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Dimension results for sample paths of operator stable Lévy processes

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Abstract

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in $\mathbb{R}^d$ with exponent B, where B is an invertible linear operator on $\mathbb{R}^d$. We determine the Hausdorff dimension and the packing dimension of the range $X([0,1])$ in terms of the real parts of the eigenvalues of B.

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1. Introduction

Let $X = \{X(t),\ t \in \mathbb{R}_+\}$ be a Lévy process in $\mathbb{R}^d$, that is, X has stationary and independent increments, $X(0) = 0$ a.s. and such that $t \mapsto X(t)$ is continuous in probability. The finite-dimensional distributions of a Lévy process X are completely determined by the distribution of $X(1)$. It is well-known that the class of possible distributions for $X(1)$ is precisely the class of infinitely divisible laws. This implies

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that for every $t > 0$ the characteristic function of $X(t)$ is given by

$$\mathbb{E}[e^{i\langle \xi, X(t)\rangle}] = e^{-t\psi(\xi)},$$

where, by the Lévy–Khintchine formula,

$$\psi(\xi) = i\langle \mathsf{a}, \xi\rangle + \frac{1}{2}\langle \xi, \Sigma\xi'\rangle + \int_{\mathbb{R}^d} \left[1 - e^{i\langle x,\xi\rangle} + \frac{i\langle x, \xi\rangle}{1 + \|x\|^2}\right] \mathsf{L}(dx), \quad \forall \xi \in \mathbb{R}^d, \tag{1.1}$$

and $\mathsf{a} \in \mathbb{R}^d$ is fixed, Σ is a non-negative definite, symmetric, $(d \times d)$ matrix, and L is a Borel measure on $\mathbb{R}^d \backslash \{0\}$ that satisfies

$$\int_{\mathbb{R}^d} \frac{\|x\|^2}{1 + \|x\|^2} \mathsf{L}(dx) < \infty.$$

The function ψ is called the *Lévy exponent* of X, and L is the corresponding *Lévy measure*. We refer to the recent books of Bertoin [2] and Sato [23] for the general theory of Lévy processes.

There has been considerable interest in studying the sample path properties of Lévy processes. Many authors have investigated the Hausdorff dimension, Hausdorff measure, packing dimension and packing measure of various random sets generated by Lévy processes. See the survey papers of Taylor [28] and Xiao [33] and the references therein for more information. For a stable Lévy process X in $\mathbb{R}^d$ with index $\alpha \in (0, 2]$, many of the results on the sample paths of X can be formulated nicely in terms of α and d. However, when X is a general Lévy process in $\mathbb{R}^d$, it is often difficult to determine explicitly the Hausdorff dimension of the range $X(E)$, where $E \subset \mathbb{R}_+$ is a Borel set. For $E = [0, 1]$, Pruitt [21] proved that $\dim_H X([0, 1]) = \gamma$ a.s., where the index γ is defined by

$$\gamma = \sup\left\{\alpha \geqslant 0 : \limsup_{r\to 0}\ r^{-\alpha} \int_0^1 \mathbb{P}\{\|X(t)\| \leqslant r\}\, dt < \infty\right\}. \tag{1.2}$$

However, Pruitt's definition of γ is usually hard to calculate. The natural question of expressing γ in terms of the Lévy exponent ψ was raised by Pruitt [21] and he obtained some partial results. This problem has recently been solved by Khoshnevisan et al. [14] who have shown that

$$\gamma = \sup\left\{\alpha < d : \int_{\xi\in\mathbb{R}^d:\ \|\xi\| > 1} \mathrm{Re}\left(\frac{1}{1 + \psi(\xi)}\right) \frac{d\xi}{\|\xi\|^{d-\alpha}} < +\infty\right\}. \tag{1.3}$$

The proof of this result relies on the potential theory for multiparameter Lévy processes and the co-dimension argument. For more historical accounts and the latest developments about the Hausdorff dimension and capacity of the range $X(E)$, we refer to Khoshnevisan and Xiao [13] and Xiao [33].

The packing dimension of the range of a Lévy process X in $\mathbb{R}^d$ was studied by Taylor [29], who proved that $\dim_P X([0, 1]) = \gamma'$ a.s., where the parameter γ' is defined by Hendricks [10] as

$$\gamma' = \sup\left\{\alpha \geqslant 0 : \liminf_{r\to 0}\ r^{-\alpha} \int_0^1 \mathbb{P}\{\|X(t)\| \leqslant r\}\, dt < \infty\right\}. \tag{1.4}$$

Similar to (1.2), this definition of γ' is also hard to use. It would be interesting to express γ' in terms of the Lévy exponent ψ. Except for subordinators, this remains to be an open problem.

The objective of this paper is to investigate the Hausdorff and packing dimensions of the range of a large class of Lévy processes, i.e., the operator stable Lévy processes in $\mathbb{R}^d$; see Section 2 for the definition and related properties of the latter. For the special case of a Lévy process X with stable components in $\mathbb{R}^d$, the Hausdorff dimension of the range $X([0,1])$ was studied by Pruitt and Taylor [22] and then extended by Hendricks [8,9] who determined the Hausdorff dimension of $X(E)$, where $E \subset \mathbb{R}_+$ is a fixed Borel set. Recently, Becker-Kern et al. [1] have obtained $\dim_{\mathrm{H}} X([0,1])$ for more general operator stable Lévy processes. Their arguments are based on the results of Pruitt [21] on $\dim_{\mathrm{H}} X([0,1])$ [cf. (1.2)] and involve several technical probability estimates of operator stable Lévy processes. In addition, they require some restrictions on the transition densities of the processes.

In this paper, by using different methods, we show that the restrictions on the transition densities of the processes in Becker-Kern et al. [1] can be removed and thus verify their conjectures on the Hausdorff and packing dimensions of $X([0,1])$. More specifically, we apply two methods to calculate the Hausdorff dimension of the range of an operator stable Lévy process in $\mathbb{R}^d$. The first method is based on the covering argument for determining the Hausdorff dimension and is closely related to the arguments of Pruitt and Taylor [22] and Hendricks [9]. The second method is more analytic and is based on (1.3) and a result of Khoshnevisan and Xiao [13]. Compared to the arguments in Becker-Kern et al. [1], our methods in this paper make use of other characteristics of an operator stable Lévy process than its transition densities and hence they are more general. In particular, the covering method allows us to obtain a formula for $\dim_{\mathrm{H}} X(E)$ for every Borel set $E \subset \mathbb{R}_+$.

The rest of the paper is organized as follows. In Section 2, we recall the definitions and some useful properties about operator stable laws, operator self-similar processes, operator stable Lévy processes, Hausdorff dimension and packing dimension. Our main results are stated and proved in Section 3. The key for the proofs is Lemma 3.4, which establishes the estimates on the expected sojourn times of X in the ball $B(0,a)$. In Section 4, we give an analytic proof of the result on $\dim_{\mathrm{H}} X([0,1])$ by using (1.3) and list some open problems.

Throughout this paper, we will use K to denote unspecified positive finite constants which may not necessarily be the same in each occurrence. More specific constants will be denoted by $K_1, K_2, \ldots$.

2. Preliminaries

A Lévy process $X = \{X(t), t \in \mathbb{R}_+\}$ in $\mathbb{R}^d$ $(d > 1)$ is called *operator stable* if the distribution ν of $X(1)$ is *full* [i.e., not supported on any $(d-1)$-dimensional hyperplane] and ν is *strictly operator stable*, i.e., there exists a linear operator B on $\mathbb{R}^d$ such that

$$\nu^t = t^B \nu \quad \text{for all } t > 0, \tag{2.1}$$

where ν^t denotes the t-fold convolution power of the infinitely divisible law ν and $t^B\nu(\mathrm{d}x) = \nu(t^{-B}\,\mathrm{d}x)$ is the image measure of ν under the linear operator t^B, which is defined by

$$t^B = \sum_{n=0}^{\infty} \frac{(\log\, t)^n}{n!}\, B^n.$$

The linear operator B is called a *stability exponent* of X. The set of all possible exponents of an operator stable law is characterized in Theorem 7.2.11 of Meerschaert and Scheffler [18].

On the other hand, a stochastic process $X = \{X(t), t \in \mathbb{R}_+\}$ with values in $\mathbb{R}^d$ is said to be *operator self-similar* if there exists a linear operator B on $\mathbb{R}^d$ such that for every $c > 0$,

$$\{X(ct), t \geqslant 0\} \stackrel{d}{=} \left\{c^B X(t), t \geqslant 0\right\},$$

where $X \stackrel{d}{=} Y$ denotes that the two processes X and Y have the same finite-dimensional distributions. Here the linear operator B is called a *self-similarity exponent* of X.

Hudson and Mason [11] proved that if X is a Lévy process in $\mathbb{R}^d$ such that the distribution of $X(1)$ is full, then X is operator self-similar if and only if $X(1)$ is strictly operator stable. In this case, every stability exponent B of X is also a self-similarity exponent of X. Hence, from now on, we will simply refer to B as an exponent of X.

