffusion equation

$$\partial_t p(x,t) = D\partial_x^2 p(x,t), \tag{3}$$

a special case of (1) with $\beta = 1$. Then [5, Theorem 3.1] shows that the solution to the time-fractional diffusion equation (1) is given by

$$m(x,t) = \int_0^\infty p(x,u)h(u,t)\,du, \tag{4}$$

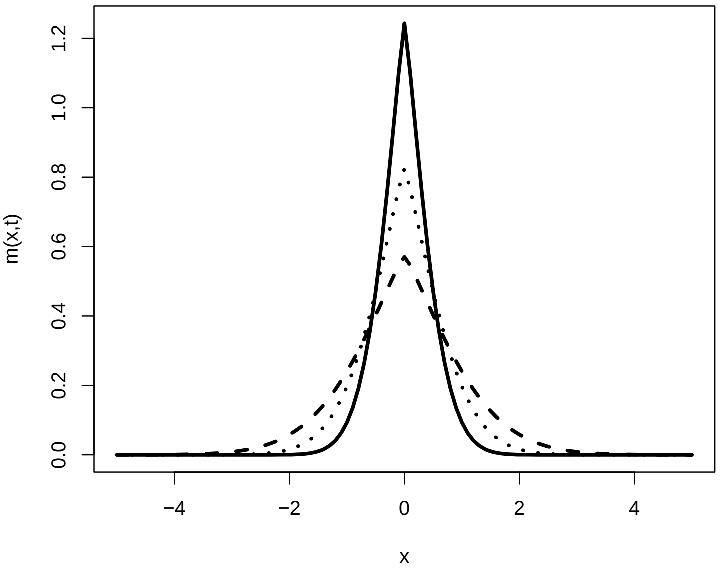

Fig. 2: Solution (4) to the time-fractional diffusion equation (1) at times $t = 0.1$ (solid line), $t = 0.3$ (dotted line), and $t = 0.8$ (dashed line) with $\beta = 0.75$ and $D = 1.0$.

where $h(u, t)$ is the pdf of the inverse stable subordinator E_t. Since the Brownian motion $B(u)$ has pdf $p(x, u)$, and the independent inverse stable subordinator $u = E_t$ has pdf $h(u, t)$, a simple conditioning argument shows that (4) is also the pdf of the time-changed process $x = B(E_t)$. Figure 2 plots the solution to (1) to show the behavior over time. Note the sharper peak and heavier tails, compared to a bell-shaped normal pdf. Since the solutions spread at the rate $t^{\beta/2}$, slower than the usual $t^{1/2}$ spreading for a classical diffusion, (1) models *subdiffusion*.

2 The inverse stable subordinator

A *subordinator* D_t is a nondecreasing Lévy process. A *Lévy process* is a stochastic process with stationary, independent increments [4, 45]. The distribution of D_t is strictly *stable* if $D_{ct} \simeq c^{1/\beta} D_t$ (same distribution) for some $0 < \beta < 1$. The pdf $g(x, t)$ of the stable subordinator D_t cannot generally be written in closed form, but computer codes to compute $g(x,t)$ are widely available [36, Chapter 5]. Due to the distributional scaling relation, one can also write $g(x,t) = t^{-1/\beta} g(xt^{-1/\beta}, 1)$. The *inverse* stable subordinator

$$E_t = \inf\{u : D(u) > t\} \tag{5}$$

is the first passage time of the stable subordinator above the level $t \geq 0$. Properties of the inverse stable subordinator are detailed in [33, Section 3] and Meerschaert and Straka [37]. An explicit formula for the moments of E_t was given by Piryatinska, Saichev and Woyczynski [41]. It follows from the definition (5) that

$$\mathbb{P}[E_t \leq u] = \mathbb{P}[D_u \geq t] = \int_t^\infty g(w,u)\,dw, \tag{6}$$

hence the inverse stable subordinator has pdf

$$h(u,t) = \frac{d}{du}\mathbb{P}[E_t \leq u] = \frac{d}{du}\left[1 - \int_0^t g(w,u)\,dw\right]. \tag{7}$$

Now a simple calculation [33, Corollary 3.1] shows that the pdf of the inverse stable subordinator E_t is

$$h(x,t) = \frac{t}{\beta} x^{-1-1/\beta} g(tx^{-1/\beta}, 1) \tag{8}$$

for all $x > 0$ and $t > 0$. This formula together with (4) was used to plot Figure 2. Figure 3 plots a typical inverse stable density $h(x,t)$. The density is supported on the positive half-line, and is discontinuous at the origin. Using asymptotic properties of stable densities, it is not hard to show (e.g., see [37, Section 4]) that

$$h(0+,t) = \lim_{x \downarrow 0} h(x,t) = \frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{9}$$

for all $t > 0$.

The Laplace transform (LT) of $g(x,t)$ is (e.g., see [36, Proposition 3.10 and p. 114])

$$\tilde{g}(s,t) = \int_0^\infty e^{-sx} g(x,t)\,dx = e^{-tcs^\beta} \tag{10}$$

and we assume $c = 1$ to get the *standard* stable subordinator. Using the fact that integration corresponds to multiplication of the Laplace transform by s^{-1}, it follows from (7) that the (standard) inverse stable subordinator pdf has Laplace transform

$$\begin{aligned} \tilde{h}(u,s) &= -\frac{d}{du}\left[s^{-1}\tilde{g}(s,u)\right] \\ &= -\frac{d}{du}\left[s^{-1}e^{-us^\beta}\right] = s^{\beta-1}e^{-us^\beta} \end{aligned} \tag{11}$$

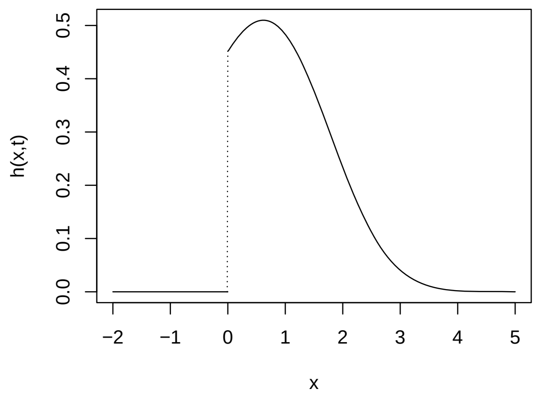

Fig. 3: Inverse stable density $h(x, t)$ with $\beta = 0.6$ and $t = 1$, from [37].

for all $s > 0$.

