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RESEARCH PAPER

NUMERICAL METHODS FOR SOLVING THE MULTI-TERM TIME-FRACTIONAL WAVE-DIFFUSION EQUATION

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Abstract

In this paper, the multi-term time-fractional wave-diffusion equations are considered. The multi-term time fractional derivatives are defined in the Caputo sense, whose orders belong to the intervals [0,1], [1,2), [0,2), [0,3), [2,3) and [2,4), respectively. Some computationally effective numerical methods are proposed for simulating the multi-term time-fractional wave-diffusion equations. The numerical results demonstrate the effectiveness of theoretical analysis. These methods and techniques can also be extended to other kinds of the multi-term fractional time-space models with fractional Laplacian.

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Key Words and Phrases: multi-term time fractional wave-diffusion equations, Caputo derivative, a power law wave equation, finite difference method, fractional predictor-corrector method

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1. Introduction

Generalized fractional partial differential equations have been used for describing important physical phenomena (see [1, 2, 10, 11, 14, 23, 24, 27]). However, studies of the multi-term time-fractional wave equations are still under development.

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The time fractional diffusion and wave-diffusion equations can be written in the following form:

$$D_t^{\alpha} u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 < x < L, \quad t > 0, \tag{1.1}$$

where x and t are the space and time variables, k is an arbitrary positive constant, $f(x,t)$ is a sufficiently smooth function, $0 < \alpha \le 2$ and D_t^{α} is a Caputo fractional derivative of order α defined as [28]

$$D_t^{\alpha} u(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{u^{(m)}(\tau)}{(t-\tau)^{1+\alpha-m}}, & m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} u(t), & \alpha = m \in N. \end{cases}$$

When $0 < \alpha < 1$, equation (1.1) is a fractional diffusion equation and when $1 < \alpha < 2$, it is the time fractional diffusion-wave equation. When $\alpha = 1$, equation (1.1) represents a traditional diffusion equation; while if $\alpha = 2$, it represents a traditional wave equation. Papers [5, 6, 37] also discussed fractional differential equations with multi-orders. However, in them multi-orders lying in $(0,2)$ are only considered.

In order to model loss, the underlying processes cannot be described by equation (1.1), but can be modelled using its generalization the multi-term time-fractional diffusion-wave and diffusion equations that are given by [22], namely

$$P_{\alpha,\alpha_1,\cdots,\alpha_n}(D_t)u(x,t) = L_x(u(x,t)) + f(x,t), \tag{1.2}$$

where

$$P_{\alpha,\alpha_1,\cdots,\alpha_n}(D_t)u(x,t) = (D_t^{\alpha} + \sum_{i=1}^{n} d_i D_t^{\alpha_i})u(x,t), \tag{1.3}$$

$0 < \alpha_n < \ldots < \alpha_1 < \alpha \le 1$, and $d_i \in R$, $i = 1,\ldots,n$, $n \in N$; $D_t^{\alpha_i}$ is a Caputo fractional derivative of order α_i with respect to t. The operator $L_x(u(x,t))$ is the well-known linear elliptic differential operator of the second order. Zhang et al. [36] considered the two-term mobile/immobile time fractional advection-dispersion equation and two-term time fractional wave-diffusion equation. Based on an appropriate maximum principle, Luchko [22] proved the uniqueness and existence results. By the attempts to describe some real processes with the equations of the fractional order, several researchers were confronted with the situation that the order α of the fractional derivative from the corresponding model equations did not remain integer and changed, say, in the intervals from 0 to 1, from 1 to 2, from 0 to 2 or even from 0 to 4, see for example [4, 15, 13, 34].

Stojanovic [32] found solutions for the fractional wave-diffusion problem in one dimension with n-term time fractional derivatives whose orders belong to the intervals $(0,1)$, $(1,2)$ and $(0,2)$, respectively, using the method of the approximation of the convolution by Laguerre polynomials in the space of tempered distributions.