Operator stable Lévy processes are scaling limits of random walks on $\mathbb{R}^d$, normalized by linear operators; see Meerschaert and Scheffler [18, Chapter 11]. Clearly, all strictly stable Lévy processes in $\mathbb{R}^d$ of index α are operator stable with exponent $B = \alpha^{-1}I$, where I is the identity operator in $\mathbb{R}^d$. More generally, let $X_1, \ldots, X_d$ be independent stable Lévy processes in $\mathbb{R}$ with indices $\alpha_1, \ldots, \alpha_d \in (0, 2]$, respectively, and define the Lévy process $X = \{X(t), t \geqslant 0\}$ by

$$X(t) = (X_1(t), \ldots, X_d(t)).$$

Then it is easy to verify that X is an operator stable Lévy process with exponent B which has $\alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_d^{-1}$ on the diagonal and 0 elsewhere. This class of Lévy processes was first studied by Pruitt and Taylor [22]. Following their terminology, we still call X a *Lévy process with stable components*. This type of Lévy processes is sometimes useful in constructing counterexamples (see [18]) and has been studied by several authors. Examples of operator stable Lévy process with dependent components can be found in Shieh [25] and Becker-Kern et al. [1]. For systematic information about operator stable laws and operator stable Lévy processes, we refer to Meerschaert and Scheffler [18].

Let $X = \{X(t), t \geqslant 0\}$ be an operator stable Lévy process in $\mathbb{R}^d$ with exponent B. Factor the minimal polynomial of B into $q_1(x) \cdots q_p(x)$, where all roots of $q_i(x)$ have real part a_i and $a_i < a_j$ for $i < j$. Let $\alpha_i = a_i^{-1}$ so that $\alpha_1 > \cdots > \alpha_p$, and note that $0 < \alpha_i \leqslant 2$ in view of Meerschaert and Scheffler [18, Theorem 7.2.1]. Define $V_i = \mathrm{Ker}(q_i(B))$ and $\dim(V_i) = d_i$. Then $d_1 + \cdots + d_p = d$ and $V_1 \oplus \cdots \oplus V_p$ is a direct sum decomposition of $\mathbb{R}^d$ into B-invariant subspaces. We may write $B = B_1 \oplus \cdots \oplus$

B_p, where $B_i : V_i \to V_i$ and every eigenvalue of B_i has real part equal to a_i. The matrix for B in an appropriate basis is then block-diagonal with p blocks, the ith block corresponding to the matrix for B_i. Write $X(t) = X^{(1)}(t) + \cdots + X^{(p)}(t)$ with respect to this direct sum decomposition, and note that by Corollary 7.2.12 of Meerschaert and Scheffler [18] we get the same decomposition for any exponent B. Since V_i is a B-invariant subspace it follows easily that $\{X^{(i)}(t), t \in \mathbb{R}_+\}$ is an operator stable Lévy process on the d_i-dimensional vector space V_i with exponent B_i. It follows from (2.1) that $X(t) \stackrel{d}{=} t^B X(1)$ and $X^{(i)}(t) \stackrel{d}{=} t^{B_i} X^{(i)}(1)$ for all $1 \leqslant i \leqslant p$. Choose an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$ such that $V_i \perp V_j$ for $i \neq j$, and let $\|x\|^2 = \langle x, x \rangle$ be the associated Euclidean norm. Then

$$\left\| t^B X(1) \right\|^2 = \left\| t^{B_1} X^{(1)}(1) \right\|^2 + \cdots + \left\| t^{B_p} X^{(p)}(1) \right\|^2. \tag{2.2}$$

The following lemma is a slight variant of Lemmas 3.3 and 3.4 in Becker-Kern et al. [1] which can also be proven directly using Corollary 2.2.5 in Meerschaert and Scheffler [18].

Lemma 2.1. *For every $i = 1, \ldots, p$ and every $\varepsilon > 0$, there exists a finite constant $K \geqslant 1$ such that*

$$K^{-1}\, t^{a_i+\varepsilon} \leqslant \left\| t^{B_i} \right\| \leqslant K\, t^{a_i-\varepsilon} \quad \text{for all } 0 < t \leqslant 1 \tag{2.3}$$

and

$$K^{-1}\, t^{-(a_i-\varepsilon)} \leqslant \left\| t^{-B_i} \right\| \leqslant K\, t^{-(a_i+\varepsilon)} \quad \text{for all } 0 < t \leqslant 1. \tag{2.4}$$

Now we recall briefly the definitions of Hausdorff and packing dimensions and refer to Falconer [4,6] Mattila [16] for more information.

Let Φ be the class of functions $\varphi : (0, \delta) \to (0, \infty)$ which are right continuous, monotone increasing with $\varphi(0+) = 0$ and such that there exists a finite constant $K > 0$ such that

$$\frac{\varphi(2s)}{\varphi(s)} \leqslant K, \quad \text{for } 0 < s < \frac{1}{2}\delta. \tag{2.5}$$

The inequality (2.5) is usually called a doubling property. A function φ in Φ is often called a *measure function*.

For $\varphi \in \Phi$, the *φ-Hausdorff measure* of $E \subseteq \mathbb{R}^d$ is defined by

$$\varphi\text{-}m(E) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_i \varphi(2r_i) : E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i < \varepsilon \right\}, \tag{2.6}$$

where $B(x, r)$ denotes the open ball of radius r centered at x. The sequence of balls satisfying the two conditions on the right-hand side of (2.6) is called an *ε-covering* of E. It is well-known that φ-m is a metric outer measure and every Borel set in $\mathbb{R}^d$ is φ-m measurable. A function $\varphi \in \Phi$ is called an *exact Hausdorff measure function* for E if $0 < \varphi\text{-}m(E) < \infty$.

The *Hausdorff dimension* of E is defined by

$$\dim_{\mathrm{H}} E = \inf\{\alpha > 0 : s^\alpha\text{-}m(E) = 0\} = \sup\{\alpha > 0 : s^\alpha\text{-}m(E) = \infty\}.$$

Packing dimension and packing measure were introduced by Tricot [31], Taylor and Tricot [30] as a dual concept to Hausdorff dimension and Hausdorff measure. For $\varphi \in \Phi$, define the set function φ-$P(E)$ on $\mathbb{R}^d$ by

$$\varphi\text{-}P(E) = \lim_{\varepsilon \to 0} \sup\left\{ \sum_i \varphi(2r_i) : \overline{B}(x_i, r_i) \text{ are disjoint, } x_i \in E,\ r_i < \varepsilon \right\}, \tag{2.7}$$

where $\overline{B}$ denotes the closure of B. A sequence of closed balls satisfying the conditions on the right-hand side of (2.7) is called an *ε-packing* of E. Unlike φ-m, the set function φ-P is not an outer measure because it fails to be countably subadditive. However, φ-P is a premeasure, so one can obtain an outer measure φ-p on $\mathbb{R}^d$ by defining

$$\varphi\text{-}p(E) = \inf\left\{ \sum_n \varphi\text{-}P(E_n) : E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}. \tag{2.8}$$

φ-$p(E)$ is called the *φ-packing measure* of E. Taylor and Tricot [30] proved that φ-$p(E)$ is a metric outer measure; hence every Borel set in $\mathbb{R}^d$ is φ-p measurable. If $\varphi(s) = s^\alpha$, s^α-$p(E)$ is called the α-dimensional packing measure of E. The *packing dimension* of E is defined by

$$\dim_{\mathrm{P}} E = \inf\{\alpha > 0 : s^\alpha\text{-}p(E) = 0\} = \sup\{\alpha > 0 : s^\alpha\text{-}p(E) = \infty\}. \tag{2.9}$$

There is an equivalent definition for $\dim_{\mathrm{P}} E$ which is sometimes more convenient to use. For any $\varepsilon > 0$ and any bounded set $E \subseteq \mathbb{R}^d$, let

$N(E, \varepsilon)$ = smallest number of balls of radius ε needed to cover E.

Then the *upper and lower box-counting dimension* of E are defined as

$$\overline{\dim}_{\mathrm{B}} E = \limsup_{\varepsilon \to 0} \frac{\log N(E, \varepsilon)}{-\log \varepsilon}$$

and

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{\varepsilon \to 0} \frac{\log N(E, \varepsilon)}{-\log \varepsilon},$$

respectively. If $\overline{\dim}_{\mathrm{B}}(E) = \underline{\dim}_{\mathrm{B}}(E)$, the common value is called the *box-counting dimension* of E. From the definitions, it is easy to verify that

$$0 \leqslant \dim_{\mathrm{H}} E \leqslant \underline{\dim}_{\mathrm{B}} E \leqslant \overline{\dim}_{\mathrm{B}} E \leqslant d \quad \text{and} \quad 0 \leqslant \dim_{\mathrm{P}} E \leqslant \overline{\dim}_{\mathrm{B}} E \leqslant d \tag{2.10}$$

for all bounded sets $E \subseteq \mathbb{R}^d$. Hence $\overline{\dim}_{\mathrm{B}} E$ and $\underline{\dim}_{\mathrm{B}} E$ can be used to determine upper bounds for $\dim_{\mathrm{H}} E$ and $\dim_{\mathrm{P}} E$.