The processes $t = D_u$ and $u = E_t$ are inverses. The graph of the inverse stable subordinator $u = E_t$ is just the graph of the stable subordinator $t = D_u$ with the axes swapped, i.e., a reflection through the diagonal line $t = u$. See Figures 4 and 5 for an illustration. Here the stable subordinator and its inverse were simulated using freely available codes [36, Chapter 5]. The stable subordinator is a strictly increasing pure jump process. Therefore, the inverse stable subordinator is continuous, and its graph has flat periods (resting times) that correspond to the jumps in the stable subordinator. The lengths of those resting periods follow a power law distribution, since they are the same as the jump distribution of the stable subordinator: Jumps larger than any given cutoff $\varepsilon > 0$ follow a Pareto distribution, where the probability of a jump length exceeding $x > \varepsilon$ is proportional to $x^{-\beta}$, e.g., see [36, Section 3.4]. Since the resting periods of the inverse stable subordinator have the same distribution, they are not exponentially distributed, and hence E_t is not a Markov process.

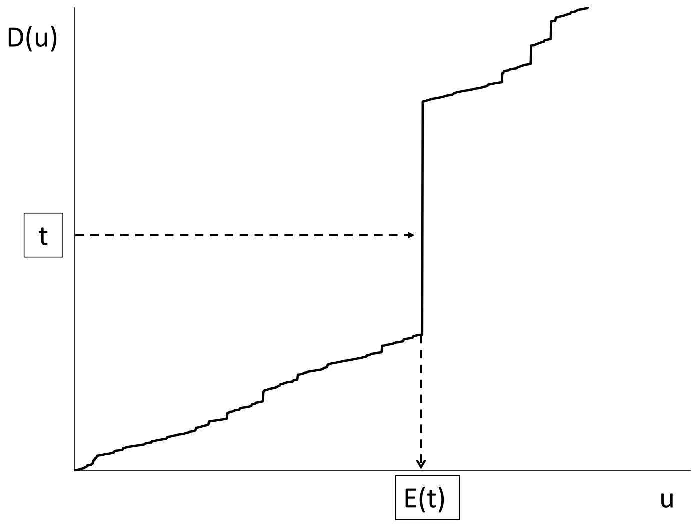

Fig. 4: A typical sample path of the stable subordinator $t = D_u$ with index $\beta = 0.8$.

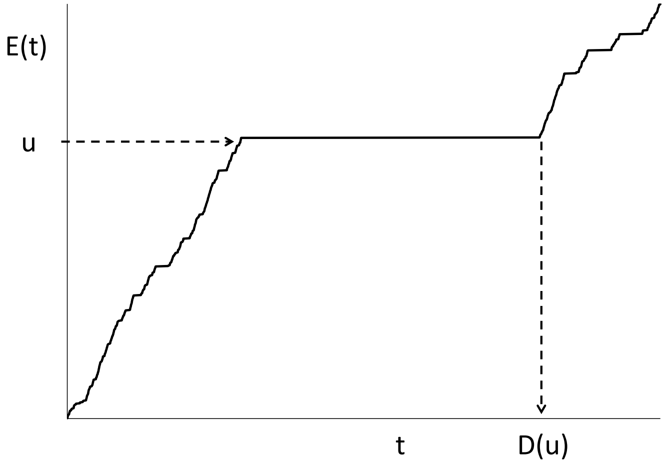

Fig. 5: The inverse stable subordinator $u = E_t$ with index $\beta = 0.8$, using the same sample path as in Figure 4. The graphs are the same, with the axes swapped.

3 Fractional Cauchy problems

The fractional diffusion equation (1) is an example of a *fractional Cauchy problem*. A *Cauchy problem* is an abstract differential equation of the form

$$\partial_t p(x,t) = L_x p(x,t); \quad p(x,0) = p_0(x) \tag{12}$$

where L_x is some spatial operator. A *Banach space* is a Cauchy complete, normed vector space. A familiar example is $L^1(\mathbb{R})$, the space of real-valued functions of one real variable, with the norm $\|f\|_1 = \int |f(x)|\,dx$. A *semigroup* is a family of linear operators $\{T_t\}$ on that space, with the property that T_0 is the identity operator, and $T_{t+s} = T_t T_s$. A C_0 semigroup is bounded and continuous in the Banach space norm. Then the generator

$$L_x f(x) = \lim_{t \to 0} \frac{T_t f(x) - T_0 f(x)}{t - 0}, \tag{13}$$

where the limit is taken in the Banach space norm, is defined on a dense subset of that space. The generator can contain ordinary derivatives as in (1), fractional derivatives in space, variable coefficients, and boundary conditions.

The fractional Cauchy problem

$$\partial_t^\beta m(x,t) = L_x m(x,t); \quad m(x,0) = p_0(x) \tag{14}$$

uses a Caputo fractional derivative (2) of order $0 < \beta < 1$. The mathematical study of fractional Cauchy problems was initiated by Kochubei [20, 21] and Schneider and Wyss [46]. Later Baeumer and Meerschaert [5, Theorem 3.1] showed that if $p(x,t)$ solves the Cauchy problem (12), then (4) solves the corresponding fractional Cauchy problem, where the function $h(x,t)$ is given by the formula (8). A few years after that, Meerschaert and Scheffler [33, Corollary 3.1] identified this function as the pdf of the inverse β-stable subordinator. Hence if L_x is the generator of some Markov process $B(t)$, it follows that the fractional Cauchy problem (14) governs the non-Markovian process $B(E_t)$. As illustrated in Figure 1, the time-fractional derivative models long resting times between motions of the original Markov process.