Time domain wave-equations for lossy media obey a frequency power-law. Frequency-dependent loss and dispersion are typically modeled with a power-law attenuation coefficient, where the power-law exponent ranges from 0 to 2. Mathematically, the power-law frequency dependence of the attenuation coefficient cannot be modeled with standard dissipative partial differential equations with integer-order derivatives. The generalized time-fractional diffusion equation corresponds to a continuous time random walk model where the characteristic waiting time elapsing between two successive jumps diverge, but the accumulated jump length variance remains finite and is proportional to t^α. The exponent α of the mean square displacement proportional to t^α often does not remain constant and changes. To adequately describe these phenomena with fractional models, multi-term time-fractional wave-diffusion equations and several approaches have been suggested in the literature (for example, [12, 13, 22]). The multi-term time-fractional wave-diffusion equations successfully capture this power-law frequency dependence.

Kelly et al. [13] modified the Szabo wave equation [4, 15, 34]:

$$\triangle p - \frac{1}{c_0^2}\partial_t^2 p - \frac{2\alpha_0}{c_0\cos(\pi y/2)}\partial_t^{y+1} p = 0, \tag{1.4}$$

adding a second time-fractional term to arrive at the power law wave equation

$$\frac{1}{c_0^2}\partial_t^2 p + \frac{2\alpha_0}{c_0\cos(\pi y/2)}\partial_t^{y+1} p + \frac{\alpha_0^2}{\cos^2(\pi y/2)}\partial_t^{2y} p = \triangle p, \tag{1.5}$$

where $0 \le y < 1$ or $1 < y \le 2$; $\triangle p = \sum_j \frac{\partial^2 p}{\partial x_j^2}$. For small α_0, the additional term is negligible, leading to an approximate solution to the Szabo wave equation (1.4). The Riemann-Liouville fractional derivative $\partial_t^y p = {}_0D_t^y p$ of order y $(0 \le m-1 < y < m)$ is defined as (see [28]):

$${}_0D_t^y p(t) = \frac{1}{\Gamma(m-y)}\frac{d^m}{dt^m}\int_0^t \frac{p(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}\,.$$

The power law wave equation is used to model sound wave propagation in anisotropic media that exhibits frequency dependent attenuation. The wave amplitude A falls off exponentially with radial distance r from the source, so that $A = e^{-\alpha(\omega)r}$ where the attenuation coefficient $\alpha(\omega)$ depends on the frequency. In human tissue, experimental evidence indicates that $\alpha(\omega) = \alpha_0|\omega|^y$ is a power law, and the exponent y lies in the interval $1 \le y \le 1.5$, see for example Duck [8]. It is easy to check, using transform methods, that the point source solution of (1.5) reproduces this power law attenuation [26]. Some additional properties of the power law wave equation are discussed in [33].

Some authors have discussed numerical approximations for fractional partial differential equations. Paper [16] considered the space fractional Fokker-Planck equation with instantaneous source and presented a fractional method of lines. [25] developed numerical methods to solve the one-dimensional equation with variable coefficients on a finite domain. [29] investigated the numerical approximation of the variational solution on bounded domains in $\mathbb{R}^2$ and presented a method for approximating the solution in two spatial dimensions using the finite element method. [20] presented a random walk model for approximating a Lévy-Feller advection-dispersion process, and proposed an explicit finite difference approximation. [17] considered a space-time fractional advection dispersion equation on a finite domain and proposed implicit and explicit difference methods to solve this equation. [18] considered a modified anomalous subdiffusion equation with a nonlinear source term. An implicit difference method is constructed. Its stability and convergence are discussed using the energy method, [18]. [38] proposed explicit and implicit Euler approximations for the variable-order fractional advection-diffusion equation with a nonlinear source term. Stability and convergence of the methods are discussed. [9] proposed an advanced implicit meshless approach for the non-linear anomalous subdiffusion equation. [21] also proposed an implicit RBF meshless approach for time fractional diffusion equations.

Numerical solutions of the multi-term time-fractional wave-diffusion equations (MT-TFWDE) with the fractional orders lying in $(0, n)(n > 2)$ are still limited. The main purpose of this paper is to derive numerical solutions of the multi-term time-fractional wave-diffusion equations with nonhomogeneous Dirichlet boundary conditions. The multi-term time-fractional derivatives are defined in the Caputo sense, whose orders belong to the intervals $[0, 1]$, $[1, 2)$, $[0, 2)$, $[0, 3)$, $[2, 3)$ and $[2, 4)$, respectively. The Caputo and Riemann-Liouville forms are related through the boundary conditions (see Remark 3). As far as we know there are no relevant research papers in the published literature that cover all of these cases.