The disadvantage of $\overline{\dim}_{\mathrm{B}}$ and $\underline{\dim}_{\mathrm{B}}$ as dimensions is that they are not σ-stable [cf. 31; 4, P. 45]. One can obtain σ-stable indices $\overline{\dim}_{\mathrm{MB}}$ and $\underline{\dim}_{\mathrm{MB}}$ by letting

$$\overline{\dim}_{\mathrm{MB}} E = \inf\left\{ \sup_n \overline{\dim}_{\mathrm{B}} E_n : E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}, \tag{2.11}$$

$$\underline{\dim}_{\mathrm{MB}} E = \inf\left\{ \sup_n \underline{\dim}_{\mathrm{B}} E_n : E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}.$$

Tricot [31] has proved that $\dim_{\mathrm{P}} E = \overline{\dim}_{\mathrm{MB}}(E)$. Hence, for all sets $E \subseteq \mathbb{R}^d$,

$$0 \leqslant \dim_{\mathrm{H}} E \leqslant \underline{\dim}_{\mathrm{MB}} E \leqslant \overline{\dim}_{\mathrm{MB}} E = \dim_{\mathrm{P}} E \leqslant d. \tag{2.12}$$

Thus, if $\dim_{\mathrm{H}} E = \dim_{\mathrm{P}} E$, then all the dimensions in (2.12) coincide.

3. Main results

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in $\mathbb{R}^d$ with exponent B. Recall from Section 2 the direct sum decomposition $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ and the associated block-diagonal representation $B = B_1 \oplus \cdots \oplus B_p$, where $d_i = \dim V_i$, $B_i : V_i \to V_i$ and every eigenvalue of B_i has real part equal to $a_i > 0$. We assume that $a_1 < a_2 < \cdots < a_p$, and we let $\alpha_i = a_i^{-1}$ so that $2 \geqslant \alpha_1 > \cdots > \alpha_p > 0$.

The following are our main results. Theorem 3.1 removes the condition on the density of $X(t)$ in Theorem 2.2 of Becker-Kern et al. [1] and extends their results to $X(E)$. This solves the problems in Remarks 3.8 and 3.9 of their paper.

Theorem 3.1. *For any Borel set $E \subset \mathbb{R}_+$, almost surely*

$$\dim_{\mathrm{H}} X(E) = \begin{cases} \alpha_1 \dim_{\mathrm{H}} E & \text{if } \dim_{\mathrm{H}} E \leqslant d_1/\alpha_1, \\ 1 + \alpha_2(\dim_{\mathrm{H}} E - 1/\alpha_1) & \text{otherwise.} \end{cases} \tag{3.1}$$

The next result shows that the range $X([0,1])$ has the same Hausdorff and packing dimensions, which confirms a conjecture of Becker-Kern et al. ([1, Remark 3.10]).

Theorem 3.2. *Let X be an operator stable Lévy process in $\mathbb{R}^d$. Then*

$$\dim_{\mathrm{H}} X([0,1]) = \dim_{\mathrm{P}} X([0,1]) = \begin{cases} \alpha_1 & \text{if } \alpha_1 \leqslant d_1, \\ 1 + \alpha_2(1 - 1/\alpha_1) & \text{otherwise.} \end{cases} \tag{3.2}$$

We break the proofs of Theorems 3.1 and 3.2 into several parts. The upper bounds in Theorems 3.1 and 3.2 are proved by using Lemmas 3.3 and 3.4 and a covering argument which goes back to Pruitt and Taylor [22] and Hendricks [8,9]; while the lower bounds are proved by using Lemma 3.7 and (2.12).

Let $K_1 > 0$ be a fixed constant. A collection $\Lambda(a)$ of cubes of side a in $\mathbb{R}^d$ is called K_1-nested if no ball of radius a in $\mathbb{R}^d$ can intersect more than K_1 cubes of $\Lambda(a)$. In this paper, we will let $\Lambda(a)$ be the collection of all cubes of the form $\prod_{j=1}^d [k_j a, (k_j+1)a]$, where $(k_1, \ldots, k_d) \in \mathbb{Z}^d$. Clearly, $\Lambda(a)$ is K_1-nested with $K_1 = 3^d$. In particular, for each integer $n \geqslant 1$ and $a = 2^{-n}$, $\Lambda(a)$ is just the collection of dyadic cubes of order n in $\mathbb{R}^d$. Another example of 3^d-nested collections of cubes is the set of all semi-dyadic cubes of order n in $\mathbb{R}^d$.

Let

$$T(a,s) = \int_0^s \mathbb{1}_{B(0,a)}(X(t))\,\mathrm{d}t$$

be the sojourn time of X in $B(0,a)$ up to time s, where $\mathbb{1}_B$ is the indicator function of the set B. The following useful covering lemma is due to Pruitt and Taylor [22].

Lemma 3.3. *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a Lévy process in $\mathbb{R}^d$ and let $\Lambda(a)$ be a fixed K_1-nested collection of cubes of side a $(0 < a \leqslant 1)$ in $\mathbb{R}^d$. For any $u \geqslant 0$, we denote by $M_u(a,s)$ the number of cubes in $\Lambda(a)$ hit by $X(t)$ at some time $t \in [u, u+s]$. Then*

$$\mathbb{E}[M_u(a,s)] \leqslant 2K_1 s[\mathbb{E}(T(a/3,s))]^{-1}.$$

The following lemma gives estimates on the expected sojourn time $T(a,s)$. Even though we only need to use the lower bounds for $\mathbb{E}[T(a,s)]$ in this paper, we also include the upper bounds which may be useful elsewhere. For example, sharp upper bounds for $\mathbb{E}[T(a,s)]$ will be useful for studying the exact Hausdorff measure functions for the range $X([0,1])$.

Lemma 3.4. *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in $\mathbb{R}^d$. For any $0 < \alpha_2' < \alpha_2 < \alpha_2'' < \alpha_1' < \alpha_1 < \alpha_1''$, there exist positive and finite constants $K_2, \ldots, K_5$ such that the following hold*:

(i) *If $\alpha_1 \leqslant d_1$, then for all $0 < a \leqslant 1$ and $a^{\alpha_1} \leqslant s \leqslant 1$,*

$$K_2\, a^{\alpha_1''} \leqslant \mathbb{E}[T(a,s)] \leqslant K_3\, a^{\alpha_1'}. \tag{3.3}$$

(ii) *If $\alpha_1 > d_1$, then for all $a > 0$ small enough, say, $0 < a \leqslant a_0$, and all $a^{\alpha_2} \leqslant s \leqslant 1$,*

$$K_4\, a^{\rho''} \leqslant \mathbb{E}[T(a,s)] \leqslant K_5\, a^{\rho'}, \tag{3.4}$$

where $\rho' = 1 + \alpha_2'(1 - 1/\alpha_1)$ and $\rho'' = 1 + \alpha_2''(1 - 1/\alpha_1)$.

Proof. We assume first $\alpha_1 \leqslant d_1$ and let $\alpha_1' < \alpha_1$ be fixed. By the operator self-similarity of X and (2.2), we have $\|X(t)\| \overset{d}{=} \|t^B X(1)\| \geqslant \|t^{B_1} X^{(1)}(1)\|$. Since $\|Ax\| \geqslant \|x\|/\|A^{-1}\|$ for any vector $x \in \mathbb{R}^{d_1}$ and any invertible linear operator A on $\mathbb{R}^{d_1}$, we use (2.4) in Lemma 2.1 to derive that

$$\left\|t^{B_1} X^{(1)}(1)\right\| \geqslant K\, t^{1/\alpha_1'} \left\|X^{(1)}(1)\right\| \quad \text{for all } 0 < t \leqslant 1.$$

Since $X^{(1)}(1)$ has a continuous and bounded density, it follows that

$$\begin{aligned}
\mathbb{E}[T(a,s)] &\leqslant \int_0^s \mathbb{P}\left(\left\|X^{(1)}(t)\right\| < a\right) \mathrm{d}t \\
&\leqslant \int_0^s \mathbb{P}\left(\left\|X^{(1)}(1)\right\| < K\, a t^{-1/\alpha_1'}\right) \mathrm{d}t \\
&\leqslant \int_0^{a^{\alpha_1'}} \mathrm{d}t + \int_{a^{\alpha_1'}}^\infty K (a t^{-1/\alpha_1'})^{d_1}\, \mathrm{d}t \\
&\leqslant K_3\, a^{\alpha_1'},
\end{aligned} \tag{3.5}$$

which gives the upper bound in (3.3). To prove the lower bound in (3.3), we fix α_i'' $(1 \leqslant i \leqslant p)$ such that $\alpha_i'' > \alpha_i > \alpha_{i+1}''$. It follows from (2.2) and (2.3) in Lemma 2.1 that

$$\begin{aligned}
\mathbb{E}[T(a,s)] &\geqslant \int_0^s \mathbb{P}\left(\|X^{(i)}(t)\| < \frac{a}{\sqrt{p}},\ 1 \leqslant i \leqslant p \right) \mathrm{d}t \\
&\geqslant \int_0^s \mathbb{P}\left(\|X^{(i)}(1)\| < K \frac{a}{\sqrt{p}}\, t^{-1/\alpha_i''},\ 1 \leqslant i \leqslant p \right) \mathrm{d}t \\
&\geqslant \int_0^{(\delta a)^{\alpha_1''}} \mathbb{P}\left(\|X^{(i)}(1)\| < K \frac{a}{\sqrt{p}}\, t^{-1/\alpha_i''},\ 1 \leqslant i \leqslant p \right) \mathrm{d}t \\
&\geqslant K_3\, a^{\alpha_1''},
\end{aligned} \tag{3.6}$$

where $0 < \delta < 1$ is a constant such that $\mathbb{P}\left(\|X(1)\| \leqslant \frac{K}{\delta\sqrt{p}} \right) > 0$. Such $\delta > 0$ exists because $X(1)$ is full. So the probability in the last integral is bounded below by a positive constant. Hence (3.3) follows from (3.5) and (3.6).