In [26, Theorem 3.6] this idea is used to show that, under some mild technical conditions, the fractional Cauchy problem (14) with

$$L_x f = \sum_{i,j=1}^{d} \frac{\partial \left(a_{ij}(x)(\partial f/\partial x_i)\right)}{\partial x_j} \tag{15}$$

and $a_{ij}(x) = a_{ji}(x)$ on a bounded domain with zero Dirichlet boundary conditions has a unique solution

$$m(x,t) = \sum_{n=1}^{\infty} \bar{f}(n) E_\beta(-\lambda_n t^\beta) \psi_n(x) \tag{16}$$

where $\{\psi_n\}$ is a complete orthonormal basis of eigenfunctions for L_x with $L_x \psi_n = \lambda_n \psi_n$, $\bar{f}(n) = \int_D f(x)\psi_n(x)\,dx$, and the Mittag-Leffler function

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)} \tag{17}$$

for any complex z. For $\beta = 1$, $E_\beta(-\lambda_n t^\beta) = e^{-\lambda_n t}$ and we recover the well-known solution to the Cauchy problem (12) obtained using separation of variables. In short, equation (16) comes from (4), using the fact that $E_\beta(-\lambda t^\beta) = \int_0^\infty e^{-\lambda u} h(u,t)\,du$. Chen et al. [16, Theorem 5.1] show that the same formula (16) yields pointwise solutions to the space-time fractional Cauchy problem (14) on a bounded domain with $L_x = -(-\Delta)^\alpha$, the fractional Laplacian. In both cases, the Cauchy problem (12) governs the probability densities of a killed Markov process $B(t)$, and the fractional Cauchy problem (14) governs the time-changed process $B(E_t)$. Note that fractional Cauchy problems have exactly the same boundary conditions as the original Cauchy problem, since these boundary conditions are part of the specification of the generator and the Banach space.

A very special case of the fractional Cauchy problem gives the governing equation of the pdf $h(x,t)$ of the inverse stable subordinator itself. Take $L_x = -\partial_x$, the generator of the *shift semigroup* $T_t f(x) = f(x-t)$ corresponding to the non-random process $B(t) = t$ (e.g., see [36, Example 3.21]). Then the pdf of $B(E_t) = E_t$ solves the fractional Cauchy problem

$$\partial_t^\beta h(x,t) = -\partial_x h(x,t) \tag{18}$$

with the point source initial condition $h(x,0) = \delta(x)$ written in terms of the Dirac delta function, reflecting the fact that $E_0 = 0$ with probability one.

Meerschaert and Straka [37] review several equivalent forms of the governing equation (14). One form uses the Riemann-Liouville fractional derivative

$$\mathbb{D}_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{f(s)\,ds}{(t-s)^\beta}, \tag{19}$$

which differs from the Caputo form in that the first derivative is placed outside the integral. Since integration and differentiation do not commute in general, these two forms are not equal. In fact, we have

$$\partial_t^\beta f(t) = \mathbb{D}_t^\beta f(t) - f(0)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{20}$$

when $0 < \beta < 1$, e.g., see [36, Eq. (2.33)]. Then we can also write the fractional Cauchy problem (14) in the form

$$\mathbb{D}_t^\beta m(x,t) = L_x m(x,t) + p_0(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{21}$$

used in the original work of [5, 33, 52]. Then the pdf $h(x,t)$ of the inverse stable subordinator also solves the fractional equation

$$\mathbb{D}_t^\beta h(x,t) = -\partial_x h(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{22}$$

for $x > 0$ and $t > 0$.

4 The fractional Poisson process

One nice application of the inverse stable subordinator is to define a *fractional Poisson process* $N(E_t)$, where $N(t)$ is the traditional Poisson process [44]. The probability mass function (pmf) of the traditional Poisson process

$$p(n,t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}; \quad n = 0, 1, 2, 3, \ldots \tag{23}$$

gives the probability that $N(t) = n$. It solves the Cauchy problem

$$\partial_t p(n,t) = -\lambda p(n,t) + \lambda p(n-1,t) \tag{24}$$

which says that particles transition from state $n-1$ to state n at rate $\lambda > 0$. Then the fractional Cauchy problem

$$\partial_t^\beta m(n,t) = -\lambda m(n,t) + \lambda m(n-1,t) \tag{25}$$

governs the pmf $m(n,t)$ of the fractional Poisson process. Now the pmf $m(n,t)$ of the fractional Poisson process is given by the formula (4) where $x = n$ is a nonnegative integer, $p(n,u)$ is given by (23), and $h(u,t)$ is the pdf (8) of the inverse stable subordinator. Laskin [24] defines the fractional Poisson process as the counting process whose pmf solves (25). Repin and Saichev [42] define the fractional Poisson process as the counting process with Mittag-Leffler waiting times between state transitions, so that the probability of waiting longer than some time $t > 0$ before the next jump equals $E_\beta(-\lambda t^\beta)$. It was shown in [27] that all these definitions are equivalent. The fractional Poisson process differs from the traditional Poisson process in that very long resting times occur more often.