The rest of this paper is organized as follows. Two implicit numerical methods for simulating the two-term mobile/immobile time-fractional diffusion equation and the two-term time-fractional wave-diffusion equation are proposed in Sections **2** and **3**, respectively. Two computationally effective fractional predictor-corrector methods for the multi-term time-fractional wave-diffusion equations are investigated in Section **4**. Finally, some examples are discussed to illustrate the application of our theoretical results.

2. A two-term mobile/immobile time-fractional advection-dispersion equation

In order to distinguish explicitly the mobile and immobile status using fractional dynamics, [30] developed the fractional-order, mobile/immobile model for the total concentration, i.e., the following two-term mobile/immobile time-fractional diffusion equation, whose orders belong to the intervals [0,1]:

$$a_2 \frac{\partial C(x,t)}{\partial t} + a_1 \frac{\partial^\gamma C(x,t)}{\partial t^\gamma} = D \frac{\partial^2 C(x,t)}{\partial x^2} + f(x,t), \tag{2.1}$$

with initial condition:

$$C(x,0) = \varphi_0(x), \tag{2.2}$$

and boundary conditions:

$$C(a,t) = \phi_1(t),\ C(b,t) = \phi_2(t), \quad 0 \le t \le T, \tag{2.3}$$

where $a_1 > 0$, $a_2 > 0$, $0 < \gamma < 1$, $D > 0$.

We define $t_k = k\tau, k = 0,1,\dots,n$; $x_i = a + ih, i = 0,1,\dots,m$, where $\tau = T/n$ and $h = (b-a)/m$ are space and time step sizes, respectively.

We discretize the Caputo time-fractional derivative as (see [31])

$$\begin{aligned} \frac{\partial^\gamma C(x,t_{k+1})}{\partial t^\gamma} &= \frac{1}{\Gamma(1-\gamma)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \frac{\frac{\partial C(x,\eta)}{\partial \eta}}{(t_{k+1}-\eta)^\gamma} d\eta \qquad (2.4) \\ = \quad & \sum_{j=0}^{k} \frac{C(x,t_{j+1}) - C(x,t_j)}{\Gamma(1-\gamma)\tau} \int_{t_j}^{t_{j+1}} \frac{d\eta}{(t_{k+1}-\eta)^\gamma} + O(\tau^{2-\gamma}) \\ = \quad & \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \sum_{j=0}^{k} b_j^\gamma [C(x,t_{k+1-j}) - C(x_i,t_{k-j})] + O(\tau^{2-\gamma}), \end{aligned}$$

where $b_j^\gamma = (j+1)^{1-\gamma} - j^{1-\gamma},\ j = 0,1,2,\cdots,n$.

Hence, we have

$$\begin{aligned} & a_2 \frac{C(x_i,t_{k+1}) - C(x_i,t_k)}{\tau} \\ & + \frac{a_1 \tau^{-\gamma}}{\Gamma(2-\gamma)} \sum_{j=0}^{k} b_j^\gamma \left[C(x_i,t_{k+1-j}) - C(x_i,t_{k-j})\right] \qquad (2.5) \\ = \quad & D \frac{C(x_{i-1},t_{k+1}) - 2C(x_i,t_{k+1}) + C(x_{i+1},t_{k+1})}{h^2} \\ & + f(x_i,t_{k+1}) + R_{i,k+1} \end{aligned}$$

where

$$|R_{i,k+1}| \le K(\tau + \tau^{2-\alpha} + h). \tag{2.6}$$

Let C_i^k be the numerical approximation to $C(x_i, t_k)$, then we obtain the following implicit difference approximation of equations (2.1)-(2.3):

$$\begin{aligned} & a_2 \frac{C_{i,k+1} - C_{i,k}}{\tau} + \frac{a_1 \tau^{-\gamma}}{\Gamma(2-\gamma)} \sum_{j=0}^{k} b_j^{\gamma} \left[C_{i,k+1-j} - C_{i,k-j}\right] \\ = \; & D \frac{C_{i-1,k+1} - 2C_{i,k+1} + C_{i+1,k+1}}{h^2} + f_{i,k+1}. \end{aligned} \tag{2.7}$$

The initial and boundary conditions are discretized as follows

$$C_{i,0} = \varphi(x_i), \; i = 0, 1, 2, \ldots, m, \tag{2.8}$$

$$C_{0,k} = \phi_1(k\tau), \; C_{m,k} = \phi_2(k\tau), \; k = 1, 2, \ldots, n. \tag{2.9}$$

Theorem 1. *The fractional implicit numerical method defined by* (2.7), (2.8) *and* (2.9) *is unconditionally stable.*

Proof. Using similar techniques as in [19], we can prove this result.

