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Operator geometric stable laws

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Abstract

Operator geometric stable laws are the weak limits of operator normed and centered geometric random sums of independent, identically distributed random vectors. They generalize operator stable laws and geometric stable laws. In this work we characterize operator geometric stable distributions, their divisibility and domains of attraction, and present their application to finance. Operator geometric stable laws are useful for modeling financial portfolios where the cumulative price change vectors are sums of a random number of small random shocks with heavy tails, and each component has a different tail index.

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1. Introduction

We introduce a new class of multivariate distributions called operator geometric stable (OGS), generalizing the geometric stable (GS) and operator stable (OS) laws. Our motivation comes from a problem in finance, where a portfolio of stocks or

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other financial instruments changes price over time, resulting in a time series of random vectors. The daily price change vectors are each accumulations of a random number of random shocks. Price shocks are typically heavy tailed with a tail parameter that is different for each stock [39]. OS models can handle the variations in tail behavior [32] while GS models [11,14,21,24,26,33,34,40,41] capture the fact that these are random sums. The combination of operator norming and geometric randomized sums pursued in this paper should provide a more useful and realistic class of distributions for portfolio modeling. The more general case where the number of summands has an arbitrary distribution is discussed in a companion paper [18]. The focus of this paper on geometric summation allows a simpler treatment, and more complete results.

Let (X_i) be a sequence of independent and identically distributed (i.i.d.) random vectors (r.v.'s) in \mathbb{R}^d . Consider a random sum $X_1 + \cdots + X_{N_p}$, where N_p is a geometric variable with mean $1/p$ independent of the X_i 's. If there exists a weak limit of

$$A_p \sum_{i=1}^{N_p} (X_i + b_p) \quad \text{as } p \rightarrow 0, \quad (1.1)$$

where A_p is a linear operator on \mathbb{R}^d and $b_p \in \mathbb{R}^d$, then we call it an OGS law. The limits of (1.1) under scalar normalization $A_p = a_p \in \mathbb{R}$ are GS vectors (see, e.g., [12,27]). The same limit of a deterministic sum (N_p replaced with positive integer n) is an OS vector (see, e.g., [10,31]).

Each component of an OGS vector may have different tail behavior, unlike GS laws where the tail behavior is the same in every coordinate. All components of an OGS law are dependent, unlike OS laws where the normal and heavy tailed components are necessarily independent. When all components have finite variance, an OGS vector has a skew Laplace distribution (see [14,23]). If the normalizing operator A_p is a scalar, the OGS law is GS. If the normalizing operator A_p is diagonal, the OGS law is marginally GS (all components have geometric stable distributions).

Geometric summation arises naturally in various fields, including biology, economics, insurance mathematics, physics, reliability and queuing theories among others (see, e.g., [11]). Thus, as limits of random sums, OGS laws will undoubtedly find numerous applications in stochastic modeling. Their infinite divisibility provides for a natural model when the variable of interest can be thought of as a random sum of small quantities, which is often the case in finance and insurance. Finally, OGS laws can be asymmetric, which further adds to their modeling applicability. Univariate and multivariate geometric stable distributions, and their special cases of skew Laplace laws compete successfully with stable and other laws in modeling financial asset returns (see, e.g., [14,21,23,24,26,28,34,39]). OGS models, which extend this class to allow different tail behavior for each vector component, should enhance their modeling potential.

The problems of geometric summation and geometric stability with operator norming have also been considered for certain (noncommutative) groups that

include finite dimensional real vector spaces as a special case. See [5–8] for details. As we develop the theory of OGS laws in this paper, we will also point out the relations between our results, and those obtained in the more abstract setting of probability on groups. For the special case of finite dimensional real vector spaces treated in this paper, our treatment using characteristic functions is simpler, and leads to several new results.

In this paper we derive fundamental properties of OGS laws. We start in Section 2 with a brief recounting of essential ideas from the theory of operator stable laws. In Section 3, we present the definition and characterization of OGS laws, their (generalized) domains of attraction, infinite divisibility and stability, and we discuss important special cases. In Section 4, we focus on marginally OGS laws with heavy tail and finite variance components and present an OGS model for financial data. All proofs are collected in Appendix A.

2. OS laws

Suppose X, X_1, X_2, \dots are independent and identically distributed random vectors on \mathbb{R}^d with common distribution μ and that Y_0 is a random vector whose distribution ω is full, i.e., not supported on any lower dimensional hyperplane. We say that ω is OS if there exist linear operators A_n on \mathbb{R}^d and nonrandom vectors $b_n \in \mathbb{R}^d$ such that

$$A_n \sum_{i=1}^n (X_i - b_n) \Rightarrow Y_0. \tag{2.1}$$

In terms of measures, we can rewrite (2.1) as

$$A_n \mu^n * \varepsilon_{s_n} \Rightarrow \omega \tag{2.2}$$

where $A_n \mu(dx) = \mu(A_n^{-1} dx)$ is the probability distribution of $A_n X$, μ^n is the n th convolution power, and ε_{s_n} is the unit mass at the point $s_n = -nA_n b_n$. In this case, we say that μ (or X) belongs to the *generalized domain of attraction* of ω (or Y_0), and we write $\mu \in \text{GDOA}(\omega)$, or $X \in \text{GDOA}(Y_0)$. Theorem 7.2.1 in [31] shows that the OS law ω is infinitely divisible and

$$\omega^t = t^E \omega * \varepsilon_{a_t} \quad \text{for all } t > 0, \tag{2.3}$$

where E is a linear operator called an exponent of ω , $t^E = \exp(E \log t)$ and $\exp(A) = I + A + A^2/2! + A^3/3! + \dots$ is the usual exponential operator. Further, the characteristic function $\hat{\omega}(x) = \mathbb{E}[e^{i\langle x, Y \rangle}]$ satisfies

$$\hat{\omega}(x)^t = \hat{\omega}(t^E x) e^{i\langle a_t, x \rangle} \quad \text{for all } t > 0, \tag{2.4}$$

see, e.g., [18]. If (2.1) holds with all $b_n = 0$ we say that μ belongs to the *strict generalized domain of attraction* of ω . In this case ω is *strictly OS*, that is (2.3) holds with all $a_t = 0$.

3. OGS laws

OGS laws arise as limits of convolutions with geometrically distributed number of terms (see [13]). Let $\{N_p, p \in (0, 1)\}$ be a family of geometric random variables with mean $1/p$, so that

$$P(N_p = n) = p(1 - p)^{n-1}, \quad n = 1, 2, \dots \tag{3.1}$$

Definition 3.1. A full random vector Y on \mathbb{R}^d is OGS if for N_p geometric with mean $1/p$ there exist i.i.d. random vectors X_1, X_2, \dots independent of N_p , linear operators A_p , and centering constants $b_p \in \mathbb{R}^d$ such that

$$A_p \sum_{i=1}^{N_p} (X_i + b_p) \Rightarrow Y \quad \text{as } p \downarrow 0. \tag{3.2}$$

If (3.2) holds we say that the distribution of X_1 is weakly geometrically attracted to that of Y , and the collection of such distributions is called the *generalized domain of geometric attraction* of Y .

The following result shows that there is a one to one correspondence between OS and OGS laws, and provides a fundamental representation of OGS vectors in terms of their OS counterparts. It is an analog of the relation between GS and stable distributions (see, e.g., [21]).

Theorem 3.2. Let Y be a full random vector on \mathbb{R}^d with distribution λ and characteristic function $\hat{\lambda}(t) = \int e^{i\langle t,x \rangle} \lambda(dx)$. Then the following are equivalent:

- (a) Y is OGS;
- (b) $Y \stackrel{d}{=} Z^E X + a_Z$ where X is OS with distribution ω satisfying (2.3), Z is standard exponential, and X, Z are independent;
- (c) $Y \stackrel{d}{=} L(Z)$ where Z is standard exponential and $\{L(s): s \geq 0\}$ is a stochastic process with stationary independent increments, independent of Z , and such that $L(1)$ is OS and $L(0) = 0$ almost surely;
- (d) The distribution λ has the form

$$\lambda(dx) = \int_0^\infty \omega(dx)^t v(dt), \tag{3.3}$$

where ω is an OS probability distribution on \mathbb{R}^d and $v(dt) = e^{-t} dt$;

- (e) The characteristic function $\hat{\lambda}$ has the form

$$\hat{\lambda}(t) = (1 - \log \hat{\omega}(t))^{-1}, \quad t \in \mathbb{R}^d, \tag{3.4}$$

where $\hat{\omega}$ is an OS characteristic function on \mathbb{R}^d .