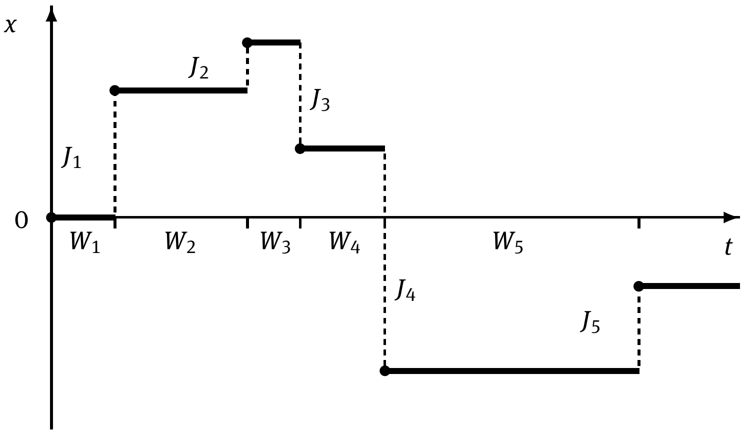

Fig. 6: A continuous time random walk, where random jumps J_n are separated by random waiting times W_n, explains the physical meaning of the time-fractional diffusion equation (1).

5 Continuous time random walks

The continuous time random walk (CTRW) is a model in statistical physics that explains the meaning of the time-fractional derivative. Start with a random walk $S(n) = J_1 + \cdots + J_n$ where the particle jumps $J_1, J_2, J_3, \ldots$ are independent and identically distributed (iid), with mean zero and finite variance $\sigma^2 > 0$. Then the central limit theorem implies that $n^{-1/2} S([nt]) \Rightarrow B(t)$ in distribution, where $B(t)$ is normal with mean zero and variance $\sigma^2 t$ [36, Section 1.1]. Now assume that the jumps J_n are separated by an iid sequence of random waiting times W_n, independent of the jumps. Then $T_n = W_1 + \cdots + W_n$ is the time of the nth jump, $N_t = \max\{n \geq 0 : T_n \leq t\}$ is the number of jumps by time $t \geq 0$, and $S(N_t)$ is the particle position at time $t \geq 0$. Figure 6 illustrates the model.

Now suppose that the random waiting times have a heavy power law tail: We assume that $\mathbb{P}[W_n > t] = Ct^{-\beta}$ where $C = 1/\Gamma(1-\beta)$. Then the *extended* central limit theorem [36, Theorem 3.41] implies that $n^{-1/\beta} T_{[nt]} \Rightarrow D_t$, a standard β-stable subordinator. The random walk T_n of waiting times and the renewal process N_t are inverse processes:

$$\{N_t \geq n\} = \{T_n \leq t\}. \tag{26}$$

Then [33, Theorem 3.2] shows that these inverse processes have inverse limits: $n^{-\beta} N_{nt} \Rightarrow E_t$, the inverse stable subordinator (5). Next an application of the continuous mapping theorem [36, Section 4.4] yields that

$$n^{-\beta/2} S(N_{nt}) = (n^\beta)^{-1/2} S(n^\beta \, n^{-\beta} N_{nt}) \approx (n^\beta)^{-1/2} S(n^\beta \, E_t) \Rightarrow B(E_t),$$

a delayed Brownian motion whose pdf solves the time-fractional diffusion equation (1). Hence the time-fractional derivative of order $0 < \beta < 1$ models long waiting times distributed according to a power law with the same index β.

Many other fractional diffusion models can be investigated using the CTRW model. Suppose for example that $X_n \geq 0$ are iid with $\mathbb{P}[X_n > x] = Cx^{-\alpha}$ for some $C > 0$ and some $1 < \alpha < 2$. Then the mean $\mu = \mathbb{E}[X_n]$ exists, and we can take $J_n = X_n - \mu$ as the particle jumps. Now $n^{-1/\alpha} S([nt]) \Rightarrow A(t)$, an α-stable Lévy process with pdf $p(x, t)$ [36, Theorem 3.41]. The governing equation of this pdf solves (12) with $p(x, 0) = \delta(x)$, the Dirac delta function, $L_x = D\partial_x^\alpha$ using a space-fractional derivative, and $D = C\Gamma(2 - \alpha)/(\alpha - 1)$ [36, p. 84]. The long-time limit of the CTRW is derived as before:

$$n^{-\beta/\alpha} S(N_{nt}) = (n^\beta)^{-1/\alpha} S(n^\beta \, n^{-\beta} N_{nt}) \approx (n^\beta)^{-1/\alpha} S(n^\beta \, E_t) \Rightarrow A(E_t).$$

The CTRW limit $A(E_t)$ has a pdf $m(x, t)$ that solves the fractional Cauchy problem (14) with the same generator $L_x = D\partial_x^\alpha$ and the same initial condition. Furthermore, the exact form of the CTRW limit pdf is given by (4) where $h(u, t)$ is the pdf (8) of the inverse stable subordinator. If

$$\mathbb{P}[X_n > x] = pCx^{-\alpha} \quad \text{and} \quad \mathbb{P}[X_n < -x] = qCx^{-\alpha}$$

we get a two-sided α-stable Lévy process with generator $L_x = pD\partial_x^\alpha + qD\partial_{-x}^\alpha$ that also involves a negative fractional derivative. If the particle jumps J_n are random vectors, the generator involves vector fractional derivatives. For example, if $J_n = R_n \Theta_n$ where $\mathbb{P}[R_n > r] = Cr^{-\alpha}$ are iid and Θ_n are iid uniformly distributed random unit vectors, independent of R_n, then $n^{-1/\alpha} S([nt]) \Rightarrow A(t)$, a spherically symmetric α-stable Lévy motion with generator $L_x = D\Delta^\alpha$, using the fractional Laplacian [36, Example 6.24]. The exact form of the constant D is given in [36, Example 6.24]. The pdf $m(x, t)$ of the time-changed process $A(E_t)$ is given by (4), where $p(x, u)$ is the pdf of the vector stable process $x = A(u)$, and $h(u, t)$ is the inverse stable subordinator (8). The pdf $m(x, t)$ solves the space-time fractional diffusion equation (14) with the generator $L_x = D\Delta^\alpha$ and a delta function initial condition. If the particle jumps are correlated, then one can obtain a delayed *fractional* Brownian motion $B_H(E_t)$ in the limit: $n^{-H} S([nt]) \Rightarrow B_H(t)$ and $n^{-\beta H} S(N_{nt}) \Rightarrow B_H(E_t)$, see [30, Theorem 2.4].