Now we consider the case when $\alpha_1 > d_1 = 1$. Note that $(X^{(1)}(1), X^{(2)}(1))$ has a continuous bounded density. Similar to (3.5), we have for any $\alpha_2' < \alpha_2$,

$$\begin{aligned}
\mathbb{E}[T(a,s)] &\leqslant \int_0^s \mathbb{P}(|X^{(1)}(t)| < a,\ \|X^{(2)}(t)\| < a)\, \mathrm{d}t \\
&\leqslant \int_0^s \mathbb{P}(|X^{(1)}(1)| < a\, t^{-1/\alpha_1},\ \|X^{(2)}(1)\| < Ka\, t^{-1/\alpha_2'})\, \mathrm{d}t \\
&\leqslant \int_0^{a^{\alpha_2'}} a\, t^{-1/\alpha_1}\, \mathrm{d}t + \int_{a^{\alpha_2'}}^\infty K\, a^{1+d_2}\, t^{-1/\alpha_1'-d_2/\alpha_2'}\, \mathrm{d}t \\
&\leqslant K_5\, a^{\rho'}.
\end{aligned}$$

On the other hand, similar to (3.6) we have

$$\mathbb{E}[T(a,s)] \geqslant \int_0^s \mathbb{P}\left(|X^{(1)}(1)| < \frac{a}{\sqrt{p}}\, t^{-1/\alpha_1},\ \|X^{(i)}(1)\| < K_6 \frac{a}{\sqrt{p}}\, t^{-1/\alpha_i''},\ 2 \leqslant i \leqslant p \right) \mathrm{d}t \tag{3.7}$$

for some constant $K_6 > 0$. Denote by $g(x_1, \ldots, x_p)$ the density function of $X(1)$. Then the density function of $X^{(1)}(1)$ is given by

$$g_1(x_1) = \int_{\mathbb{R}^{d-1}} g(x_1, x_2, \cdots, x_p)\, \mathrm{d}x_2 \cdots \mathrm{d}x_p.$$

Since $X^{(1)}(1)$ is a strictly stable random variable with index $\alpha_1 > 1$, by Theorem 1 of Taylor [27] its distribution is of *type A*, i.e., $g_1(0) > 0$. Combining this with the continuity of g, we see that there exist a super-rectangle $I = [-m, m] \times J$ in $\mathbb{R}^d$, where $m > 0$ is a constant and J is a cube in $\mathbb{R}^{d-1}$, and a constant $\ell > 0$ such that $g(x_1, \ldots, x_p) \geqslant \ell$ for all $(x_1, \ldots, x_p) \in I$. Now we choose a constant $\delta \in (0, 1)$ such that

$$J \subset \left\{ (x_2, \ldots, x_p) \in \mathbb{R}^{d-1} :\ \|x_i\| \leqslant \frac{K_6}{\delta\sqrt{p}},\ 2 \leqslant i \leqslant p \right\}$$

and let $\eta = 1/(m\sqrt{p})$. Note that $t \geqslant (\eta a)^{\alpha_1}$ implies $\frac{a}{\sqrt{p}} t^{-1/\alpha_1} \leqslant m$. Furthermore, since $\alpha_1 > \alpha_2''$, there exists a constant $0 < a_0 \leqslant 1$ such that for all $0 < a \leqslant a_0$, we have $(\eta a)^{\alpha_1} < (\delta a)^{\alpha_2''}$. Hence, it follows from (3.7) that

$$\begin{aligned}
&\mathbb{E}[T(a,s)] \\
&\geqslant \int_{(\eta a)^{\alpha_1}}^{(\delta a)^{\alpha_2''}} \mathbb{P}\left(|X^{(1)}(1)| < \frac{a}{\sqrt{p}}\, t^{-1/\alpha_1},\ \|X^{(i)}(1)\| < K_6\, \frac{a}{\sqrt{p}}\, t^{-1/\alpha_i''},\ 2 \leqslant i \leqslant p \right) \mathrm{d}t \\
&\geqslant \int_{(\eta a)^{\alpha_1}}^{(\delta a)^{\alpha_2''}} \int_{[-a/\sqrt{p}\, t^{-1/\alpha_1},\, \frac{a}{\sqrt{p}} t^{-1/\alpha_1}]\times J} g(x_1, x_2, \ldots, x_p)\, \mathrm{d}x_1 \cdots \mathrm{d}x_p\, \mathrm{d}t \\
&\geqslant K\,\ell \int_{(\eta a)^{\alpha_1}}^{(\delta a)^{\alpha_2''}} \frac{a}{\sqrt{p}}\, t^{-1/\alpha_1}\, \mathrm{d}t \\
&\geqslant K_4 a^{\rho''}
\end{aligned}$$

for some constant $K_4 > 0$ that may depend on the constants m, δ, p and the cube J. This finishes the proof of (3.4). $\square$

Now we can prove the upper bounds in Theorems 3.1 and 3.2.

Lemma 3.5. *For any Borel set $E \subset \mathbb{R}_+$, almost surely*

$$\dim_{\mathrm{H}} X(E) \leqslant \begin{cases} \alpha_1 \dim_{\mathrm{H}} E & \text{if } \dim_{\mathrm{H}} E \leqslant d_1/\alpha_1, \\ 1 + \alpha_2(\dim_{\mathrm{H}} E - 1/\alpha_1) & \text{otherwise} \end{cases} \tag{3.8}$$

and

$$\dim_{\mathrm{P}} X(E) \leqslant \begin{cases} \alpha_1 \dim_{\mathrm{P}} E & \text{if } \dim_{\mathrm{P}} E \leqslant d_1/\alpha_1, \\ 1 + \alpha_2(\dim_{\mathrm{P}} E - 1/\alpha_1) & \text{otherwise.} \end{cases} \tag{3.9}$$

Remark 3.6. It should be pointed out that, unlike (3.8), the upper bounds for $\dim_{\mathrm{P}} X(E)$ in (3.9) may not be sharp even when X is a Brownian motion, cf. Talagrand and Xiao [26]. The problem for determining $\dim_{\mathrm{P}} X(E)$ for operator stable Lévy processes is still open (cf. Problem 4.3).

Proof. We only prove (3.8). A similar argument also yields that for every bounded set $E \subset \mathbb{R}_+$, almost surely

$$\overline{\dim}_{\mathrm{B}} X(E) \leqslant \begin{cases} \alpha_1 \overline{\dim}_{\mathrm{B}} E & \text{if } \overline{\dim}_{\mathrm{B}} E \leqslant d_1/\alpha_1, \\ 1 + \alpha_2(\overline{\dim}_{\mathrm{B}} E - 1/\alpha_1) & \text{otherwise.} \end{cases} \tag{3.10}$$

Then (3.9) follows from (2.11) and (3.10).