Theorem 2. *Let $C_{i,k}$ be the numerical solution computed by use of the implicit numerical methods (2.7)-(2.9), $C(x,t)$ is the exact solution of the problem* (2.1)-(2.3). *Then there is a positive constant $\bar{C}$, such that*

$$|C_{i,k} - C(x_i, t_k)| \le \bar{C}(\tau + \tau^{2-\alpha} + h), \tag{2.10}$$

where $i = 1, 2, \cdots, m-1; k = 1, 2, \ldots, n$.

Proof. Using similar techniques in [19], we can prove this result.

3. A two-term time-fractional wave-diffusion equation

A two-term time fractional wave-diffusion model with damping with index $1 < \gamma = 1 + \bar{\gamma} < 2$ and whose orders belong to the intervals [1,2) can be written as the following form:

$$a_2 \frac{\partial^{\gamma} C(x,t)}{\partial t^{\gamma}} + a_1 \frac{\partial C(x,t)}{\partial t} = D \frac{\partial^2 C(x,t)}{\partial x^2} + f(x,t), \tag{3.1}$$

with initial condition:

$$C(x,0) = \varphi_0(x), \; \frac{\partial C(x,0)}{\partial t} = \varphi_1(x), \; a \le x \le b, \tag{3.2}$$

and boundary conditions:

$$C(a,t) = \phi_1(t), \; C(b,t) = \phi_2(t), \quad 0 \le t \le T, \tag{3.3}$$

where $a_1 > 0$, $a_2 > 0$ and $D > 0$.

This partial differential equation with $\gamma = 2$ is called the telegraph equation which governs electrical transmission in a telegraph cable. It can also be characterized as a fractional diffusion-wave equation (which governs wave motion in a string) with a damping effect due to the terms $1 < \gamma < 2$ and $a_1 \frac{\partial C(x,t)}{\partial t}$ in equation (3.1).

Let $U(x,t) = \frac{\partial C(x,t)}{\partial t}$, then the equation (3.1) can be rewritten as

$$\frac{\partial C}{\partial t} = U, \tag{3.4}$$

$$a_2 \frac{\partial^{\bar{\gamma}} U}{\partial t^{\bar{\gamma}}} + a_1 U = D \frac{\partial^2 C(x,t)}{\partial x^2} + f(x,t), \tag{3.5}$$

where $0 < \bar{\gamma} < 1$. Hence, we can obtain the following difference scheme

$$\frac{C_{i,k+1} - C_{i,k}}{\tau} = U_{i,k+1}, \tag{3.6}$$

$$a_2 \frac{\tau^{-\bar{\gamma}}}{\Gamma(2-\bar{\gamma})} \sum_{j=0}^{k} b_j^{\bar{\gamma}} [U_{i,k-j+1} - U_{i,k-j}] + a_1 U_{i,k+1}$$
$$= D \frac{C_{i-1,k+1} - 2C_{i,k+1} + C_{i-1,k+1}}{h^2} + f_{i,k+1}. \tag{3.7}$$

Thus, the above equations can be rewritten as

$$(a_2 + a_1\mu + 2r) C_{i,k+1} - rC_{i-1,k+1} - rC_{i+1,k+1}$$
$$= (a_2 + a_1\mu) C_{i,k} + \tau \left[b_k^{\bar{\gamma}} U_{i,0} + \sum_{j=0}^{k-1} (b_j^{\bar{\gamma}} - b_{j+1}^{\bar{\gamma}}) U_{i,k-j} \right] + \mu\tau f_{i,k+1}, \tag{3.8}$$

$$U_{i,k+1} = \frac{C_{i,k+1} - C_{i,k}}{\tau}, \tag{3.9}$$

where $\mu = \tau^{\bar{\gamma}} \Gamma(2-\bar{\gamma})$, $r = D\mu\tau/h^2$.