6 Fractal properties

Blumenthal and Getoor [8] showed that the range of a stable subordinator is a random fractal with dimension β. Hence the same number β describes the power law waiting times, the order of the fractional derivative, and the fractal dimension of the stable subordinator. Fractal properties of the inverse stable subordinator, and time-changed processes $B(E_t)$, are derived in [31]. The graph of a Brownian motion $B(t)$ is a random fractal with dimension $3/2$. Meerschaert, Nane and Xiao [31, Proposition 2.3] show that the graph of the delayed Brownian motion $B(E_t)$ is a random fractal with dimension $1 + \beta/2$, which reduces to $3/2$ in the limit case $\beta = 1$. The graph of a scalar-valued α-stable Lévy process $A(t)$ with index $1 < \alpha \leq 2$ is a random fractal with dimension $2 - 1/\alpha$. This reduces to $3/2$ in the case $\alpha = 2$, since an α-stable Lévy process with $\alpha = 2$ is a Brownian motion. The graph of the CTRW limit $A(E_t)$ is a random fractal with dimension $1 + \beta(1 - 1/\alpha)$ [31, Proposition 2.3], which reduces to $2 - 1/\alpha$ in the limit case $\beta = 1$. The graph of a fractional Brownian motion with Hurst index $0 < H < 1$ is a random fractal with dimension $2 - H$, which reduces to $3/2$ in the special case $H = 1/2$ of a Brownian motion. The graph of a delayed fractional Brownian motion $B_H(E_t)$ is a random fractal with dimension $\beta + 1 - H\beta$, which reduces to $2 - H$ in the limit case $\beta = 1$. Hence, even though the sample paths of the inverse stable subordinator E_t are continuous and nondecreasing, they are sufficiently irregular as to influence the fractal dimension of a time-changed process.

7 Higher order equations

In many cases, fractional partial differential equations can be written in equivalent higher order forms, some of which do not involve any fractional derivatives. The inverse stable subordinator explains the equivalence between these equations.

One interesting example involves Brownian subordinators. Given a Brownian motion $B(t)$, let B'_t be another independent Brownian motion. Allouba and Zheng [1, 2] consider the time-changed process $X_t = B(|B'_t|)$, which they call “Brownian time Brownian motion.” Burdzy [9] considers a closely related process called “iterated Brownian motion” where $B(t)$ is a two-sided Brownian motion on $-\infty < t < \infty$ and $Y_t = B(B'_t)$. Both processes have the same pdf, and hence the same governing equation

$$\partial_t m(x, t) = \frac{\partial_x^2 p_0(x)}{\sqrt{\pi t}} + \partial_x^4 m(x, t); \quad m(x, 0) = p_0(x) \tag{27}$$

for $t > 0$ and x real [2]. They also consider a vector equation, but we will focus here on the scalar case. Note that (27) is *not* a Cauchy problem, due to the presence of the first term on the right-hand side, which depends on $t > 0$. Baeumer, Meerschaert and Nane [7, Corollary 3.2] show that in fact equation (27) is equivalent to the time-fractional diffusion equation (1) with $\beta = 1/2$. One way to see this is to apply $\partial_t^{1/2}$ to both sides of (1), or to the equivalent form (21) with $L_x = -\partial_x^2$. Another approach is to note that the absolute value $|B'_t|$ has the same pdf as the maximum $M_t = \max\{B'_u : 0 \le u \le t\}$ by the reflection principle. But since the stable subordinator D_t with index $\beta = 1/2$ is the first passage time of the Brownian motion B'_t, and since the maximum process is the inverse of the first passage time, $|B'_t|$ has the same pdf as the inverse stable subordinator E_t. Since (1) governs the delayed Brownian motion $B(E_t)$ and (27) governs the Brownian time Brownian motion $B(|B'_t|)$, and since both processes have the same pdf, the governing equations must be equivalent.

More generally, Allouba and Zheng [2] show that if $B(t)$ is a Markov process in one or more dimensions with generator L_x, then the pdf of $B(E_t)$ with $\beta = 1/2$ also solves the higher order equation

$$\partial_t m(x,t) = \frac{L_x p_0(x)}{\sqrt{\pi t}} + L_x{}^2 m(x,t); \quad m(x,0) = p_0(x). \tag{28}$$

Then the pdf of the Brownian time process $B(|B'_t|)$ solves the same equation. The higher order equation (28) is equivalent to the fractional Cauchy problem (14) with $\beta = 1/2$.

Another iterated equation comes from the theory of medical ultrasound. Kelly et al. [19] propose a time-fractional wave equation

$$\frac{1}{c_0^2}\partial_t^2 m(x,t) + \frac{2\alpha_0}{c_0 b}\mathbb{D}_t^{\beta+1} m(x,t) + \frac{\alpha_0^2}{b^2}\mathbb{D}_t^{2\beta} m(x,t) = \Delta_x m(x,t) \tag{29}$$

to model the variations in pressure $m(x,t)$ for acoustic wave conduction in a complex medium (e.g., human tissue), where c_0 is the speed of sound in a homogeneous medium, and the constant $b = \cos(\pi y/2)$. This equation models power law attenuation, which is commonly seen in applications: An input sound wave attenuates according to a power law, and in particular, the amplitude decays like $e^{-\alpha(\omega)t}$ where the attenuation coefficient $\alpha(\omega) = \alpha_0|\omega|^\beta$ depends on the frequency ω of the input wave according to a power law with index β. Straka et al. [48] show that the higher order equation (29) on one dimension is equivalent to a lower order time-fractional equation that involves an inverse stable subordinator. Start with the stable subordinator $t/c_0 + (\alpha_0/b)^{1/\beta} D_t$ where D_t is the standard stable subordinator. The pdf $h_0(x,t)$ of the corresponding inverse stable subordinator solves the governing equation