Remark 3.3. Theorem 3.2 complements known characterizations of OGS laws that follow from more general results on geometric stability on groups. To be more specific, the equivalence of (a), (b) and (d) in Theorem 3.2 in the special case $a_t = 0$ in (2.3) for all $t > 0$ follows from Theorem 2.10 in [7]. See also Proposition 5.3 in [6] and Chapter 2.13 in [8].

3.1. Special cases

Below we list important special cases of OGS laws.

3.1.1. Strictly OGS laws

If the OS law given by the characteristic function $\hat{\omega}$ in (3.4) is strictly OS, then the distribution given by $\hat{\lambda}$ is called *strictly OGS*. For the strictly OGS laws, convergence in (3.2) holds with $b_p = 0$. We also have the representation

$$Y \stackrel{d}{=} Z^E X, \tag{3.5}$$

where Z is a standard exponential variable and X is strictly OS with exponent E (and independent of Z).

3.1.2. GS laws

When the operators in (3.2) are of the form $A_p = a_p I_d$, where $a_p > 0$ and I_d is a d -dimensional identity matrix, then the limiting distributions are called *GS laws* (see, e.g., [16,41], and also [27] for a summary of their properties, applications, and references). The characteristic function of a GS law is of form (3.4) where $\hat{\omega}$ is the characteristic function of some α -stable distribution in \mathbb{R}^d , so that

$$\hat{\lambda}(t) = (1 + I_\alpha(t) - i \langle t, m \rangle)^{-1}, \quad t \in \mathbb{R}^d, \tag{3.6}$$

where $m \in \mathbb{R}^d$ is the location parameter (the mean if $\alpha > 1$) and

$$I_\alpha(t) = \int_{S_d} \omega_{\alpha,1}(\langle t, s \rangle) \Gamma(ds). \tag{3.7}$$

Here, S_d is the unit sphere in \mathbb{R}^d , Γ is a finite measure on S_d , called the *spectral measure*, and

$$\omega_{\alpha,\beta}(u) = \begin{cases} |u|^\alpha \left(1 - i\beta \operatorname{sign}(u) \tan\left(\frac{\pi\alpha}{2}\right) \right) & \text{for } \alpha \neq 1, \\ |u| \left(1 + i\beta \frac{2}{\pi} \operatorname{sign}(u) \log|u| \right) & \text{for } \alpha = 1. \end{cases} \tag{3.8}$$

When $\alpha = 2$ we obtain the special case of multivariate Laplace distribution (see below), while when $\alpha < 2$, the probability $P(Y_j > x)$ associated with each component of a GS random variable Y decreases like the power function $x^{-\alpha}$ as x increases to infinity. As in the stable case, the spectral measure controls the dependence among the components of Y (which are dependent even if Γ is discrete and concentrated on the intersection of S_d with the coordinate axes, in which case the coordinates of the

corresponding stable vector are independent). Note that (3.5) still holds with $E = 1/\alpha$. In one dimension the characteristic function (3.6) reduces to

$$\hat{\lambda}(t) = \mathbb{E}e^{itY} = (1 + \sigma^\alpha \omega_{\alpha,\beta}(t) - i\mu t)^{-1}, \tag{3.9}$$

where the parameter α is the index of stability as before, $\beta \in [-1, 1]$ is a skewness parameter, the parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ control the location and the scale, respectively, and $\omega_{\alpha,\beta}$ is given by (3.8). Although GS distributions have the same type of tail behavior as stable laws, their densities are more peaked near the mode. Since such sharp peaks and heavy tails are often observed in financial data, GS laws have found applications in the area of financial modeling (see, e.g., [21,26,34,35,42]).

3.1.3. Skew Laplace distributions

When the variables X_i in (3.2) are in the (classical) domain of attraction of the normal law (for example, if they have finite second moments), then the characteristic function $\hat{\omega}$ in (3.4) corresponds to a multivariate normal distribution, so that

$$\hat{\lambda}(t) = \frac{1}{1 + \frac{1}{2}\langle t, \Sigma t \rangle - i\langle m, t \rangle}, \quad t \in \mathbb{R}^d, \tag{3.10}$$

where $m \in \mathbb{R}^d$ and Σ is a $d \times d$ nonnegative definite symmetric matrix. These are *multivariate Laplace distributions* (see [23]). In the symmetric case ($m = 0$), we obtain an elliptically contoured distribution with the density

$$g(y) = 2(2\pi)^{-d/2} |\Sigma|^{-1/2} (\langle y, \Sigma^{-1}y \rangle / 2)^{v/2} K_v(\sqrt{2\langle y, \Sigma^{-1}y \rangle}),$$

where $v = (2 - d)/2$ and $K_v(u)$ is the modified Bessel function of the third kind (see, e.g., [1]). In one dimension we obtain a univariate skew Laplace distribution (with an explicit density) studied in [22]. More information on theory and applications of Laplace laws can be found in [14].

3.1.4. Marginally GS laws

If the operators A_p in (3.2) are diagonal matrices $\text{diag}(a_{p1}, \dots, a_{pd})$ for some positive a_{pi} 's, then the one-dimensional marginals of the limiting OGS vector Y are geometric stable with characteristic function (3.9) and possibly different values of α . The characteristic function of Y is given by (3.4) where this time $\hat{\omega}$ corresponds to a marginally stable OS random vector X introduced in [43] and studied in [3,30] (see also [37]). If the values of α for all marginal distributions are strictly less than 2, then the characteristic function of Y takes the form:

$$\hat{\lambda}(t) = \left(1 + \int_{S_d} \int_0^\infty \left(e^{i\langle t, r^E s \rangle} - 1 - \frac{i\langle t, r^E s \rangle}{1 + \|r^E s\|^2} \right) \frac{dr}{r^2} \Gamma(ds) - i\langle t, m \rangle \right)^{-1}, \tag{3.11}$$

where E is a diagonal matrix

$$E = \text{diag}(1/\alpha_1, \dots, 1/\alpha_d), \quad 0 < \alpha_i < 2, \quad i = 1, \dots, d, \tag{3.12}$$

called the *exponent* of Y , the power r^E is a diagonal matrix $\text{diag}(|r|^{1/\alpha_1}, \dots, |r|^{1/\alpha_d})$, the vector $m \in \mathbb{R}^d$ is the shift parameter, and as for the GS vectors, the spectral measure Γ is a finite measure on the unit sphere S_d in \mathbb{R}^d . As in the stable and GS cases, the spectral measure determines the dependence structure among the components of a marginally GS vector. The fact that these distributions allow for a different tail behavior for their marginals makes them, along with marginally stable laws, attractive in financial portfolio analysis (see [35,39]).

3.2. Divisibility and stability properties

Since OS laws are infinitely divisible, and so is exponential distribution ν , in view of (3.3) we conclude that OGS laws are infinitely divisible as well (see, e.g., Property (e), XVII.4 of [4]). Their Lévy representation can be obtained as a special case of the result below, where ν is any infinitely divisible law on \mathbb{R}_+ . In this general case, it follows from Theorem 2, XIII.7 of [4] that the Laplace transform $\tilde{\nu}$ of ν has the form

$$\tilde{\nu}(z) = \int_0^\infty e^{-zt} \nu(dt) = \exp\left(\int_{[0,\infty)} \frac{e^{-zs} - 1}{s} dK(s)\right) \tag{3.13}$$

where K is nondecreasing, continuous from the right, and fulfills

$$\int_1^\infty \frac{1}{s} dK(s) < \infty. \tag{3.14}$$

(When $s = 0$ the integrand in (3.13) is extended by continuity to equal $-z$.) Using this representation, we obtain the following result, which is an extension of one-dimensional cases studied in [9,25].