Assume first that $\dim_{\mathrm{H}} E \leqslant d_1/\alpha_1$. For any $\gamma > \dim_{\mathrm{H}} E$, we choose $\alpha_1'' > \alpha_1$ such that $\gamma' \hat{=} 1 - \frac{\alpha_1''}{\alpha_1} + \gamma > \dim_{\mathrm{H}} E$. Then for every $\varepsilon > 0$, there exists a sequence $\{I_i\}$ of intervals in $\mathbb{R}_+$ with length $|I_i| < \varepsilon$ such that

$$E \subseteq \bigcup_{i=1}^{\infty} I_i \quad \text{and} \quad \sum_{i=1}^{\infty} |I_i|^{\gamma'} < 1. \tag{3.11}$$

For each interval I_i, let $s_i = |I_i|$ and $b_i = |I_i|^{1/\alpha_1}$. It follows from Lemmas 3.3 and 3.4 that $X(I_i)$ can be covered by M_i cubes $C_{i,j} \in \Lambda(b_i)$ of sides b_i in $\mathbb{R}^d$ and

$$\mathbb{E}(M_i) \leqslant K\,|I_i| \cdot |I_i|^{-\alpha_1''/\alpha_1}. \tag{3.12}$$

Note that

$$X(E) \subset \bigcup_i \bigcup_{j=1}^{M_i} C_{i,j}$$

and the diameter of $C_{i,j}$ is $\sqrt{d}\,b_i$. That is, $\{C_{i,j}\}$ is a $(\sqrt{d}\,\varepsilon^{1/\alpha_1})$-covering of $X(E)$. It follows from (3.11) and (3.12) that

$$\begin{aligned}\mathbb{E}\left(\sum_{i=1}^{\infty} M_i\, b_i^{\alpha_1\gamma}\right) &\leqslant K \sum_{i=1}^{\infty} |I_i|^{1-\alpha_1''/\alpha_1} \cdot |I_i|^{\gamma}\\ &= K \sum_{i=1}^{\infty} |I_i|^{\gamma'} < K.\end{aligned}$$

Letting $\varepsilon \to 0$ and using Fatou's lemma, we have $\mathbb{E}(s^{\alpha_1\gamma}\text{-}m(X(E))) \leqslant K$. Thus $s^{\alpha_1\gamma}$-$m(X(E)) < \infty$ a.s. which implies that $\dim_{\mathrm{H}} X(E) \leqslant \alpha_1\gamma$ a.s. Since $\gamma > \dim_{\mathrm{H}} E$ is arbitrary, we obtain (3.8) in the case when $\dim_{\mathrm{H}} E \leqslant d_1/\alpha_1$.

Now we consider the case when $\dim_{\mathrm{H}} E > d_1/\alpha_1$. This implies that $\alpha_1 > 1$ and $d_1 = 1$. For any $\gamma > \dim_{\mathrm{H}} E$, we choose $\alpha_2'' > \alpha_2$ such that

$$\gamma' \hat{=} 1 - \frac{\alpha_2''}{\alpha_2} + \frac{\alpha_2''}{\alpha_2}\,\gamma > \dim_{\mathrm{H}} E. \tag{3.13}$$

So there exists a sequence $\{I_i\}$ of intervals in $\mathbb{R}_+$ such that (3.11) holds. Let $s_i = |I_i|$ and $b_i = |I_i|^{1/\alpha_2}$. Denote by M_i the number of cubes $C'_{i,j} \in \Lambda(b_i)$ of side b_i in $\mathbb{R}^d$ that meet $X(I_i)$. Then by Lemmas 3.3 and 3.4,

$$\mathbb{E}(M_i) \leqslant K\,|I_i| \cdot |I_i|^{-\rho''/\alpha_2}, \tag{3.14}$$

where we recall that $\rho'' = 1 + \alpha_2''(1 - \frac{1}{\alpha_1})$. It follows from (3.14) and (3.11) that

$$\begin{aligned}\mathbb{E}\left(\sum_{i=1}^{\infty} M_i\, b_i^{1+\alpha_2''(\gamma-\alpha_1^{-1})}\right) &\leqslant K \sum_{i=1}^{\infty} |I_i|^{1-\rho''/\alpha_2} \cdot |I_i|^{\alpha_2''\gamma/\alpha_2+(1-\alpha_2''\alpha_1^{-1})/\alpha_2}\\ &= K \sum_{i=1}^{\infty} |I_i|^{1-\alpha_2''/\alpha_2+\alpha_2''\gamma/\alpha_2}\\ &= K \sum_{i=1}^{\infty} |I_i|^{\gamma'} < K.\end{aligned}$$

The same argument as in the first part yields $\dim_{\mathrm{H}} X(E) \leqslant 1 + \alpha_2(\dim_{\mathrm{H}} E - 1/\alpha_1)$ a.s. Thus we have proven (3.8). $\square$

Lemma 3.7 below proves the lower bounds of $\dim_{\mathrm{H}} X(E)$ in Theorem 3.1. Similar results under more restrictive conditions [such as either $d = 1$ or independence among the components of X] can be found in Falconer [5] and Lin and Xiao [15]. By taking $E=[0,1]$ and using (2.12), we obtain the desired lower bound for $\dim_{\mathrm{P}} X([0,1])$ in Theorem 3.2.

Lemma 3.7. *Let $X=\{X(t), t\in\mathbb{R}_+\}$ be an operator stable Lévy process in $\mathbb{R}^d$. Then for any Borel set $E\subset\mathbb{R}_+$, almost surely*

$$\dim_{\mathrm{H}} X(E) \geqslant \begin{cases} \alpha_1 \dim_{\mathrm{H}} E & \text{if } \dim_{\mathrm{H}} E \leqslant d_1/\alpha_1, \\ 1+\alpha_2(\dim_{\mathrm{H}} E - 1/\alpha_1) & \text{otherwise.} \end{cases} \tag{3.15}$$

Proof. For the proof of (3.15), we use a standard capacity argument; see e.g., Kahane [12, Chapter 10], Falconer [4, Chapter 4], Mattila [16, Chapter 8], Taylor [28] or Xiao [33]. Note that Frostman's lemma and theorem are only proved for compact sets in Kahane [12]. Both of them are still valid for all Borel sets as shown in Falconer [4, Chapter 4] and Mattila [16, Chapter 8].

First consider the case when $\dim_{\mathrm{H}} E \leqslant d_1/\alpha_1$. If $\dim_{\mathrm{H}} E = 0$, there is nothing to prove. So we assume $\dim_{\mathrm{H}} E>0$. For any $0<\gamma<\alpha_1\dim_{\mathrm{H}} E$, we choose $0<\alpha_1'<\alpha_1$ such that $\gamma<\alpha_1'\dim_{\mathrm{H}} E$. Then, it follows from Frostman's lemma [cf. 12,16] that there exists a probability measure σ on E such that

$$\int_E\int_E \frac{\sigma(\mathrm{d}s)\sigma(\mathrm{d}t)}{|s-t|^{\gamma/\alpha_1'}} < \infty. \tag{3.16}$$

By Frostman's theorem [cf. 12,16], we know that, in order to prove $\dim_{\mathrm{H}} X(E)\geqslant\gamma$ almost surely, is suffices to show

$$\int_E\int_E \mathbb{E}(\|X(s)-X(t)\|^{-\gamma})\sigma(\mathrm{d}s)\sigma(\mathrm{d}t) < \infty. \tag{3.17}$$

It follows from (2.2) that for all $s,t\in\mathbb{R}_+$ such that $|s-t|\leqslant 1$,

$$\begin{aligned} \mathbb{E}(\|X(s)-X(t)\|^{-\gamma}) &= \mathbb{E}(\||s-t|^B X(1)\|^{-\gamma}) \\ &\leqslant \mathbb{E}(\||s-t|^{B_1} X^{(1)}(1)\|^{-\gamma}) \\ &\leqslant K|s-t|^{-\gamma/\alpha_1'}, \end{aligned} \tag{3.18}$$

where in deriving the last inequality, we have used Lemma 2.1 and the elementary fact that if a random variable X in $\mathbb{R}^{d_1}$ has a bounded density, then for any $0<\gamma<d_1$, $\mathbb{E}(\|X\|^{-\gamma})<\infty$. Also, a simple argument using Lemma 2.1 shows that

$$\sup_{|s-t|\geqslant 1} \mathbb{E}(\|X(s)-X(t)\|^{-\gamma}) < \infty.$$

Now it is clear that (3.17) follows from (3.18) and (3.16).

Now we consider the case when $\dim_{\mathrm{H}} E > 1/\alpha_1$ and $d_1=1$. Let $1<\gamma<1+\alpha_2(\dim_{\mathrm{H}} E - 1/\alpha_1)$ be fixed. Note that since $\rho=\gamma/\alpha_2-(1/\alpha_2-1/\alpha_1)<\dim_{\mathrm{H}} E$, we can choose $0<\alpha_2'<\alpha_2$ such that $\rho'=\gamma/\alpha_2'-(1/\alpha_2'-1/\alpha_1)<\dim_{\mathrm{H}} E$. Then there exists a probability measure σ on E such that

$$\int_E\int_E \frac{\sigma(\mathrm{d}s)\sigma(\mathrm{d}t)}{|s-t|^{\rho'}} < \infty. \tag{3.19}$$

Similar to (3.18), we use (2.2) to deduce that for all $s, t \in \mathbb{R}_+$ such that $|s-t| \leqslant 1$,

$$\begin{aligned}
&\mathbb{E}(\|X(s)-X(t)\|^{-\gamma})\\
&\quad = \mathbb{E}(\||t-s|^{B}X(1)\|^{-\gamma})\\
&\quad \leqslant \mathbb{E}([|t-s|^{2/\alpha_1}|X^{(1)}(1)|^2 + \||t-s|^{B_2}X^{(2)}(1)\|^2]^{-\gamma/2})\\
&\quad \leqslant K\int_{\mathbb{R}^{1+d_2}} \frac{1}{|s-t|^{\gamma/\alpha_1}|x_1|^{\gamma} + \||s-t|^{B_2}x_2\|^{\gamma}}\, g(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&\quad \leqslant K\int_{\mathbb{R}^{1+d_2}} \frac{1}{|s-t|^{\gamma/\alpha_1}|x_1|^{\gamma} + |s-t|^{\gamma/\alpha_2'}\|x_2\|^{\gamma}}\, g(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&\quad = K|s-t|^{-\gamma/\alpha_1}\int_{\mathbb{R}^{1+d_2}} \frac{1}{|x_1|^{\gamma} + |s-t|^{\gamma/\alpha_2'-\gamma/\alpha_1}\|x_2\|^{\gamma}}\, g(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2,
\end{aligned}\tag{3.20}$$

where $g(x_1, x_2)$ is the density function of $(X^{(1)}(1), X^{(2)}(1))$ which is bounded and continuous. We will use integration by parts to derive an upper bound for the integral J in (3.20). To this end, let

$$F(r_1,r_2) = \mathbb{P}(|X^{(1)}(1)| \leqslant r_1,\ \|X^{(2)}(1)\| \leqslant r_2).$$