$$\frac{1}{c_0}\partial_t h_0(x,t) + \frac{\alpha_0}{b}\mathbb{D}_t^\beta h_0(x,t) = -\partial_x h_0(x,t) \tag{30}$$

for $x > 0$ and $t > 0$. Then [48, Section 3] shows that the function $h_0(x,t)$ also solves the higher order equation (29) in one dimension. The key is to note that the operator on the left-hand side of (29) is the same as the operator on the left-hand side of (30) applied twice. A closely related argument in Meerschaert et al. [38] shows that if $p(x,t)$ solves the traditional wave equation (12) with $L_x = \Delta_x$, then the function $m(x,t)$ given by (4) with h replaced by h_0 solves (29) in three dimensions. The inverse stable subordinator also leads to a useful CTRW model for the time-fractional wave equation (29), see [38, Section 5].

8 Subordinators and inverse subordinators

The standard β-stable subordinator is one example of a subordinator, i.e., a non-decreasing Lévy process. More generally, we can consider a wide array of subordinators and their inverses, both of which can be useful in applications. Any subordinator D_t with pdf $g(x,t)$ has a Laplace transform

$$\tilde{g}(s,t) = \int_0^\infty e^{-sx} g(x,t)\,dx = e^{-t\psi_D(s)} \tag{31}$$

where the *Laplace symbol* can be written in the form

$$\psi_D(s) = as + \int_0^\infty \left(1 - e^{-sy}\right)\phi_D(dy) \tag{32}$$

using the Lévy-Khintchine formula [4, 45]. Here $\phi_D(dy)$ is the Lévy measure, which governs the jumps of the process [36, p. 51]. For a standard stable process we have $a = 0$ and $\phi_D(y,\infty) = y^{-\beta}/\Gamma(1-\beta)$, which leads to $\psi_D(s) = s^\beta$, e.g., see [36, p. 114]. For the rest of this section we assume that $a = 0$, but note that an example with $a > 0$ was already discussed in Section 7.

Meerschaert and Scheffler [35, Theorem 3.1] shows that, under some mild technical conditions, the inverse subordinator E_t defined by (5) has a pdf

$$h(x,t) = \int_0^t \phi_D(t-u,\infty) g(u,x)\,dy. \tag{33}$$

The formula (8) can be obtained as a special case. They also show that the inverse subordinator pdf has Laplace transform $\tilde{h}(x,s) = s^{-1}\psi_D(s)e^{-x\psi_D(s)}$ [35, Eq. (3.130], which reduces to (11) in the case of a standard β-stable subordinator. Veillette and Taqqu [51] develop numerical methods for computing the pdf $h(x,t)$ of a general inverse subordinator.

Now suppose that $A(t)$ is a Lévy process with pdf $p(x,t)$. Since this process can take both positive and negative values, we apply the Fourier transform (FT)

$$\hat{p}(k,t) = \int_{-\infty}^{\infty} e^{-ikx} p(x,t)\,dx.$$

The Lévy-Khintchine formula [36, Theorem 3.4] implies that

$$\hat{p}(k,t) = e^{t\psi_A(k)} \tag{34}$$

where the *Fourier symbol* can be written in the form

$$\psi_A(k) = -ikb - Dk^2 + \int_{-\infty}^{\infty} \left(e^{-iky} - 1 + \frac{iky}{1+y^2}\right)\phi_A(dy). \tag{35}$$

The time-changed process $B(E_t)$ has pdf $m(x,t)$ given by (4) [33, Corollary 3.8]. Taking LT and FT in this equation, we can see that

$$\begin{aligned}
\bar{m}(k,s) &= \int_{-\infty}^{\infty} e^{-ikx} \int_0^{\infty} e^{-st} m(x,t)\,dt\,dx \\
&= \int_0^{\infty} \left(\int_{-\infty}^{\infty} e^{-ikx} p(x,u)\,dx \right) \left(\int_0^{\infty} e^{-st} h(u,t)\,dt \right) du \\
&= \int_0^{\infty} e^{u\psi_A(k)} s^{-1}\psi_D(s) e^{-u\psi_D(s)}\,du \\
&= \frac{s^{-1}\psi_D(s)}{\psi_D(s) - \psi_A(k)}
\end{aligned} \tag{36}$$

which we can rewrite in the form

$$\psi_D(s)\bar{m}(k,s) = \psi_A(k)\bar{m}(k,s) + s^{-1}\psi_D(s). \tag{37}$$

The Laplace and Fourier symbols correspond to pseudo-differential operators (e.g., see Jacob [18]). Using the functional calculus, since $(ik)\hat{f}(k)$ is the FT of the weak derivative $\partial_x f(x)$, the FT $\psi_A(k)\hat{f}(k)$ inverts to $\psi_A(-i\partial_x)f(x)$. Similarly, since $s\tilde{f}(s)$ is the LT of the weak derivative $\partial_t f(t)$, the LT $\psi_D(s)\tilde{f}(s)$ inverts to $\psi_D(\partial_t)f(t)$.