Theorem 3.4. *Let ω be a full OS law with exponent E and let $g(s, x)$ denote the Lebesgue density of ω^s for any $s > 0$. Assume further that ν is infinitely divisible and that (3.13) holds, where $dK(s)$ has no atom at zero. Then*

$$\lambda = \int_0^\infty \omega^t \nu(dt)$$

is infinitely divisible with Lévy representation $[a, 0, \phi]$, where

$$a = \int_0^\infty \int_{\mathbb{R}^d} \frac{x}{1 + \|x\|^2} g(s, x) dx \frac{1}{s} dK(s) \tag{3.15}$$

and $d\phi(x) = h(x) dx$ with

$$h(x) = \int_0^\infty g(s, x) \frac{1}{s} dK(s). \tag{3.16}$$

Remark 3.5. To obtain the Lévy measure of an OGS distribution λ , use the above result with standard exponential distribution ν , so that $dK(s) = e^{-s} ds$ (no atom at zero!)

Remark 3.6. Under the conditions of Theorem 3.4, λ has no normal component. If $dK(s)$ has an atom $b > 0$ at zero then (3.13) becomes

$$\tilde{v}(z) = \exp\left(\int_{(0,\infty)} \frac{e^{-zs} - 1}{s} dK(s)\right) = \exp\left(-bz + \int_{(0,\infty)} \frac{e^{-zs} - 1}{s} dK(s)\right),$$

so that $\hat{\lambda}(\xi) = \tilde{v}(\psi(\xi)) = e^{-b\psi(\xi)} \hat{\lambda}_1(\xi)$ where $-\psi(\xi)$ is the log-characteristic function of ω and λ_1 is infinitely divisible with Lévy representation $[a, 0, \phi]$ as described in Theorem 3.4. If ω has Lévy representation $[a_2, Q_2, \phi_2]$ then λ is infinitely divisible with Lévy representation $[a + ba_2, bQ_2, \phi + b\phi_2]$. For example, take Z standard exponential, $b > 0$, v the distribution of $b + Z$, ω strictly operator stable with exponent E and X, X_1 i.i.d. with distribution ω . Then the mixture λ defined by (3.3) is the distribution of $b^E X + Z^E X_1$, the sum of two independent infinitely divisible laws.

Remark 3.7. Theorem 3.4 can be obtained as a special case of Theorem 30.1 in Sato [45] for subordinated Lévy processes. Since Sato uses a different form of the Lévy representation, his formula for the centering constant is different.

3.2.1. Geometric infinite divisibility

A random vector Y (and its probability distribution) is said to be *geometric infinitely divisible* if for all $p \in (0, 1)$ we have

$$Y \stackrel{d}{=} \sum_{i=1}^{N_p} Y_{pi}, \tag{3.17}$$

where N_p is geometrically distributed random variable given by (3.1), the variables Y_{pi} are i.i.d. for each p , and N_p and (Y_{pi}) are independent (see, e.g., [12]). Since geometric infinitely divisible laws arise as the weak limits of triangular arrays with geometric number of terms in each row, it follows that the OGS distributions are geometric infinitely divisible.

Proposition 3.8. *Let Y be OGS given by the characteristic function (3.4). Then, Y is geometric infinitely divisible and relation (3.17) holds where the Y_{pi} 's have the characteristic function of the form*

$$\hat{\lambda}_p(t) = (1 - \log \hat{\omega}^p(t))^{-1}. \tag{3.18}$$

3.2.2. Stability with respect to geometric summation

The following characterization of strictly OGS distributions extends similar properties of GS and Laplace distributions (see, e.g., [12,14,16]).

Theorem 3.9. *Let Y, Y_1, Y_2, \dots be i.i.d. random variables in \mathbb{R}^d , and let N_p be a geometrically distributed random variable independent of the sequence (Y_i) . Then*

$$S_p = A_p \sum_{i=1}^{N_p} Y_i \stackrel{d}{=} Y, \quad p \in (0, 1), \tag{3.19}$$

with some operators A_p on \mathbb{R}^d if and only if Y is strictly OGS, in which case Y admits the representation (3.5) for some OS random variable X with exponent E and $A_p = p^E$ for $p \in (0, 1)$.

Remark 3.10. Theorem 3.9 also follows from a more general result on strict geometric stability on nilpotent Lie groups, see Theorem 2.12 in [7] and Theorem 4.3 in [6]. We give a simpler proof using characteristic functions.

The above result can be somewhat strengthened if the operators in (3.19) correspond to diagonal matrices. The following result follows from Theorem 3.9 combined with similar result for GS distributions (see [15, Theorem 3.2]), when we take into account that the stability relation (3.20) holds for each coordinate of Y .

Theorem 3.11. Let Y, Y_1, Y_2, \dots be i.i.d. random variables in \mathbb{R}^d , and let N_p be a geometrically distributed random variable independent of the sequence (Y_i) . Then

$$S_p = A_p \sum_{i=1}^{N_p} (Y_i + b_p) \stackrel{d}{=} Y, \quad p \in (0, 1), \tag{3.20}$$

with some diagonal A_p 's and $b_p \in \mathbb{R}^d$ if and only if Y is marginally strictly GS with representation (3.5), where E is the diagonal matrix (3.12), Z is standard exponential variable, and X is marginally strictly stable with indices $\alpha_1, \dots, \alpha_d$. Moreover, we must necessarily have $b_p = 0$ and $A_p = p^E$ for each p .

4. Infinitely divisible laws with Laplace and Linnik marginals and an application in financial modeling

Here we consider marginally GS laws discussed in Section 3.1.4, whose characteristic exponent (3.12) contains some α_i 's less than two and some equal to two. For simplicity, we focus on a bivariate symmetric case with $\alpha_1 = 2$ and $0 < \alpha_2 < 2$. It is well known that a symmetric bivariate OS r.v. $X = (X_1, X_2)$ with the characteristic exponent (3.12) and the above α_i 's has independent components (see [43]) with characteristic functions

$$\mathbb{E}e^{itX_1} = e^{-\sigma^2 t^2}, \quad t \in \mathbb{R}, \tag{4.1}$$

(normal distribution with mean 0 and variance $2\sigma^2$) and

$$\mathbb{E}e^{isX_2} = e^{-\eta^\alpha |s|^\alpha}, \quad s \in \mathbb{R}, \tag{4.2}$$

(symmetric α stable with scale parameter $\eta > 0$), respectively. Consequently, the ch.f. of $X = (X_1, X_2)$ is

$$\hat{\omega}(t, s) = \mathbb{E}e^{i(tX_1 + sX_2)} = e^{-\sigma^2 t^2 - \eta^\alpha |s|^\alpha}, \quad (t, s) \in \mathbb{R}^2, \tag{4.3}$$

and the corresponding OGS ch.f. (3.4) takes the form

$$\hat{\lambda}(t, s) = \frac{1}{1 + \sigma^2 t^2 + \eta^\alpha |s|^\alpha}, \quad (t, s) \in \mathbb{R}^2. \tag{4.4}$$

The marginal distributions of the OGS r.v. $Y = (Y_1, Y_2)$ with the above ch.f. are classical Laplace and symmetric GS (also called Linnik) distributions with characteristic functions

$$\hat{\lambda}_1(t) = \mathbb{E}e^{itY_1} = \frac{1}{1 + \sigma^2 t^2}, \quad t \in \mathbb{R} \tag{4.5}$$

and

$$\hat{\lambda}_2(s) = \mathbb{E}e^{isY_2} = \frac{1}{1 + \eta^\alpha |s|^\alpha}, \quad s \in \mathbb{R}, \tag{4.6}$$

respectively (see, e.g., [14]). The respective densities are

$$f_1(x) = \frac{1}{2\sigma} e^{-|x|/\sigma}, \quad x \in \mathbb{R}, \tag{4.7}$$

and

$$\begin{aligned} f_2(y) &= \frac{1}{\eta} \int_0^\infty z^{-1/\alpha} p_\alpha\left(\frac{y}{\eta z^{1/\alpha}}\right) e^{-z} dz \\ &= \frac{\sin \frac{\pi\alpha}{2}}{\pi\eta} \int_0^\infty \frac{v^\alpha \exp(-v|y|/\eta) dv}{1 + v^{2\alpha} + 2v^\alpha \cos \frac{\pi\alpha}{2}}, \quad y \neq 0, \end{aligned} \tag{4.8}$$

where p_α is the density of standard symmetric stable law (with ch.f. (4.2) where $\eta = 1$). We shall refer to the above distribution as an OGS law with marginal Laplace and Linnik distributions (in short: MLL distribution), denoting it by $\mathcal{MLL}_\alpha(\sigma, \eta)$.