Remark 1. It should be possible to prove stability and convergence by a method similar to [19]. However, we leave this as an open problem for future research.

4. Multi-term time-fractional wave-diffusion equations

In this section, we are concerned with providing good quality methods for solution of the $n+1$ term time-fractional wave-diffusion of the general form:

$$a_n D_t^{\beta_n} y(x,t) + \cdots + a_1 D_t^{\beta_1} y(x,t) + a_0 D_t^{\beta_0} y(x,t)$$
$$= D \frac{\partial^2 y(x,t)}{\partial x^2} + f(x,t), \tag{4.1}$$

subject to initial conditions

$$y^{(i)}(x,0) = y_0^{(i)}(x), \ i = 0, 1, \cdots, m-1, \tag{4.2}$$

and boundary conditions

$$y(a,t) = \phi_1(t), \ y(b,t) = \phi_2(t), \quad 0 \le t \le T, \tag{4.3}$$

where $m-1 < \beta_n \leq m$, $\beta_n > \beta_{n-1} > \cdots > \beta_1 > \beta_0 = 0$, $0 < \alpha_j = \beta_j - \beta_{j-1} \leq 1$, $0 < \beta_1 < 1$, $a_n > 0$, a_k $(k = 0, 1, 2, \cdots, n-1)$ is a arbitrary constant, $D_t^{\beta_i}$ is a Caputo fractional derivative of order β_i with respect to t or a Riemann-Liouville fractional derivative of order β_i with respect to t.

Remark 2. The relationship between the Caputo fractional derivative and the Riemann-Liouville fractional derivative (see [28]) is

$$D_t^{\alpha} y(t) =_0 D_t^{\alpha} y(t) - \sum_{k=0}^{m-1} \frac{y^{(k)}(0^+) t^{k-\alpha}}{\Gamma(k-\alpha+1)}.$$

It is found that these models can be recast with the Riemann-Liouville fractional derivative or Caputo fractional derivative for noninteger power. Then, the basic strategy of this study is that the Riemann-Liouville fractional derivative is replaced by the Caputo fractional derivative to derive a modified model, where the initial conditions can be easily prescribed. In this paper, we will focus on the multi-term time-fractional derivatives defined in the Caputo sense.

Firstly, we want to rewrite the given multi-term time fractional differential equation in the form of a system of single-term equations. We assume that all of the integers that are contained in the interval $(0, \beta_n]$ are also members of the finite sequence $(\beta_j)_{j=1}^{k}$. In other words, it is impossible for two consecutive elements of the finite sequence (β_j) to lie on opposite sides of an integer number.

Now we rewrite the multi-term time fractional wave-diffusion equations (4.1), (4.2) and (4.3) in the form of a system of fractional differential equations:

$$\begin{cases} D_t^{\alpha_1} z_1(x,t) & = D_t^{\beta_1} z_1(x,t) = z_2(x,t), \\ D_t^{\alpha_2} z_2(x,t) & = D_t^{\beta_2-\beta_1} z_2(x,t) = z_3(x,t), \\ & \vdots \\ D_t^{\alpha_{n-1}} z_{n-1}(x,t) & = D_t^{\beta_{n-1}-\beta_{n-2}} z_{n-1}(x,t) = z_n(x,t), \\ D_t^{\alpha_n} z_n(x,t) & = D_t^{\beta_n-\beta_{n-1}} z_n(x,t) \\ & = \frac{1}{a_n}[D\frac{\partial^2 z_1(x,t)}{\partial x^2} + f(x,t) \\ & \quad -a_0 z_1(x,t) - \cdots - a_{n-1} z_n(x,t)], \\ y(x,t) & = z_1(x,t). \end{cases} \tag{4.4}$$

These are subject to the initial conditions

$$z_i(x,0) = z_0^{(i)}(x) = \begin{cases} y_0^{(1)}(x), & \text{if } i = 1 \\ y_0^{(l)}(x), & \text{if } \alpha_i = l \in N, \\ 0, & \text{else.} \end{cases} \tag{4.5}$$

THEOREM 3. *The multi-term time-fractional differential equation* (4.1) *with initial conditions* (4.2) *is equivalent to the system of time-fractional differential equations* (4.4) *with the initial conditions* (4.5).