Then by using spherical coordinates, we can write

$$\begin{aligned}
F(r_1,r_2) &= \int_{|x_1|\leqslant r_1}\int_{\|x_2\|\leqslant r_2} g(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1\\
&= \int_{-r_1}^{r_1}\int_0^{r_2}\int_{S_{d_2-1}} \tilde{g}(\rho_1,\rho_2\theta)\,\rho_2^{d_2-1}\mu(\mathrm{d}\theta)\,\mathrm{d}\rho_2\,\mathrm{d}\rho_1,
\end{aligned}\tag{3.21}$$

where $\tilde{g}(y_1, y_2\theta)$ is bounded and continuous in $(y_1, y_2, \theta) \in \mathbb{R}\times\mathbb{R}_+\times S_{d_2-1}$ and μ is the surface measure on the unit sphere S_{d_2-1} in $\mathbb{R}^{d_2}$. Note that there also exists a finite constant $K_7 > 0$ such that

$$F(r_1,r_2) \leqslant (1\wedge K_7 r_1)(1\wedge K_7 r_2) \quad \text{for all } r_1, r_2 \geqslant 0. \tag{3.22}$$

For simplicity of notation, we denote $c = |s-t|^{1/\alpha_2'-1/\alpha_1}$. By using Fubini's theorem and integration by parts when integrating $\mathrm{d}r_1$, we deduce

$$\begin{aligned}
J &= \int_0^\infty\int_0^\infty \frac{1}{r_1^{\gamma}+c^{\gamma}r_2^{\gamma}}\, F(\mathrm{d}r_1,\mathrm{d}r_2)\\
&= \int_0^\infty \mathrm{d}r_2\int_0^\infty\left[\frac{\gamma r_1^{\gamma-1}}{(r_1^{\gamma}+c^{\gamma}r_2^{\gamma})^2}\int_0^{r_1}\int_{S_{d_2-1}}\tilde{g}(\rho_1,r_2\theta)r_2^{d_2-1}\,\mu(\mathrm{d}\theta)\mathrm{d}\rho_1\right]\mathrm{d}r_1\\
&= \int_0^1 \mathrm{d}r_2\int_0^\infty[\cdots]\,\mathrm{d}r_1 + \int_1^\infty \mathrm{d}r_2\int_0^\infty[\cdots]\,\mathrm{d}r_1\\
&\triangleq J_1 + J_2.
\end{aligned}\tag{3.23}$$

Now we estimate J_1 and J_2 separately. Since $\tilde{g}$ is bounded, we have

$$
\begin{aligned}
J_1 &\leqslant K \int_0^1 r_2^{d_2-1}\, \mathrm{d}r_2 \int_0^\infty \frac{\gamma r_1^{\gamma-1}}{(r_1^\gamma + c^\gamma r_2^\gamma)^2}\, r_1\, \mathrm{d}r_1 \\
&\leqslant K \int_0^1 r_2^{d_2-1} \frac{1}{(cr_2)^{\gamma-1}}\, \mathrm{d}r_2 \int_0^\infty \frac{\gamma s_1^\gamma}{(s_1^\gamma + 1)^2}\, \mathrm{d}s_1 \\
&= \frac{K_8}{c^{\gamma-1}} = K_8\, |s-t|^{-(\gamma-1)(1/\alpha_2'-1/\alpha_1)},
\end{aligned}
\tag{3.24}
$$

In getting the second inequality above, we have used the change of variable $r_1 = cr_2\, s_1$. Also note that since $1 < \gamma < \alpha_1 \leqslant 2 \leqslant d_2 + 1$, the last two integrals are convergent and K_8 is a positive and finite constant.

On the other hand, it follows from Fubini's theorem and integration by parts for $\mathrm{d}r_2$ that

$$
\begin{aligned}
J_2 = &- \int_{S_{d_2-1}} \mu(\mathrm{d}\theta) \int_0^\infty \mathrm{d}r_1 \frac{\gamma r_1^{\gamma-1}}{(r_1^\gamma + c^\gamma)^2} \left[\int_0^1 \int_0^{r_1} \tilde{g}(\rho_1, \rho_2\theta)\rho_2^{d_2-1}\, \mathrm{d}\rho_1\, \mathrm{d}\rho_2 \right] \\
&+ \int_{S_{d_2-1}} \mu(\mathrm{d}\theta) \int_0^\infty \mathrm{d}r_1 \int_1^\infty \frac{2\gamma^2 c^\gamma r_1^{\gamma-1}\, r_2^{\gamma-1}}{(r_1^\gamma + c^\gamma r_2^\gamma)^3} \\
&\times \left[\int_0^{r_2} \int_0^{r_1} \tilde{g}(\rho_1, \rho_2\theta)\rho_2^{d_2-1}\, \mathrm{d}\rho_1\, \mathrm{d}\rho_2 \right] \mathrm{d}r_2 \\
\leqslant &\int_0^\infty \mathrm{d}r_1 \int_1^\infty \frac{2\gamma^2 c^\gamma r_1^{\gamma-1}\, r_2^{\gamma-1}}{(r_1^\gamma + c^\gamma r_2^\gamma)^3} \\
&\times \left[\int_0^{r_2} \int_0^{r_1} \int_{S_{d_2-1}} \tilde{g}(\rho_1, \rho_2\theta)\rho_2^{d_2-1}\, \mu(\mathrm{d}\theta)\mathrm{d}\rho_1\, \mathrm{d}\rho_2 \right] \mathrm{d}r_2.
\end{aligned}
$$

Note that the triple integral in the brackets is $F(r_1, r_2)$, thus (3.22) together with a change of variables $r_1 = cr_2\, s_1$ implies that

$$
\begin{aligned}
J_2 &\leqslant K \int_1^\infty \mathrm{d}r_2 \int_0^\infty \frac{c^\gamma r_1^{\gamma-1}\, r_2^{\gamma-1}}{(r_1^\gamma + c^\gamma r_2^\gamma)^3}\, r_1\, \mathrm{d}r_1 \\
&\leqslant \frac{K}{c^{\gamma-1}} \int_1^\infty \frac{1}{r_2^\gamma}\mathrm{d}r_2 \int_0^\infty \frac{s_1^\gamma}{(s_1^\gamma + 1)^3}\, \mathrm{d}s_1 \\
&= K_9\, |s-t|^{-(\gamma-1)(1/\alpha_2'-1/\alpha_1)}.
\end{aligned}
\tag{3.25}
$$

Here we have used again the fact that $\gamma > 1$.

Combining (3.20), (3.23), (3.24) and (3.25), we have proven that for $|s-t| \leqslant 1$

$$
\mathbb{E}(\|X(s) - X(t)\|^{-\gamma}) \leqslant K\, |s-t|^{(1/\alpha_2'-1/\alpha_1)-\gamma/\alpha_2'} = K\, |s-t|^{-\rho'}. \tag{3.26}
$$

Again a simple argument using (2.2) and Lemma 2.1 shows that

$$
\sup_{|s-t| \geqslant 1} \mathbb{E}(\|X(s) - X(t)\|^{-\gamma}) \leqslant \mathbb{E}\Big[(|X^{(1)}(1)|^2 + \|X^{(2)}(1)\|^2)^{-\gamma/2} \Big] < \infty.
$$

Therefore, it follows from (3.19) and (3.26) that (3.17) holds. Using Frostman's theorem again, we have $\dim_{\mathrm{H}} X(E) \geqslant \gamma$ a.s. This finishes the proof of Lemma 3.7.

4. Further remarks and open questions

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a Lévy process in $\mathbb{R}^d$ with Lévy exponent ψ. Recently, Khoshnevisan et al. [14] have proved the following formula for $\dim_{\mathrm{H}} X([0,1])$ in terms of ψ: almost surely

$$\dim_{\mathrm{H}} X([0,1]) = \sup\left\{\alpha < d : \int_{\xi \in \mathbb{R}^d:\ \|\xi\| > 1} \mathrm{Re}\left(\frac{1}{1+\psi(\xi)}\right) \frac{\mathrm{d}\xi}{\|\xi\|^{d-\alpha}} < +\infty\right\}. \tag{4.1}$$

This gives a different, analytic way to study the Hausdorff dimension of $X([0,1])$ for Lévy processes. We refer to Khoshnevisan and Xiao [13] for further developments on Hausdorff dimension and capacity. The following result is an extension of Proposition 7.7 (see also Remark 7.8) of Khoshnevisan and Xiao [13], as well as the result of Pruitt and Taylor [22] for Lévy processes with stable components.