Since both Laplace and Linnik distributions have been found useful in modeling univariate data (see, e.g., [14] and references therein), multivariate laws with these marginals will also be valuable for modeling data with both power and exponential tail behavior of one-dimensional components. Many financial data exhibit features characteristic of Laplace and Linnik laws—high peak at the mode and relatively slowly converging tail probabilities. We first collect basic properties of bivariate MLL distributions, some of which illustrate results of previous sections, and then fit a bivariate MLL model to foreign currency exchange rates and compare its fit with that of an OS model.

4.1. Basic properties

The following representation that follows from our Theorem 3.2(b) plays an important role in studying bivariate MLL distributions.

Theorem 4.1. *If $Y = (Y_1, Y_2)$ has an $\mathcal{MLL}_\alpha(\sigma, \eta)$ distribution given by the ch.f. (4.4), then*

$$Y \stackrel{d}{=} (Z^{1/2} X_1, Z^{1/\alpha} X_2), \tag{4.9}$$

where the variables Z , X_1 , X_2 are mutually independent, Z is standard exponential, and X_1 , X_2 have normal and symmetric stable distributions with ch.f.'s (4.1) and (4.2), respectively.

Our next result gives the joint density of an MLL random vector.

Theorem 4.2. *The distribution function and density of $Y = (Y_1, Y_2) \sim \mathcal{MLL}_\alpha(\sigma, \eta)$ are, respectively,*

$$F(y_1, y_2) = \int_0^\infty \Phi\left(\frac{y_1}{\sqrt{2z}\sigma}\right) \Psi_\alpha\left(\frac{y_2}{z^{1/\alpha}\eta}\right) e^{-z} dz, \quad (y_1, y_2) \in \mathbb{R}^2, \quad (4.10)$$

and

$$f(y_1, y_2) = C_{\sigma, \eta} \int_0^\infty z^{-1/2-1/\alpha} e^{-z-\frac{y_1^2}{4\sigma^2 z}} p_\alpha\left(\frac{y_2}{z^{1/\alpha}\eta}\right) dz, \quad (y_1, y_2) \neq (0, 0), \quad (4.11)$$

where Φ is the standard normal distribution function, Ψ_α and p_α are the distribution function and the density of standard symmetric α -stable distribution with ch.f. $\exp(-|t|^\alpha)$, and

$$C_{\sigma, \eta} = \frac{1}{2\sqrt{\pi}\sigma\eta} \quad (4.12)$$

The Lévy representation of MLL ch.f. follows from Theorem 3.4.

Theorem 4.3. *The ch.f. of $Y = (Y_1, Y_2) \sim \mathcal{MLL}_\alpha(\sigma, \eta)$ admits Lévy representation $[(a_1, a_2), 0, \phi]$, where*

$$a_i = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{x_i}{1+x_1^2+x_2^2} g(s, x_1, x_2) dx_1 dx_2 \frac{1}{s} e^{-s} ds, \quad i = 1, 2, \quad (4.13)$$

and $d\phi(x) = h(x) dx$, where

$$h(x_1, x_2) = \int_0^\infty g(s, x_1, x_2) \frac{1}{s} e^{-s} ds. \quad (4.14)$$

Here,

$$g(s, x_1, x_2) = \left(2\sqrt{\pi}\sigma\eta s^{1/2+1/\alpha} e^{\frac{x_1^2}{4\sigma^2 s}}\right)^{-1} p_\alpha\left(\frac{x_2}{s^{1/\alpha}\eta}\right), \quad (4.15)$$

where p_α is the density of standard symmetric α -stable distribution with ch.f. $\exp(-|t|^\alpha)$.

MLL distributions have the stability property (3.20) with $A_p = \text{diag}(p^{1/2}, p^{1/\alpha})$ and $b_p = 0$. The following result is an extension of corresponding stability properties of univariate and multivariate Laplace and Linnik distributions (see, e.g., [14]).

Theorem 4.4. Let Y, Y_1, Y_2, \dots be i.i.d. symmetric bivariate random vectors whose first components have finite variance, and let N_p be a geometrically distributed random variable independent of the sequence (Y_i) . Then

$$S_p = A_p \sum_{i=1}^{N_p} (Y_i + b_p) \stackrel{d}{=} Y, \quad p \in (0, 1), \tag{4.16}$$

with some diagonal A_p 's and $b_p \in \mathbb{R}^2$ if and only if Y has an MLL distribution given by the ch.f. (4.4). Moreover, we must necessarily have $b_p = 0$ and $A_p = \text{diag}(p^{1/2}, p^{1/\alpha})$ for each p .

Our next result shows that like Laplace and Linnik laws, the conditional distributions of $Y_2|Y_1 = y$ and of $Y_1|Y_2 = y$ are scale mixtures of stable and normal distributions, respectively.

Theorem 4.5. Let $Y = (Y_1, Y_2) \sim \mathcal{ML}\mathcal{L}_\alpha(\sigma, \eta)$.

(i) The conditional distribution of $Y_2|Y_1 = y \neq 0$ is the same as that of

$$(U + V_y)^{1/\alpha} S, \tag{4.17}$$

where the variables U, V_y , and S are mutually independent, U is gamma distributed with density

$$f_U(x) = \frac{1}{\Gamma(1/2)} x^{-1/2} e^{-x}, \quad x > 0, \tag{4.18}$$

V_y has inverse Gaussian distribution with density

$$f_{V_y}(x) = \frac{|y|e^{|y|/\sigma}}{2\sqrt{\pi}\sigma} x^{-3/2} e^{-\left(\frac{y^2}{4\sigma^2 x} + x\right)}, \quad x > 0, \tag{4.19}$$

and S is symmetric stable with the ch.f. (4.2).

(ii) The conditional distribution of $Y_1|Y_2 = y \neq 0$ is the same as that of

$$Z_y^{1/2} X, \tag{4.20}$$

where X and Z_y are independent, X is normally distributed with mean zero and variance $2\sigma^2$, and Z_y has a weighted exponential distribution with density

$$f_{Z_y}(x) = \frac{\omega(x)e^{-x}}{\int_0^\infty \omega(x)e^{-x} dx}, \quad x > 0. \tag{4.21}$$

The weight function in (4.21) is

$$\omega(x) = x^{-1/\alpha} p_\alpha \left(\frac{y}{\eta x^{1/\alpha}} \right), \quad x > 0, \tag{4.22}$$

where p_α is the density of standard symmetric α -stable distribution.

Remark 4.6. Using the results of [17], we can obtain the weighted exponential r.v. Z_y with density (4.21) from a standard exponential r.v. Z via the transformation

$Z_y \stackrel{d}{=} q(Z)$, where $q = q(z)$ is the unique solution of the equation

$$\int_q^\infty \omega(x)e^{-x} dx = e^{-z} \int_0^\infty \omega(x)e^{-x} dx. \tag{4.23}$$

The following two results deal with tail behavior and joint moments of MLL variables. In the first result we present the exact tail behavior of linear combinations of the components of an MLL r.v., showing that they are heavy tailed with the same tail index α .