P r o o f. Using a similar technique in [7], we can prove the result.

Secondly, we consider the following system of fractional differential equations:

$$\begin{cases} D_t^{\alpha_1} z_1(x,t) &= g_1(x,t,z_1(x,t)), \\ D_t^{\alpha_2} z_2(x,t) &= g_2(x,t,z_2(x,t)), \\ & \vdots \\ D_t^{\alpha_n} z_n(x,t) &= g_n(x,t,z_n(x,t)). \end{cases} \tag{4.6}$$

Thirdly, we propose a fractional predictor-corrector method for solving the following initial-value problem:

$$D_t^{\alpha_i} z_i(x,t) = g_i(x,t,z_i(x,t)), \tag{4.7}$$

$$z_i(x,0) = z_0^{(i)}(x), \ i = 1,2,\cdots,n, \tag{4.8}$$

where $0 < \alpha_i \leq 1$.

It is well know that the initial-value problem (4.7) with (4.8) is equivalent to the Volterra integral equation:

$$z_l(x,t) = z_0^{(l)}(x) + \frac{1}{\Gamma(\alpha_l)} \int_0^t (t-\tau)^{\alpha_l - 1} g_l(x,\tau,z_l(x,\tau)) d\tau. \tag{4.9}$$

For the sake of simplicity, we assume that we are working on a uniform grid $t_j = j\tau, j = 0,1,\ldots,M$; $M\tau = T$; $x_i = ih, i = 0,1,\ldots,N$; $Nh = (b-a)$. It is known that the classical Adams-Bashforth-Moulton method for first order ordinary differential equations is a reasonable and practically useful compromise in the sense that its stability properties allow for a safe application to mildly stiff equations without undue propagation of rounding error, whereas the implementation does not require extremely time consuming elements. Thus, a fractional Adams-Bashforth method and a fractional Adams-Moulton method are chosen as our predictor and corrector formulas.

The predictor $z_{l,P}^{i,k+1}$ is determined by the fractional Adams-Bashforth method (shown by [35]):

$$z_{l,P}^{i,k+1} = z_l^{i,0} + \frac{1}{\Gamma(\alpha_l)} \sum_{j=0}^{k} b_{j,k+1}^{\alpha_l} g_l(x_i, t_j, z_l^{i,j}), \tag{4.10}$$

where

$$b_{j,k+1}^{\alpha_l} = \frac{\tau^{\alpha_l}}{\alpha_l} [(k+1-j)^{\alpha_l} - (k-j)^{\alpha_l}]. \tag{4.11}$$

The corrector formula is determined by the fractional Adams-Moulton method (shown by [35]):

$$z_l^{i,k+1} = z_l^{i,0} + \frac{1}{\Gamma(\alpha_l)} \tag{4.12}$$
$$\times \quad \left(\sum_{j=0}^{k} a_{j,k+1}^{\alpha_l} g_l(x_i, t_j, z_l^{i,j}) + a_{k+1,k+1}^{\alpha_l} g_l(x_i, t_{k+1}, z_{l,P}^{i,k+1}) \right),$$

where

$$a_{j,k+1}^{\alpha_l} = \frac{\tau^{\alpha_l}}{\alpha_l(\alpha_l+1)}$$
$$\times \begin{cases} k^{\alpha_l+1} - (k-\alpha_l)(k+1)^{\alpha_l}, & j=0, \\ (k-j+2)^{\alpha_l+1} + (k-j)^{\alpha_l+1} \\ \qquad -2(k-j+1)^{\alpha_l+1}, & 1 \le j \le k, \\ 1, & j=k+1. \end{cases} \tag{4.13}$$

The above numerical techniques are used to solve the system of fractional differential equations in ((4.4) and (4.5).