For example, if $A(t)$ is a Brownian motion then $\psi_A(k) = -Dk^2 = D(ik)^2$ and $\psi_A(k)\hat{f}(k)$ is the FT of $D\partial_x^2 f(x)$. If D_t is the standard β-stable subordinator, then $\psi_D(s) = s^\beta$, and $\psi_D(s)\tilde{f}(s)$ is the LT of $\mathbb{D}_t^\beta f(t)$. For a discussion of weak versus strong derivatives, and how this relates to the formula $s\tilde{f}(s) - f(0)$ for the LT of the traditional derivative $\partial_t f(t)$, see [37, Section 3]. In short, the extra term $f(0)$ comes from the weak derivative of the Heaviside function at $t = 0$. Now invert the LT and FT in (37) to obtain the governing equation of $A(E_t)$:

$$\psi_D(\partial_t)m(x,t) = \psi_A(-i\partial_x)m(x,t) + \delta(x)\phi_D(t,\infty), \tag{38}$$

using the fact [35, Eq. (3.12)] that $\phi_D(t,\infty)$ has LT $s^{-1}\psi_D(s)$. More generally, we can consider generalized Cauchy problems of the form (38) with $\psi_A(-i\partial_x)$ replaced by the generator L_x of some semigroup. Chen [15] develops solution to generalized Cauchy problems, extending [5, Theorem 3.1] to the case of a general time operator. This provides a governing equation for the time-changed process $B(E_t)$ where $B(t)$ is a Markov process with generator L_x, and E_t is an independent general inverse subordinator. See Toaldo [49] for some related results.

If $A(t)$ is a Brownian motion and D_t is the standard stable subordinator, then (38) reduces to (21) with $L_x = D\partial_x^2$, since $\phi_D(t,\infty) = t^{-\beta}/\Gamma(1-\beta)$. If $A(t) = t$ and D_t is the standard stable subordinator, then $p(x,t) = \delta(x-t)$, $\hat{p}(k,t) = e^{\psi_A(k)t}$ with $\psi_A(k) = -ik$, and (38) reduces to (22) with $L_x = -\partial_x$. This generator is also a weak derivative, a fact that has caused some confusion in the literature [37, Section 5]: Weak and traditional derivatives are the same for differentiable functions, but since the inverse stable pdf $h(x,t)$ has a jump discontinuity at $x = 0$, the weak derivative has an extra term $h(0+,t)H'(x) = \delta(x)t^{-\beta}/\Gamma(1-\beta)$ at $x = 0$. This cancels the last term in (38), and then (20) yields the alternative governing equation found in Hahn, Kobayashi and Umarov [17]:

$$\partial_t^\beta h(x,t) = -\partial_x h(x,t) - \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{39}$$

where now ∂_x is the traditional derivative, which is only defined on $x > 0$.

8.1 Tempered stable subordinator

One issue with the stable subordinator pdf is that its mean and variance are undefined, since $g(x,t) \approx x^{-\beta-1}$ for x large. A useful idea [10, 43] for handling this is to “temper” the heavy tail of the pdf so that all moments exist. The function $e^{-\lambda x}g(x,t)$ has a light tail for any $\lambda > 0$, and if the tempering parameter λ is sufficiently small, then the difference will not be noticeable for moderate x. Hence tempering is a mathematical construct to avoid diverging moments. Of course $e^{-\lambda x}g(x,t)$ is no longer a probability density, but we can apply (10) to see that

$$g_\lambda(x,t) = e^{-\lambda x}g(x,t)e^{tc\lambda^\beta} \tag{40}$$

is a pdf, with LT

$$\tilde{g}_\lambda(s,t) = e^{-tc\psi_\lambda(s)} \tag{41}$$

where $\psi_\lambda(s) = (\lambda+s)^\beta - \lambda^\beta$. It is not clear from this calculation that $g_\lambda(x,t)$ should be the pdf of a Lévy subordinator, but another calculation using (32) with an exponentially tempered Lévy measure $\phi_\lambda(t,\infty) = ce^{\lambda y}t^{-\beta}/\Gamma(1-\beta)$ shows that $\tilde{g}(s,t)$ is indeed the Laplace symbol of a subordinator. Taking $c = 1$ yields the pdf of the standard tempered stable subordinator, which we will denote by D_t^λ. Substituting D_t^λ for D_t in (5) yields the inverse tempered stable subordinator, which we will denote by E_t^λ. Since the jump intensity (Lévy measure) is an exponentially tempered power law, the effect of the tempering is to "cool" the big jumps.

Alrawashdeh et al. [3] give an explicit formula for the inverse tempered stable pdf

$$h_\lambda(x,t) = e^{x\lambda^\beta}\left[e^{-\lambda t}h(x,t) + \lambda\int_0^t e^{-\lambda\tau}h(x,\tau)\,d\tau - \lambda^\beta\int_0^t e^{-\lambda\tau}g(\tau,x)\,d\tau\right] \tag{42}$$

using the inverse stable pdf (8) and the standard β-stable pdf. Plots of the pdf are quite similar to Figure 3, see [3, Fig. 1]. Some alternative forms for the inverse tempered stable pdf $h_\lambda(x,t)$ are given in Kumar and Vellaisamy [23], along with an explicit formula for $h_\lambda(0+,t)$ in terms of the incomplete gamma function, asymptotic behavior of the moments of E_t^λ, and higher order governing equations for the case $\beta = 1/n$ for some integer n. See Stanislavsky et al. [47] and Veillette and Taqqu [50] for additional information.

If $A(t)$ is a Lévy process with pdf $p(x,t)$ and Fourier symbol $\psi_A(k)$, then (4) with h replaced by h_λ gives the pdf of $B(D_t^\lambda)$. This pdf solves (38) with ψ_D replaced by ψ_λ. We call $\mathbb{D}_t^{\beta,\lambda} = \psi_\lambda(\partial_t)$ the *tempered* fractional derivative. A calculation using the shift property of the LT [36, p. 209] shows that

$$\mathbb{D}_t^{\beta,\lambda}f(t) = e^{-\lambda t}\mathbb{D}_t^\beta\left[e^{\lambda t}f(t)\right] - \lambda^\beta f(t). \tag{43}$$

If $B(u)$ is a Brownian motion, then (38) reduces to the tempered fractional diffusion equation

$$\mathbb{D}_t^{\beta,\lambda}m(x,t) = \partial_x^2 m(x,t) + \delta(x)\phi_\lambda(t,\infty). \tag{44}$$

This model exhibits transient anomalous diffusion, resembling anomalous subdiffusion (1) at small time scales and traditional diffusive behaviour at large time scales. More generally, (14) with ∂_t^β replaced by $\psi_\lambda(\partial_t)$ represents a *tempered* fractional Cauchy problem. Classical solutions to tempered fractional Cauchy problems on bounded domains, similar to (16), are developed in [29].