Theorem 4.7. *Let $Y = (Y_1, Y_2) \sim \mathcal{MLL}_\alpha(\sigma, \eta)$ and let $(a, b) \in \mathbb{R}^2$ with $a^2 + b^2 > 0$. Then, as $x \rightarrow \infty$, we have*

$$P(aY_1 + bY_2 > x) \sim \begin{cases} \frac{1}{\pi} \eta^\alpha |b|^\alpha \Gamma(\alpha) \sin \frac{\pi\alpha}{2} x^{-\alpha} & \text{for } a \in \mathbb{R}, b \neq 0, \\ \frac{1}{2} e^{-\frac{x}{|a|\sigma}} & \text{for } a \neq 0, b = 0. \end{cases} \tag{4.24}$$

Finally we give condition for the existence of joint moments of MLL random vectors.

Theorem 4.8. *Let $Y = (Y_1, Y_2) \sim \mathcal{MLL}_\alpha(\sigma, \eta)$ and let $\alpha_1, \alpha_2 \geq 0$. Then the joint moment $\mathbb{E}|Y_1|^{\alpha_1} |Y_2|^{\alpha_2}$ exists if and only if $\alpha_2 < \alpha$, in which case we have*

$$\mathbb{E}|Y_1|^{\alpha_1} |Y_2|^{\alpha_2} = \frac{2^{\alpha_1} \sigma^{\alpha_1} \eta^{\alpha_2} (1 - \alpha_2) \Gamma(\frac{\alpha_1}{2} + \frac{\alpha_2}{\alpha} + 1) \Gamma(\frac{\alpha_1}{2} + \frac{1}{2}) \Gamma(1 - \frac{\alpha_2}{\alpha})}{\sqrt{\pi} (2 - \alpha_2) \cos \frac{\pi\alpha_2}{2}}, \tag{4.25}$$

where for $\alpha_2 = 1$ we set $(1 - \alpha_2) / \cos \frac{\pi\alpha_2}{2} = 2/\pi$.

Remark 4.9. Note that for $\alpha_2 = 0$ we obtain absolute moments of classical Laplace distribution (see, e.g., [14]) and for $\alpha_1 = 0$ we get fractional absolute moments of a symmetric Linnik distribution:

$$e(\alpha_2) = \mathbb{E}|Y_2|^{\alpha_2} = \frac{\eta^{\alpha_2} (1 - \alpha_2) \Gamma(\frac{\alpha_2}{\alpha} + 1) \Gamma(1 - \frac{\alpha_2}{\alpha})}{(2 - \alpha_2) \cos \frac{\pi\alpha_2}{2}}. \tag{4.26}$$

The above formula is useful in estimating the parameters of Linnik laws (see, e.g., [14]).

4.2. MLL model for financial asset returns

To illustrate the modeling potential of OGS laws, we fit a portfolio of foreign exchange rates with an MLL model and compare the fit with that of an OS model. We use the data set of foreign exchange rates presented in [32]. The data contains 2853 daily exchange rate logs for the US Dollar versus the German Deutsch Mark

(X_1) and the Japanese Yen (X_2). In order to unmask the variations in tail behavior, we transform the original vector of log-returns using the same linear transformation as in [32] to obtain

$$Z_1 = 0.69X_1 - 0.72X_2 \quad \text{and} \quad Z_2 = 0.72X_1 + 0.69X_2.$$

The tail parameters were estimated in [32] to be 1.998 for Z_1 and 1.656 for Z_2 , indicating that Z_1 fits a finite variance model, whereas Z_2 is heavy tailed. The operator stable model assumes Z_1 normal and Z_2 stable. The OGS model will fit Z_1 with a Laplace and Z_2 with a Linnik law. We estimate all parameters of normal and Laplace models using standard maximum likelihood techniques. The Linnik and stable models are estimated using moment type estimators.

Since we assume symmetry around zero, we only need to estimate the scale of the normal and Laplace (σ in (4.5)) fit. These were 0.999 and 0.7296, respectively. Scale parameters for the stable and Linnik (η in (4.6)) distributions are estimated to be 1.404 and 1.571, respectively. The scale estimator for the Linnik distribution is based on formula (4.26). Because of symmetry, the location and skewness for the stable model are taken as zero. The scale for the stable model (η in (4.2)) is estimated based on the moment formula 1.2.17 in [44].

The goodness-of-fit was assessed using the Kolmogorov distance (KD) and the Anderson-Darling (AD) statistics for the fitted marginals. The former is defined as

$$KD = \sup_x |F(x) - F_n(x)|,$$

where F_n and F are the empirical and the fitted distribution functions, respectively. The latter statistic is

$$AD = \sup_x \frac{|F(x) - F_n(x)|}{\sqrt{F(x)(1 - F(x))}},$$

and measures the goodness-of-fit in the tails. The results are summarized in Table 1.

The KD statistics for Z_1 and Z_2 under the OGS model are about half of those under the operator stable model. The AD statistics are also smaller under the OGS model, indicating that the OGS distribution provides a better fit to this data. Figs. 1 and 2 compare the fit of both distributions to the data. It is clear from the graphs that the OGS model fits the sharp central peak better than the OS model. In

Table 1
The Goodness-of-fit statistics (KD—Kolmogorov distance and AD—Anderson-Darling) for OS and OGS models

	Operator stable		OGS model	
	Z_1	Z_2	Z_1	Z_2
KD	0.04620	0.08825	0.02538	0.05091
AD	2.167	0.183	0.064	0.124

In the stable model, the variables Z_1 and Z_2 have normal and stable distributions, respectively. In the OGS model, they have Laplace and Linnik laws, respectively.

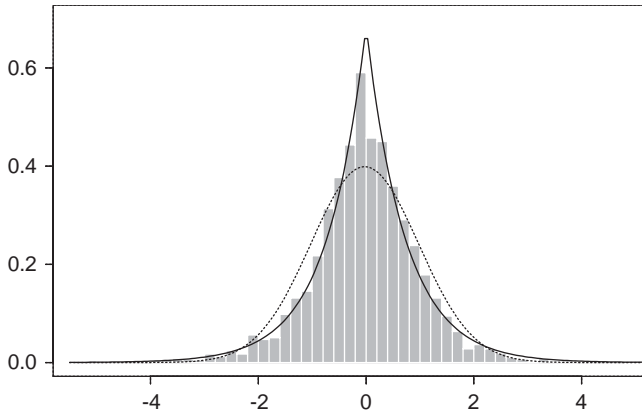


Fig. 1. Histogram of Z_1 with pdf's of normal (dashed line) and Laplace (solid line) models.

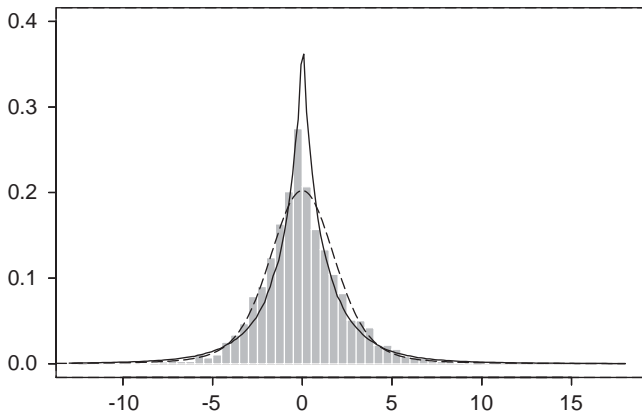


Fig. 2. Histogram of Z_2 with pdf's of stable (dashed line) and Linnik (solid line) models.

conclusion, both quantitative and graphical evidence show that the OGS laws have high potential in modeling and in this case outperform the best OS models.

Remark 4.10. From the view point of risk management, the investor is interested in the distribution of the original log returns X_1 and X_2 . Since the X_i 's are linear functions of the Z_i 's, we can recover their density by change of variables in the density of the Z_i 's. In the above MLL model, the joint density of X_1 and X_2 is estimated as

$$g(x_1, x_2) = f(0.69x_1 - 0.72x_2, 0.72x_1 + 0.69x_2),$$

where f is the MLL density (4.11) with $\alpha = 1.656$, $\sigma = 0.7296$, and $\eta = 1.571$.