The predictor $z_{l,P}^{i,k+1}$ is determined by the fractional Adams-Bashforth method ($l = 1, \cdots, n-1$):

$$z_{l,P}^{i,k+1} = z_l^{i,0} + \frac{1}{\Gamma(\alpha_l)} \sum_{j=0}^{k} b_{j,k+1}^{\alpha_l} z_{l+1}^{i,j}, \tag{4.14}$$

$$z_{n,P}^{i,k+1} = z_n^{i,0} + \frac{1}{\Gamma(\alpha_n)} \sum_{j=0}^{k} b_{j,k+1}^{\alpha_n} [\frac{z_1^{i+1,j} - 2z_1^{i,j} + z_1^{i-1,j}}{h^2}$$
$$+ f_{i,j} - a_1 z_2^{i,j} - \cdots, -a_{n-1} z_n^{i,j}]. \tag{4.15}$$

The corrector formula is determined by the fractional Adams-Moulton method ($l = 1, \cdots, n-1$):

$$z_l^{i,k+1} = z_l^{i,0} + \frac{1}{\Gamma(\alpha_l)} \left(\sum_{j=0}^{k} a_{j,k+1}^{\alpha_l} z_{l+1}^{i,j} + a_{k+1,k+1}^{\alpha_i} z_{l+1,P}^{i,k+1}, \right), \tag{4.16}$$

$$z_n^{i,k+1} = z_n^{i,0} + \frac{1}{a_n \Gamma(\alpha_n)} \{ \sum_{j=0}^{k} a_{j,k+1}^{\alpha_n} [\frac{z_1^{i+1,j} - 2z_1^{i,j} + z_1^{i-1,j}}{h^2}$$
$$+ f_{i,j} - a_1 z_2^{i,j} - \cdots - a_{n-1} z_n^{i,j}]$$
$$+ a_{k+1,k+1}^{\alpha_n} [\frac{z_{1,P}^{i+1,k+1} - 2z_{1,P}^{i,k+1} + z_{1,P}^{i-1,k+1}}{h^2}$$
$$+ f_{i,k+1} - a_1 z_{2,P}^{i,k+1} - \cdots - a_{n-1} z_{n,P}^{i,k+1}]\}. \tag{4.17}$$

We note that

$$\frac{\partial^2 z(x_i,t_{k+1})}{\partial x^2} = \frac{z(x_{i+1},t_{k+1}) - 2z(x_i,t_{k+1}) + z(x_{i-1},t_{k+1})}{h^2} + O(h^2), \quad (4.18)$$

$$\frac{\partial z(x_i,t_{k+1})}{\partial t} = \frac{z(x_i,t_{k+1}) - z(x_i,t_k)}{\tau} + O(\tau). \quad (4.19)$$

If all $\alpha_l \in (0,1)$, we call the fractional predictor-corrector methods (4.14)-(4.16) with (4.17) as FPCM-1. If some $\alpha_l = 1$, then we replace the fractional predictor-corrector methods (4.14)-(4.16) with the following method

$$z_l^{i,k+1} = z_l^{i,k} + \tau z_{l+1}^{i,k+1}, \quad (4.20)$$

which is called FPCM-2.

Thus, we have the following results.

THEOREM 4. *For FPCM-1, we have the following error estimation:*

$$\max_{\substack{0<i<N \\ 1<k<M}} |z_l(x_i,t_k) - z_l^{i,k}| = O(\tau^q + h^2), \quad (4.21)$$

where $q = 1 + \min \alpha_l$.

THEOREM 5. *For FPCM-2, we have the following error estimation:*

$$\max_{\substack{0<i<N \\ 1<k<M}} |z_l(x_i,t_k) - z_l^{i,k}| = O(\tau^q + \tau + h^2). \quad (4.22)$$

5. Numerical results

In order to illustrate the practical application of our numerical methods, some examples are presented.

EXAMPLE 1. Consider the following two-term time fractional diffusion equation:

$$\frac{\partial C(x,t)}{\partial t} + \frac{\partial^\gamma C(x,t)}{\partial t^\gamma} = \frac{\partial^2 C(x,t)}{\partial x^2} + f(x,t), \quad (5.1)$$

together with the following boundary and initial conditions

$$\begin{cases} C(0,t) = t^2,\ C(1,t) = et^2,\ 0 \le t \le 1, \\ C(x,0) = 0,\ 0 \le x \le 1 \end{cases} \quad (5.2)$$

where $0 < \gamma < 1$ and $f(x,t) = (2t - t^2 + \frac{2t^{2-\gamma}}{\Gamma(3-\gamma)})e^x$.