Proposition 4.1. *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a Lévy process in $\mathbb{R}^d$ with Lévy exponent ψ. If ψ satisfies the following condition: there are constants $2 \geqslant \beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_d > 0$ such that for every $\varepsilon > 0$, there exists a constant $\tau > 1$ such that*

$$\frac{K^{-1}}{\|\xi\|^{\varepsilon}\sum_{j=1}^{d}|\xi_j|^{\beta_j}} \leqslant \mathrm{Re}\left(\frac{1}{1+\psi(\xi)}\right) \leqslant \frac{K\|\xi\|^{\varepsilon}}{\sum_{j=1}^{d}|\xi_j|^{\beta_j}}, \quad \forall \xi \in \mathbb{R}^d \text{ with } \|\xi\| \geqslant \tau, \tag{4.2}$$

where $K \geqslant 1$ is a constant which may depend on ε and τ. Denote $n_1 = \max\{j : \beta_j = \beta_1\}$. Then almost surely,

$$\dim_{\mathrm{H}} X([0,1]) = \begin{cases} \beta_1 & \text{if } \beta_1 \leqslant n_1, \\ 1 + \beta_2(1 - 1/\beta_1) & \text{otherwise.} \end{cases} \tag{4.3}$$

Proof. The proof, based on (4.1), is a slight modification of that of Proposition 7.7 of Khoshnevisan and Xiao [13]. Hence it is omitted. $\square$

Proposition 4.1 leads to a completely different proof of the Hausdorff dimension of $X([0,1])$ for operator stable Lévy processes.

Theorem 4.2. *Let X be an operator stable Lévy process in $\mathbb{R}^d$ as in Theorem 3.2. Then (3.2) holds almost surely.*

Proof. In the notations of Section 3, we will show that for every $\varepsilon > 0$, there exists a constant $K \geqslant 1$ such that (4.2) holds for $\beta_1 \geqslant \cdots \geqslant \beta_d$ defined by $\beta_j = \alpha_\ell$ if $\sum_{i=0}^{\ell-1} d_i < j \leqslant \sum_{i=\ell-1}^{\ell} d_i$, where $d_0 = 0$. Once this is proved, the theorem will follow from Proposition 4.1 with $n_1 = d_1$.

The proof is based on asymptotic inverses, a method first used in Meerschaert [17] to get sharp bounds on the probability tails of operator stable random vectors. Use the Jordan decomposition (see, e.g., [18, Theorem 2.1.16]) to obtain a basis $b_1, \ldots, b_d$ for $\mathbb{R}^d$ in which B is block-diagonal where every block is of the form

$$\begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 1 & a & 0 & \cdots & 0 \\ 0 & 1 & a & & \vdots \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & 1 & a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} C & 0 & 0 & \cdots & 0 \\ I & C & 0 & \cdots & 0 \\ 0 & I & C & & \vdots \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & I & C \end{pmatrix}, \tag{4.4}$$

where a is a real eigenvalue of B in the first case, and in the second case

$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.5}$$

where $a \pm ib$ is a complex conjugate pair of eigenvalues of B. Define $||x|| = \sqrt{\langle x, x\rangle}$ using the inner product associated with this basis (see, e.g., [18, Proposition 1.1.20 (b)]), so that $\langle b_i, b_j\rangle = I\ (i = j)$. In these coordinates, the matrix power t^B can be explicitly computed (see [18, Lemma 2.2.3]) as well as the norm $\|t^B x\|$ for every $x \in \mathbb{R}^d$ (see [18, Proof of Theorem 2.2.4]). This follows easily from the (unique) decomposition $B = S + N$ where S is semi-simple (diagonalizable over the complex numbers) and N is nilpotent ($N^m = 0$ for some positive integer m). In the first case [i.e., B is a $(k+1) \times (k+1)$ block as the first matrix in (4.4)], if $x = (x_1, \ldots, x_{k+1})$ are the coordinates for one block and $z(t) = t^B x = (z_1(t), \ldots, z_{k+1}(t))$, then

$$z_j(t) = \sum_{n=0}^{j-1} \frac{t^a (\log t)^n}{n!} x_{j-n} \tag{4.6}$$

for all $j = 1, \ldots, k+1$. In the second case, if $u = (x_1, y_1, \ldots, x_{k+1}, y_{k+1})$ are coordinates for one block and $t^B u = (z_1(t), w_1(t), \ldots, z_{k+1}(t), w_{k+1}(t))$, then

$$\begin{aligned} z_j(t) &= \sum_{n=0}^{j-1} \frac{t^a (\log t)^n}{n!} (\cos(b \log t)\, x_{j-n} - \sin(b \log t)\, y_{n-j}), \\ w_j(t) &= \sum_{n=0}^{j-1} \frac{t^a (\log t)^n}{n!} (\sin(b \log t)\, x_{j-n} + \cos(b \log t)\, y_{n-j}) \end{aligned} \tag{4.7}$$

for all $j = 1, \ldots, k+1$. Recall from Section 2 the direct sum decomposition $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ and the associated block-diagonal representation $B = B_1 \oplus \cdots \oplus B_p$, where $B_i : V_i \to V_i$ and every eigenvalue of B_i has real part equal to a_i. Now apply Theorem 3.1 in Meerschaert and Veeh [19] to obtain a further direct sum decomposition $V_i = U_{i1} \oplus \cdots \oplus U_{iq(i)}$ where U_{ij} is a B-invariant subspace and every non-zero vector $x \in U_{ij}$ is of order j, so that $N^j x = 0$ and $N^{j-1} x \neq 0$. Note that every basis element $b_1, \ldots, b_d$ lies in one of these subspaces. Write $x = \sum_i \sum_j x_{ij}$ with respect to this direct sum decomposition, so that $x_{ij} \in U_{ij}$. Then it follows from (4.6) and (4.7) that

$$\|t^{-B^*}x\|^2 = \sum_{i=1}^{p}\sum_{j=1}^{q(i)} \frac{t^{-2a_i}(\log t)^{2(j-1)}}{((j-1)!)^2}\|x_{ij}\|^2 + o_{ij}(t,x), \tag{4.8}$$

where B^* is the transpose of B, $o_{ij}(t,x)$ is a linear combination of terms of the form $t^{-2a_i}(\log t)^k \times |x_{ijr}|\,|x_{ijs}|$ with $k < 2(j-1)$, x_{ijr} is one of the coordinates of x_{ij} in the basis $b_1,\dots,b_d$, and the coefficients of this linear combination are independent of both x and t. Then clearly, $R(t) = 1/\|t^{-B^*}x\|$ is a regularly varying function of $t > 0$ at infinity with index $a = \min\{a_i : x_i \neq 0\}$ where $x = \sum_i x_i$ with respect to the direct sum decomposition $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$. Of course the function $R(t) = 1/\|t^{-B^*}x\|$ (as well as $t(r)$ below) also depends on x. We have suppressed x so that the notation will not get too heavy.

Since $a > 0$, the function $R(t)$ has an asymptotic inverse $t(r)$, regularly varying at infinity with index $\alpha = 1/a$, such that $R(t(r)) \sim r$ as $r \to \infty$ (see, e.g., [3, p. 28] or [24, p. 21]). In fact, we can take

$$t(r) = \sum_{i=1}^{p}\sum_{j=1}^{q(i)} K_{ij} r^{\alpha_i}(\log r)^{\alpha_i(j-1)}\|x_{ij}\|^{\alpha_i}, \tag{4.9}$$

where $K_{ij} = (\alpha_i^{j-1}/(j-1)!)^{\alpha_i}$, and the convergence of $R(t(r))/r \to 1$ as $r \to \infty$ is uniform in x on compact sets of $\mathbb{R}^d\backslash\{0\}$. To see this, let i be the index such that $a_i = a$, and let $j = k+1$ be the order of x_i. Then by (4.8) we can write

$$\|t^{-B^*}x\|^2 = \frac{t^{-2a}(\log\ t)^{2k}}{(k!)^2}\|x_{ij}\|^2 + o(t^{-2a}(\log\ t)^{2k}) \quad \text{as } t \to \infty, \tag{4.10}$$

and the convergence is uniform in x on compact sets of $\mathbb{R}^d\backslash\{0\}$. Similarly, it follows from (4.9) that as $r \to \infty$

$$t(r) = K_{ij}\, r^{\alpha}(\log r)^{\alpha k}\|x_{ij}\|^{\alpha} + o(r^{\alpha}(\log r)^{\alpha k}) \tag{4.11}$$

uniformly for x on compact sets of $\mathbb{R}^d\backslash\{0\}$.