Remark 4.11. To compare the fits of stable and Linnik models, for consistency we used the same method of moments to estimate the parameters, although for stable parameters, maximum likelihood estimators (MLEs) are also available (see, e.g., [2,29] for the symmetric case and [36,38] for the skew case). Numerical routines for stable MLEs are available at John P. Nolan website (<http://academic2.american.edu/~jpnolan/>), and also from Bravo Risk Management Group. We used MLEs to estimate stable parameters of Z_2 under the operator stable case as well, obtaining essentially the same results as above; the goodness-of-fit statistics for Z_2 were $KD = 0.0774$ and $AD = 0.183$.

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Appendix A

A.1. Auxiliary results

The following two results taken from [18] will be needed to prove Theorem 3.2.

Theorem A.1. *Suppose that $X \in \text{GDOA}(Y_0)$ and (2.1) holds. If N_n are positive integer-valued random variables independent of (X_i) with $N_n \rightarrow \infty$ in probability, and if $N_n/k_n \Rightarrow Z$ for some random variable $Z > 0$ with distribution ν and some sequence of positive integers (k_n) tending to infinity, then*

$$A_{k_n} \sum_{i=1}^{N_n} (X_i - b_{k_n}) \Rightarrow Y \quad (\text{A.1})$$

where Y has distribution

$$\lambda(dx) = \int_0^\infty \omega(dx)' \nu(dt) \quad (\text{A.2})$$

and ω is the distribution of Y_0 .

Theorem A.2. *Suppose that (X_i) are independent, identically distributed random vectors on \mathbb{R}^d , M_n are positive integer-valued random variables independent of (X_i) with $M_n \rightarrow \infty$ in probability, and*

$$B_n \sum_{i=1}^{M_n} (X_i - a_n) \Rightarrow Y \quad (\text{A.3})$$

for some random vector Y with distribution λ and some linear operators B_n on \mathbb{R}^d and centering constants $a_n \in \mathbb{R}^d$. Then there exists a sequence of positive integers (k_n)

tending to infinity such that for any subsequence (n') there exists a further subsequence (n'') , a random variable $Z > 0$ with distribution ν , and a random vector Y_0 with distribution ω such that $M_{n''}/k_{n''} \Rightarrow Z$,

$$B_{n''} \sum_{i=1}^{k_{n''}} (X_i - a_{n''}) \Rightarrow Y_0, \tag{A.4}$$

and (A.2) holds.

Remark A.3. Theorem A.1 is also called *Gnedenko’s transfer theorem* and has been generalized to various algebraic structures including locally compact groups. See Section 1 in [5] and Chapter 2.12 in [8] for details. The assertion of Theorem A.2 is also known as *Szasz’s compactness theorem*, see [46] for the real valued case. It has been generalized to nilpotent Lie groups in [7].

A.2. Proof of Theorem 3.2

If (d) holds, take Z a random variable with distribution ν and let N_p be a geometric random variables with mean $1/p$. Then $N_p \rightarrow \infty$ a.s. and $pN_p \Rightarrow Z$ as $p \rightarrow 0$. Take Y_0, X_1, X_2, \dots i.i.d. as ω . Since ω is OS, (2.3) shows that

$$(X_1 + \dots + X_n) \stackrel{d}{=} n^E Y_0 + a_n$$

so that (2.1) holds with $A_n = n^{-E}$ and $b_n = n^{-1}a_n$. Given a sequence $p_n \rightarrow 0$ let $k_n = [1/p_n]$ and write $N_n = N_{p_n}$ so that $k_n \rightarrow \infty$ and

$$\frac{N_n}{k_n} = \frac{p_n N_{p_n}}{p_n [1/p_n]} \Rightarrow Z$$

where Z is standard exponential. Then Theorem A.1 shows that (A.1) holds where the limit Y has distribution (A.2). Condition on the value of Z and use (2.3) to show that (b) holds. Since this is true for any sequence $p_n \rightarrow 0$, (a) also holds.

If (d) holds, take Z standard exponential and $\{L(s) : s > 0\}$ a stochastic process with stationary independent increments independent of Z such that $L(s)$ has characteristic function $\hat{\omega}(t)^s$. Then $L(Z)$ has characteristic function

$$\mathbb{E}[\mathbb{E}[e^{i\langle t, L(Z) \rangle} | Z]] = \int_0^\infty \hat{\omega}(t)^s e^{-s} ds = \hat{\lambda}(t) \tag{A.5}$$

so that (c) and (d) are equivalent. Let ψ be the log-characteristic function of $\hat{\omega}(t)$ as in Definition 3.1.1 of [32], so that $\hat{\omega}(t)^s = e^{s\psi(t)}$. Then (A.5) implies

$$\hat{\lambda}(t) = \int_0^\infty e^{s\psi(t)} e^{-s} ds = \frac{1}{1 - \psi(t)} = \frac{1}{1 - \log \hat{\omega}(t)}, \tag{A.6}$$

so that (d) and (e) are equivalent. Note that $\text{Re}(\psi(t)) \leq 0$ since $|e^{\psi(t)}| \leq 1$. To see that (b) implies (d) choose any Borel set $M \subset \mathbb{R}^d$ and use (2.4) to compute

$$\begin{aligned} P\{Y \in M\} &= P\{Z^E X + a_Z \in M\} \\ &= \int_0^\infty P\{Z^E X + a_Z \in M | Z = t\} \nu(dt) \\ &= \int_0^\infty (t^E \omega * a_t)(M) \nu(dt) \\ &= \int_0^\infty \omega^t(M) \nu(dt) \end{aligned}$$

so that (d) holds.

Finally we show that (a) implies (e). The proof is similar to the special case of geometric stable laws (see [33]). Condition on N_p in (3.2) and take characteristic functions to obtain

$$\frac{p\hat{\mu}_p(t)}{1 - (1-p)\hat{\mu}_p(t)} \rightarrow \hat{\lambda}(t) \quad \text{as } p \downarrow 0, \tag{A.7}$$

where $\hat{\mu}_p(t) = \hat{\mu}(A_p^* t) * \varepsilon_{A_p b_p}$ is the characteristic function of $A_p(X + b_p)$ and $\hat{\lambda}$ is the characteristic function of Y . Relation (A.7) can be written equivalently as

$$1 - \frac{1 - (1-p)\hat{\mu}_p(t)}{p\hat{\mu}_p(t)} \rightarrow 1 - \frac{1}{\hat{\lambda}(t)} \quad \text{as } p \downarrow 0 \tag{A.8}$$

assuming that $\hat{\lambda}(t) \neq 0$, which we will verify at the end of the proof. Simplifying we obtain

$$\frac{\hat{\mu}_p(t) - 1}{p\hat{\mu}_p(t)} \rightarrow 1 - \frac{1}{\hat{\lambda}(t)} \quad \text{as } p \downarrow 0, \tag{A.9}$$

and also

$$\left| \frac{\hat{\mu}_p(t) - 1}{\hat{\mu}_p(t)} \right| \rightarrow 0, \quad \text{as } p \downarrow 0. \tag{A.10}$$

Now, by (A.10), we conclude that $\hat{\mu}_p(t) \rightarrow 1$, which combined with (A.9) produces

$$\frac{\hat{\mu}_p(t) - 1}{p} \rightarrow 1 - \frac{1}{\hat{\lambda}(t)} \quad \text{as } p \downarrow 0. \tag{A.11}$$

Letting $p = 1/n$, (A.11) takes the form

$$n(\hat{\mu}(A_{1/n}^* t) * \varepsilon_{A_{1/n} b_{1/n}} - 1) \rightarrow 1 - \frac{1}{\hat{\lambda}(t)} \tag{A.12}$$

which implies that

$$(\hat{\mu}(A_{1/n}^* t) * \varepsilon_{A_{1/n} b_{1/n}})^n \rightarrow \exp\left(1 - \frac{1}{\hat{\lambda}(t)}\right). \tag{A.13}$$