The exact solution of the equations (5.1) and (5.2) is $u(x,t) = t^2 e^x$.
From Table 1, we can observe that

$$\|E\|_{\max} \le C(\tau + h^2).$$

h	$\tau = h^2$	$\|E\|_{\max}$
1/4	1/16	2.0112e-2
1/8	1/64	4.7622e-3
1/16	1/256	1.1347e-3
1/32	1/1024	2.7067e-4

TABLE 1. Numerical maximum errors using the implicit numerical method (2.7)-(2.9) for the two-term mobile/immobile time fractional diffusion equations (5.1)-(5.2) with $\gamma = 0.85$ and $T = 1$ in Example 1.

EXAMPLE 2. Consider the following two-term wave-diffusion equation with damping with $1 < \gamma < 2$:

$$\frac{\partial^\gamma C(x,t)}{\partial t^\gamma} + \frac{\partial C(x,t)}{\partial t} = \frac{\partial^2 C(x,t)}{\partial x^2} + f(x,t), \tag{5.3}$$

where $f(x,t) = (6t^{3-\gamma}/\Gamma(4-\gamma) + 3t^2 - t^3)e^x$, and the initial and boundary conditions are given by

$$\begin{cases} C(x,0) = 0, \ C_t(x,0) = 0, & 0 < x < 1, \\ C(0,t) = t^3, \ C(1,t) = et^3, & 0 \le t \le 1. \end{cases} \tag{5.4}$$

The exact solution of the equations (5.3) and (5.4) is $u(x,t) = t^3 e^x$.

h	$\tau = h^2$	$\|E\|_{\max}$
1/4	1/16	0.1098
1/8	1/64	2.7616e-2
1/16	1/256	6.7214e-3
1/32	1/1024	1.6341e-3

TABLE 2. Numerical maximum errors using the implicit numerical method (3.8)-(3.9) for the two-term wave-diffusion-wave equations (5.2)-(5.3) with $\gamma = 1.85$ and $T = 1$ in Example 2.

From Table 2, we can conclude that

$$\|E\|_{\max} \le C(\tau + h^2).$$

EXAMPLE 3. Consider the following power law wave equation ($3/2 < y = 1.6 < 2$):

$$a_5 \partial_t^{2y} u + a_3 \partial_t^{y+1} u + a_2 \partial_t^2 u = \triangle u + f(x,t),$$

where $a_2 = \frac{1}{c_0^2}$, $a_3 = \frac{2\alpha_0}{c_0 \cos(\pi y/2)}$, $a_5 = \frac{\alpha_0^2}{\cos^2(\pi y/2)}$,

$$f(x,t) = [\frac{a_5\Gamma(6)}{\Gamma(6-2y)} t^{5-2y} + \frac{a_3\Gamma(6)}{\Gamma(5-y)} t^{4-y} + 20a_2 t^3 - t^5]e^x.$$

Initial and boundary conditions:

$$u(x,0) = u_t(x,0) = u_{tt}(x,0) = 0, \tag{5.5}$$

$$u(0,t) = t^5,\ u(1,t) = et^5. \tag{5.6}$$

The exact solution is $u(x,t) = t^5 e^x$.

h	$\tau = h^2$	FPCM-2	FPCM-1
1/4	1/16	0.3664	0.1711
1/8	1/64	0.1110	4.9093e-2
1/16	1/256	3.3093e-2	1.0323e-2
1/32	1/1024	9.3205e-3	2.01235e-3

TABLE 3. Numerical maximum errors using the Predictor-Corrector method (FPCM-1 and FPCM-2) for the power low equation $y = 1.6$ in Example 3.

From Table 3, we can conclude that

$$\|E\|_{\max} \leq C(\tau + \tau^{1+\min\alpha_l} + h^2), \quad \text{in } FPCM-2;$$

$$\|E\|_{\max} \leq C(\tau^{1+\min\alpha_l} + h^2), \quad \text{in } FPCM-1.$$

6. Conclusion

In this paper, some computationally effective numerical methods are proposed for simulating the two-term mobile/immobile time-fractional diffusion equation, the two-term time-fractional wave-diffusion equation and two computationally effective fractional predictor-corrector methods for the multi-term time-fractional wave-diffusion equations. The numerical results demonstrate the effectiveness of theoretical analysis.

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