Now it suffices to show that $R(t(r))^{-2} = \|t(r)^{-B^*}x\|^2 \sim r^{-2}$ as $r \to \infty$ uniformly for x on compact sets of $\mathbb{R}^d\backslash\{0\}$. This follows from (4.10), (4.11) and an elementary computation:

$$\begin{aligned} R(t(r))^{-2} &= \frac{\alpha^{2k}}{(k!)^2}\left[K_{ij}\, r^{\alpha}(\log r)^{\alpha k}\|x_{ij}\|^{\alpha}\right]^{-2a}(\log r)^{2k}\,\|x_{ij}\|^2 + \cdots \\ &= r^{-2} + o(r^{-2}) \quad \text{as } r \to \infty, \end{aligned} \tag{4.12}$$

where the convergence is uniform in x on compact sets of $\mathbb{R}^d\backslash\{0\}$. This establishes our claim.

Since $X(t)$ and $t^B X(1)$ are identically distributed we have

$$t\psi(\xi) = \psi(t^{B^*}\xi) \tag{4.13}$$

for all $\xi \in \mathbb{R}^d$ and all $t > 0$. Let $F(\xi) = \mathrm{Re}\,(\psi(\xi))$ so that $tF(\xi) = F(t^{B^*}\xi)$ for all $t > 0$ and $\xi \in \mathbb{R}^d$. Moreover, $F(\xi)$ is bounded away from zero and infinity on compact subsets of $\mathbb{R}^d\backslash\{0\}$ since $X(t)$ is full (see, e.g., [18, Corollary 7.1.12]).

Given $x \in \mathbb{R}^d\backslash\{0\}$ and $r > 0$, we define $\theta_r = t(r)^{-B^*}(rx)$. Then it follows from the above that as $r \to \infty$, $\|\theta_r\| = r\|t(r)^{-B^*}x\| = r/R(t(r)) \to 1$ uniformly for x on compact sets in $\mathbb{R}^d$. Consequently for every $0 < \eta < 1$, there exists some $r_0 > 0$ such that $1 - \eta < \|\theta_r\| < 1 + \eta$ for all $r \geqslant r_0$ and all $x \in S_d$. Here $S_d = \{x : \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^d$.

For any $\xi \in \mathbb{R}^d\backslash\{0\}$, let $r = \|\xi\|$ and $x = \xi/r \in S_d$ so that $\xi = rx$. It follows that

$$F(rx) = F(t(r)^{B^*}\theta_r) = t(r)F(\theta_r) \tag{4.14}$$

and $F(\theta_r)$ is bounded away from zero and infinity for all $r \geqslant r_0$ and $x \in S_d$. On the other hand, for any $\varepsilon > 0$, there is a constant $\tau \geqslant \max\{r_0, e\}$ such that for all $r \geqslant \tau$, $(\log r)^{\alpha_i(q(i)-1)} \leqslant r^{\varepsilon/2}$ for every $1 \leqslant i \leqslant p$. Therefore, it follows from (4.14) and (4.9) that

$$\begin{aligned} F(\xi) &\leqslant K \sum_{i=1}^{p} \sum_{j=1}^{q(i)} r^{\alpha_i} (\log r)^{\alpha_i(j-1)} \|x_{ij}\|^{\alpha_i} \\ &\leqslant K\, r^{\varepsilon/2} \sum_{i=1}^{p} (r\,\|x_i\|)^{\alpha_i} = K\,\|\xi\|^{\varepsilon/2} \sum_{i=1}^{p} \|\xi_i\|^{\alpha_i} \end{aligned} \tag{4.15}$$

for all $\|\xi\| \geqslant \tau$. Similarly, we derive from (4.14) and (4.9) that for all $\|\xi\| \geqslant \tau$,

$$\begin{aligned} F(\xi) &\geqslant K' \sum_{i=1}^{p} \sum_{j=1}^{q(i)} r^{\alpha_i} (\log r)^{\alpha_i(j-1)} \|x_{ij}\|^{\alpha_i} \\ &\geqslant K' \sum_{i=1}^{p} \|\xi_i\|^{\alpha_i}. \end{aligned} \tag{4.16}$$

Now we consider $G(\xi) = \mathrm{Im}\,(\psi(\xi))$. Note that (4.13) implies $tG(\xi) = G(t^{B^*}\xi)$ for all $t > 0$ and $\xi \in \mathbb{R}^d$. By the continuity of $\psi(\xi)$, $G(\xi)$ is bounded on compact subsets of $\mathbb{R}^d$. Hence, similar to (4.15), we have that for all $\|\xi\| \geqslant \tau$,

$$|G(\xi)| \leqslant K \sum_{i=1}^{p} \sum_{j=1}^{q(i)} r^{\alpha_i} (\log r)^{\alpha_i(j-1)} \|x_{ij}\|^{\alpha_i} \leqslant K\,\|\xi\|^{\varepsilon/2} \sum_{i=1}^{p} \|\xi_i\|^{\alpha_i}. \tag{4.17}$$

Combining (4.15), (4.16) and (4.17) with the following identity:

$$\mathrm{Re}\left(\frac{1}{1+\psi(\xi)}\right) = \frac{1 + \mathrm{Re}\,\psi(\xi)}{(1 + \mathrm{Re}\,\psi(\xi))^2 + (\mathrm{Im}\,\psi(\xi))^2}, \tag{4.18}$$

we obtain

$$\frac{K_{10}}{\|\xi\|^{\varepsilon}\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}} \leqslant \mathrm{Re}\left(\frac{1}{1+\psi(\xi)}\right) \leqslant \frac{K_{11}}{\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}}, \quad \forall \xi \in \mathbb{R}^d \text{ with } \|\xi\| \geqslant \tau. \tag{4.19}$$

Thus (4.2) holds. This completes our proof. $\square$

We end this section with some open questions. Since $\dim_{\mathrm{H}} X([0,1]) = \dim_{\mathrm{P}} X([0,1])$ is given by Theorem 3.2, it would be interesting to further investigate the following natural question.

Problem 4.3. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in $\mathbb{R}^d$. Find exact Hausdorff and packing measure functions for the range $X([0,1])$.

We also mention that, if X is a Lévy process with stable components in $\mathbb{R}^d$ or an operator stable Lévy process in $\mathbb{R}^d$, no general formula for the packing dimension of $X(E)$ has yet been established. When X is a one-dimensional Brownian motion, the packing dimension of $X(E)$ was studied by Talagrand and Xiao [26] who showed that the inequality $\dim_{\mathrm{P}} X(E) < 2 \dim_{\mathrm{P}} E$ holds for some Cantor-type set $E \subset [0,1]$. Hence the formula analogous to that for $\dim_{\mathrm{H}} X(E)$ does not hold for the packing dimension $\dim_{\mathrm{P}} X(E)$. Xiao [32] proved a formula for $\dim_{\mathrm{P}} X(E)$ in terms of the packing dimension profile of E introduced by Falconer and Howroyd [7]. We believe a result analogous to that in Xiao [32] for Brownian motion still holds for all stable Lévy processes in $\mathbb{R}$ with stability index $\alpha > 1$ [This is the only remaining problem for $\dim_{\mathrm{P}} X(E)$, where X is a stable Lévy process X in $\mathbb{R}^d$ with index α, since Perkins and Taylor [20] have shown that if $\alpha \leqslant d$, then a.s. $\dim_{\mathrm{P}} X(E) = \alpha \dim_{\mathrm{P}} E$ for all Borel sets $E \subset \mathbb{R}_+$]. However, for Lévy processes with stable components in $\mathbb{R}^d$ or operator stable Lévy processes, the packing dimension profile introduced by Falconer and Howroyd [7] does not seem to be appropriate for characterizing $\dim_{\mathrm{P}} X(E)$. One may need to introduce a corresponding concept of packing dimension profile that can capture different growths in different directions.

Shieh [25] has investigated the Hausdorff dimension of the multiple points of a class of operator stable processes including Lévy processes with stable components. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in $\mathbb{R}^d$ with exponent B which has $a_1 \leqslant \cdots \leqslant a_d$ on its diagonal and 0 elsewhere. Let

$$L_k = \left\{x \in \mathbb{R}^d : \exists \text{ distinct } t_1, \dots, t_k \text{ such that } X(t_1) = \cdots = X(t_k) = x\right\}.$$

be the set of k-multiple points of X. Under certain conditions, Shieh [25] proved that for $k \geqslant 2$ almost surely,

$$\dim_{\mathrm{H}} L_k = \min\left\{\alpha_1\left(k - (k-1)\sum_{i=1}^{d} \alpha_i^{-1}\right), d - k\alpha_d\left(\sum_{i=1}^{d} \alpha_i^{-1} - 1\right)\right\}, \qquad (4.20)$$

where $\alpha_i = a_i^{-1}$ $(i = 1, \dots, d)$ and negative dimension means that the set L_k is empty. We believe his result may still be true for all operator stable Lévy processes, where now a_i are the real parts of the eigenvalues of B as described at the beginning of Section 3 and each $\alpha_i = a_i^{-1}$ is repeated $d_i = \dim V_i$ times. It would be interesting to solve the following problem:

Problem 4.4. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in $\mathbb{R}^d$. Let L_k be the set of k-multiple points. Show that (4.20) holds.

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