Since the left-hand side of (A.13) is the characteristic function of the centered and operator normalized partial sums of a sequence of i.i.d. random vectors, the right-hand side of (A.13) must be the characteristic function $\hat{\omega}$ of some operator stable law. Solving

$$\exp\left(1 - \frac{1}{\hat{\lambda}(t)}\right) = \hat{\omega}(t) \tag{A.14}$$

for $\hat{\lambda}(t)$ shows that (e) holds. To verify that $\hat{\lambda}(t) \neq 0$, take an arbitrary sequence $p_n \rightarrow 0$ and apply (3.2) to see that (A.3) holds with $M_n = N_{p_n}$, $B_n = A_{p_n}$ and $a_n = -b_{p_n}$. Then Theorem A.2 implies that there exists a sequence of positive integers (k_n) tending to infinity such that for any subsequence (n') there exists a further subsequence (n'') , a random variable $Z > 0$ with distribution ν , and a random vector Y_0 with distribution ω such that $N_{p_{n''}}/k_{n''} \Rightarrow Z$ and (A.4) holds where ω related to λ via (A.2). Since the limit Y_0 in (A.4) is the weak limit of a triangular array, ω is infinitely divisible (cf. Theorem 3.3.4 in [31]). Since $N_{p_{n''}}/k_{n''} \Rightarrow Z$, the law ν of Z is infinitely divisible as the weak limit of infinitely divisible laws. Since (A.2) holds with both ω and ν infinitely divisible, λ is also infinitely divisible (see, e.g., Property (e), XVII of [4]). Then it follows from the Lévy representation (cf. Theorem 3.1.11 in [31]) that $\hat{\lambda}(t) \neq 0$ so that the right-hand side in (A.8) is well-defined.

A.3. Proof of Theorem 3.4

Let $\hat{\omega}(\xi) = e^{-\psi(\xi)}$, where $-\psi(\xi)$ is the log-characteristic function of ω . Then

$$\hat{\lambda}(\xi) = \int_0^\infty \hat{\omega}(\xi)^t \nu(dt) = \int_0^\infty e^{-t\psi(\xi)} \nu(dt) = \tilde{\nu}(\psi(\xi)). \tag{A.15}$$

Moreover, since $e^{-s\psi(\xi)}$ is the characteristic function of ω^s , we have

$$e^{-s\psi(\xi)} = \int_{\mathbb{R}^d} e^{i\langle x, \xi \rangle} g(s, x) dx.$$

Hence, by (3.13) and (A.15), we have

$$\begin{aligned} \hat{\lambda}(\xi) &= \exp\left(\int_0^\infty (e^{-s\psi(\xi)} - 1) \frac{1}{s} dK(s)\right) \\ &= \exp\left(\int_0^\infty \int_{\mathbb{R}^d} [e^{i\langle x, \xi \rangle} - 1] g(s, x) dx \frac{1}{s} dK(s)\right). \end{aligned}$$

Note that

$$F(\xi) = \int_0^\infty (e^{-s\psi(\xi)} - 1) \frac{1}{s} dK(s) = \int_0^\infty \int_{\mathbb{R}^d} [e^{i\langle x, \xi \rangle} - 1] g(s, x) dx \frac{1}{s} dK(s)$$

exists and is the log-characteristic function of the infinitely divisible law λ . Write

$$F(\xi) = \int_0^\infty \int_{\mathbb{R}^d} \left[e^{i\langle x, \xi \rangle} - 1 - \frac{i\langle x, \xi \rangle}{1 + \|x\|^2} \right] g(s, x) dx \frac{1}{s} dK(s) + i\langle a, \xi \rangle$$

$$= I(\xi) + i\langle a, \xi \rangle$$

where a is given by (3.15). We will show below that $I(\xi)$ exists for all $\xi \in \mathbb{R}^d$. Then, since $F(\xi)$ exists, it follows that $a \in \mathbb{R}^d$ exists.

Now let

$$h(x, \xi) = e^{i\langle x, \xi \rangle} - 1 - \frac{i\langle x, \xi \rangle}{1 + \|x\|^2}$$

and note that $|h(x, \xi)| \leq C_1 \|x\|^2$ for $\|x\| \leq 1$ and $|h(x, \xi)| \leq C_2$ for all $x \in \mathbb{R}^d$, where C_1 and C_2 are some constants. In order to show that $I(\xi)$ exists, it suffices to show that

$$\int_0^\infty \int_{\mathbb{R}^d} |h(x, \xi)| g(s, x) dx \frac{1}{s} dK(s) < \infty. \tag{A.16}$$

For $\delta > 0$ write the LHS of (A.16) as

$$\int_0^\delta \int_{\mathbb{R}^d} |h(x, \xi)| g(s, x) dx \frac{1}{s} dK(s) + \int_\delta^\infty \int_{\mathbb{R}^d} |h(x, \xi)| g(s, x) dx \frac{1}{s} dK(s) = I_1 + I_2.$$

In view of (3.14), we have

$$I_2 \leq \int_\delta^\infty \int_{\mathbb{R}^d} C_2 g(s, x) dx \frac{1}{s} dK(s) = C_2 \int_\delta^\infty \frac{1}{s} dK(s) < \infty.$$

On the other hand,

$$\frac{1}{s} \int_{\mathbb{R}^d} |h(x, \xi)| g(s, x) dx = \frac{1}{s} \int_{\mathbb{R}^d} |h(x, \xi)| d\omega^s(x)$$

$$\leq \frac{1}{s} \int_{\mathbb{R}^d} f(x) d\omega^s(x),$$

where f is a bounded C^∞ -function such that $f(0) = 0$ and $|h(x, \xi)| \leq f(x)$ for all $x \in \mathbb{R}^d$. Note that $f \in \mathcal{D}(A)$, where A is the generator of the continuous convolution semigroup $(\omega^t)_{t>0}$. Hence

$$\lim_{s \downarrow 0} \frac{1}{s} \int_{\mathbb{R}^d} f(x) d\omega^s(x) = A(f).$$

Therefore, for some $\delta > 0$, we have

$$\frac{1}{s} \int_{\mathbb{R}^d} f(x) d\omega^s(x) \leq M$$

for all $0 < s \leq \delta$. Consequently,

$$I_1 \leq M \int_0^\delta dK(s) = M(K(\delta) - K(0)) < \infty,$$

and (A.16) follows.

Since $I(\xi)$ exists, it follows from Fubini’s theorem that

$$\begin{aligned} I(\xi) &= \int_{\mathbb{R}^d} \left[e^{i\langle x, \xi \rangle} - 1 - \frac{i\langle x, \xi \rangle}{1 + \|x\|^2} \right] \left(\int_0^\infty g(s, x) \frac{1}{s} dK(s) \right) dx \\ &= \int_{\mathbb{R}^d} \left[e^{i\langle x, \xi \rangle} - 1 - \frac{i\langle x, \xi \rangle}{1 + \|x\|^2} \right] h(x) dx, \end{aligned}$$

where

$$h(x) = \int_0^\infty g(s, x) \frac{1}{s} dK(s)$$

exists. Therefore, the log-characteristic function F of λ has the form

$$F(\xi) = i\langle a, \xi \rangle + \int_{\mathbb{R}^d} \left[e^{i\langle x, \xi \rangle} - 1 - \frac{i\langle x, \xi \rangle}{1 + \|x\|^2} \right] d\phi(x),$$

where $d\phi(x) = h(x) dx$. This concludes the proof.

A.4. Proof of Proposition 3.8

Writing relation (3.17) in terms of the characteristic functions, we obtain

$$\frac{p\hat{\lambda}_p(t)}{1 - (1 - p)\hat{\lambda}_p(t)} = \hat{\lambda}(t), \quad p \in (0, 1), \quad t \in \mathbb{R}^d, \tag{A.17}$$

where $\hat{\lambda}$ and $\hat{\lambda}_p$ are the characteristic functions of Y and Y_{pi} , respectively. Substituting (3.18) into (A.17) and noting that $\hat{\lambda}(t) = (1 - \log \hat{\omega}(t))^{-1}$ we immediately obtain the validity of (A.17).