It has been suggested [3, 35] that a tempered Caputo fractional derivative can be defined as $\partial_t^{\beta,\lambda} f(t) = \mathbb{D}_t^{\beta,\lambda} f(t) - f(0)\phi_\lambda(t,\infty)$, to simplify (44) into a form resembling (1). Then the inverse tempered stable pdf solves the tempered fractional Cauchy problem (18) with ∂_t^β replaced by $\partial_t^{\beta,\lambda}$ [3, Proposition 3.1]. Some other forms of the tempered fractional derivative have been suggested [6, 25, 32].

A CTRW model for tempered fractional diffusion was developed in Chakrabarty and Meerschaert [11]: The waiting times are exponentially tempered power laws. Several applications of tempered fractional diffusion to problems in geophysics are outlined in Meerschaert, Zhang and Baeumer [39]. Another interesting application is the tempered fractional Poisson process $N(E_t^\lambda)$, see [3, Section 7]. The pmf $m(x,t)$ of this process is given by (4) where $p(n,t)$ is given by (23) and h is replaced by h_λ. It solves the tempered fractional Cauchy problem (25) with ∂_t^β replaced by $\partial_t^{\beta,\lambda}$ [3, Eq. (7.4)].

8.2 Distributed order and ultraslow diffusion

Chechkin et al.[12, 13, 14] consider a model where the fractional order β of the time derivative in (1) is randomized. They define the *distributed order* fractional diffusion equation

$$\partial_t^{p(\beta)} m(x,t) = D\partial_x^2 m(x,t), \tag{45}$$

where the distributed order fractional derivative is defined as a mixture of Caputo derivatives of different order:

$$\partial_t^{p(\beta)} f(t) = \int_0^1 \partial_t^\beta f(t)\, p(\beta) d\beta. \tag{46}$$

If the pdf $p(\beta)$ gives positive probability to values of β near zero, then (45) models *ultraslow* diffusion, where a plume of particles spreads at a logarithmic rate. Meerschaert and Scheffler [34, Theorem 3.4] show that the model (45) corresponds to a subordinator D_t with Lévy measure

$$\phi_D(y,\infty) = \int_0^1 \frac{y^{-\beta}}{\Gamma(1-\beta)} p(\beta) d\beta. \tag{47}$$

Hence the jumps of this subordinator are power law, with an index β governed by the pdf $p(\beta)$. A calculation using (32) with $a = 0$ shows that

$$\psi_D(s) = \int_0^1 s^\beta \, p(\beta) d\beta. \tag{48}$$

Now let E_t be the distributed order inverse subordinator defined by (5). Its pdf $h(x, t)$ is given by (33), and so if $B(t)$ is an independent Brownian motion with pdf $p(x, t)$, then the pdf $m(x, t)$ of $B(E_t)$ solves the distributed order fractional diffusion equation (45). This pdf is given by (4), where $p(x, t)$ is the pdf of $B(t)$, and $h(x, t)$ is the pdf of the distributed order inverse subordinator. If the pdf $p(\beta)$ gives positive probability to values of β near zero, then the subordinator D_t is ultrafast, and hence its inverse E_t is ultraslow: The presence of very large jumps in D_t translates to very long waiting times for E_t, see Kovács and Meerschaert [22] for more details.

The distributed order fractional diffusion equation (45) is a special case of (38). To see this, recall that $s^{-1}\psi_D(s)$ is the LT of $\phi_D(y, \infty)$, and substitute (48) into (37) to get

$$\int_0^1 s^\beta \bar{m}(k, s) \, p(\beta) d\beta = -Dk^2 \bar{m}(k, s) + \int_0^1 s^{\beta-1} \, p(\beta) d\beta. \tag{49}$$

Rearrange to get

$$\int_0^1 \left(s^\beta \bar{m}(k, s) - s^{\beta-1} \right) \, p(\beta) d\beta = -Dk^2 \bar{m}(k, s)$$

and note that $\hat{m}(k, 0) = 1$ since $B(E_0) = 0$. Invert the FT and LT to arrive at (45). A CTRW model for distributed order fractional diffusion is developed in [34]: Take B_n iid with pdf $p(\beta)$, and conditional on the value of B_n, take waiting times iid with $\mathbb{P}[W_n > t | B_n = \beta] = Ct^{-\beta}$ for some $C > 0$. Classical solutions to more general distributed order Cauchy problems, defined by (14) with ∂_t^β replaced by $\partial_t^{p(\beta)}$, were developed in [28].

9 Summary

Time-fractional equations like (1) model time-changed stochastic processes $B(E_t)$ where E_t in the inverse of a subordinator D_t. The waiting times between particle jumps follow a power law distribution, and the power law index equals the order of the time-fractional derivative. The inverse stable time change produces many interesting stochastic models, including delayed Brownian motion, and the fractional Poisson process. The range of the β-stable subordinator is a random fractal of dimension β, the same number that describes the power law jumps and the order of the fractional derivative. The inverse stable time change also changes the fractal dimension of the graph of $B(E_t)$. A continuous time random walk model with power law waiting times gives a physical interpretation to the time-fractional derivative. Higher order governing equations for $B(E_t)$ exist, and in some cases, do not involve any fractional derivatives. More general inverse subordinators lead to tempered fractional diffusion, a tempered fractional Poisson process, and ultra-slow diffusion, where particles spread at a logarithmic rate.

Acknowledgment: M. M. Meerschaert was partially supported by ARO MURI grant W911NF-15-1-0562.

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