A.5. Proof of Theorem 3.9

Assume that Y is strictly OGS so that the representation (3.5) holds for some OS random variable X with exponent E . Conditioning on N_p we write the characteristic function of the LHS in (3.19) as follows:

$$\begin{aligned} \mathbb{E}[e^{i\langle t, S_p \rangle}] &= \sum_{n=1}^\infty \mathbb{E}[e^{i\langle t, A_p \sum_{i=1}^n Y_i \rangle}] (1 - p)^{n-1} p \\ &= \sum_{n=1}^\infty [\hat{\lambda}(A_p^* t)]^n (1 - p)^{n-1} p \\ &= \frac{p\hat{\lambda}(A_p^* t)}{1 - (1 - p)\hat{\lambda}(A_p^* t)}, \end{aligned} \tag{A.18}$$

where $\hat{\lambda}$ is the characteristic function of Y . Now, since $\hat{\lambda}(t) = (1 - \log \hat{\omega}(t))^{-1}$, where $\hat{\omega}$ is the characteristic function of a strictly OS law, the above equals:

$$\frac{p\hat{\lambda}(A_p^*t)}{1 - (1 - p)\hat{\lambda}(A_p^*t)} = \frac{p[1 - \log \hat{\omega}(A_p^*t)]^{-1}}{1 - (1 - p)[1 - \log \hat{\omega}(A_p^*t)]^{-1}} = \frac{1}{1 - p^{-1} \log \hat{\omega}(A_p^*t)}. \tag{A.19}$$

Substituting $A_p = p^E$ into (A.19) we obtain the characteristic function of Y , since the OS characteristic function $\hat{\omega}$ satisfies the relation $\hat{\omega}(p^{E^*}t) = [\hat{\omega}(t)]^p$ for each $p > 0$.

Conversely, assume that relation (3.19) holds. Then, by definition, Y must be OGS with the characteristic function $\hat{\lambda}$ of the form (3.4) with some OS characteristic function $\hat{\omega}$. Following the above calculations, we write relation (3.19) in terms of the characteristic functions as follows:

$$\frac{p\hat{\lambda}(A_p^*t)}{1 - (1 - p)\hat{\lambda}(A_p^*t)} = \hat{\lambda}(t), \quad p \in (0, 1), \quad t \in \mathbb{R}^d. \tag{A.20}$$

Substituting $\hat{\lambda}(t) = (1 - \log \hat{\omega}(t))^{-1}$ we obtain the following relation for $\hat{\omega}$:

$$\hat{\omega}(A_p^*t) = [\hat{\omega}(t)]^p, \quad p \in (0, 1), \quad t \in \mathbb{R}^d, \tag{A.21}$$

which essentially holds only for strictly OS characteristic function $\hat{\omega}$ with some exponent E and $A_p = p^E$.

A.6. Proof of Theorem 4.2

To obtain (4.10) use representation (4.9) coupled with independence of X_1 and X_2 , and apply a simple conditioning argument:

$$\begin{aligned} F(y_1, y_2) &= \int_0^\infty P(Y_1 \leq y_1, Y_2 \leq y_2 \mid Z = z) e^{-z} dz \\ &= \int_0^\infty P\left(X_1 \leq \frac{y_1}{\sqrt{z}}\right) P\left(X_2 \leq \frac{y_2}{z^{1/\alpha}}\right) e^{-z} dz. \end{aligned} \tag{A.22}$$

Since X_1 and X_2 have normal and stable laws with ch.f.'s (4.1), (4.2), respectively, we obtain (4.10). To obtain (4.11), differentiate the above function with respect to y_1 and y_2 . Alternatively, apply standard transformation theorem for functions of random vectors to obtain the density of Y directly from the joint density of Z , X_1 , and X_2 .

A.7. Proof of Theorem 4.5

We start with Part (i). Proceeding as in the proof of relation (5.1.7) from [44], we obtain the following expression for the ch.f. of the conditional distribution of $Y_2 \mid Y_1 = y$:

$$\hat{\lambda}_{2|1}(s) = \mathbb{E}(e^{isY_2} \mid Y_1 = y) = \frac{\int_{\mathbb{R}} e^{-ity} \hat{\lambda}(t, s) dt}{2\pi f_1(y)}, \tag{A.23}$$

where $\hat{\lambda}$ is the joint ch.f. (4.4) and f_1 is the marginal (Laplace) density of Y_1 given by (4.7). Upon factoring $\hat{\lambda}$

$$\hat{\lambda}(t, s) = \frac{1}{1 + \eta^\alpha |s|^\alpha} \frac{1}{1 + \sigma_s^2 t^2}, \tag{A.24}$$

where

$$\sigma_s = \frac{\sigma}{\sqrt{1 + \eta^\alpha |s|^\alpha}}, \tag{A.25}$$

we apply Fourier inversion formula

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-ity} (1 + \sigma_s^2 t^2)^{-1} dt = \frac{1}{2\sigma_s} e^{-|y|/\sigma_s}, \tag{A.26}$$

obtaining after some algebra

$$\hat{\lambda}_{2|1}(s) = \frac{1}{\sqrt{1 + \eta^\alpha |s|^\alpha}} e^{-\frac{|y|}{\sigma}(\sqrt{1 + \eta^\alpha |s|^\alpha} - 1)}. \tag{A.27}$$

To finish the proof apply the representation of ν -stable random variables (see, e.g., [20, Theorem 3.1]) noting that

$$\hat{\lambda}_{2|1}(s) = g(-\log \hat{\phi}(s)), \tag{A.28}$$

where

$$g(s) = \frac{1}{\sqrt{1 + s}} e^{-\frac{|y|}{\sigma}(\sqrt{1 + s} - 1)} \tag{A.29}$$

is the Laplace transform of the distribution of $U + V_y$ while $\hat{\phi}$ is the ch.f. (4.2) of the α -stable r.v. X_2 .

We now move to Part (ii). The characteristic function of the conditional distribution of Y_1 given $Y_2 = y$ is

$$\hat{\lambda}_{1|2}(t) = \mathbb{E}(e^{itY_1} | Y_2 = y) = \frac{\int_{\mathbb{R}} e^{itu} f(u, y) du}{f_2(y)}, \tag{A.30}$$

where f is the joint density (4.11) of Y_1 and Y_2 and f_2 is the marginal Linnik density (4.8) of Y_2 . Substituting these into (A.30) and changing the order of integration we obtain after some elementary algebra:

$$\hat{\lambda}_{1|2}(t) = \frac{\int_{\mathbb{R}} e^{-zt^2\sigma^2} \omega(z) e^{-z} dz}{\int_{\mathbb{R}} \omega(z) e^{-z} dz}, \tag{A.31}$$

with ω as in (4.22). Thus, we have

$$\hat{\lambda}_{1|2}(t) = h(-\log \hat{\phi}(t)), \tag{A.32}$$

where h is the Laplace transform of the positive r.v. with density (4.21) and $\hat{\phi}$ is the normal ch.f. (4.1). By the representation of ν -stable r.v.'s cited above, we obtain the variance mixture (4.20) of normal distributions. This concludes the proof.

A.8. Proof of Theorem 4.7

For $b = 0$ and $a \neq 0$ we obtain Laplace variable with the survival function

$$P(aY_1 > x) = \frac{1}{2} e^{-\frac{x}{|a|\sigma}}, \quad x > 0.$$

For $b \neq 0$ we note that the power tail of the Linnik variable bY_2 dominates the exponential tail of aY_1 :

$$\lim_{x \rightarrow \infty} \frac{P(aY_1 > x)}{P(bY_2 > x)} = 0.$$

Consequently, the tail behavior of $aY_1 + bY_2$ is the same as that of bY_2 (see Lemma 4.4.2 of [44]). The latter follows from more general results for univariate ν -stable laws (see, e.g., [19]).

A.9. Proof of Theorem 4.8

Apply representation (4.9) to obtain

$$|Y_1|^{\alpha_1} |Y_2|^{\alpha_2} \stackrel{d}{=} Z^{\frac{\alpha_1 + \alpha_2}{2} + \frac{\alpha_2}{\alpha}} |X_1|^{\alpha_1} |X_2|^{\alpha_2}. \quad (\text{A.33})$$

Since all positive absolute moments of Z and X_1 exists, it is clear that the joint absolute moment of Y_1 and Y_2 exists if and only if the absolute moment of X_2 of order α_2 exists. The latter exists if and only if $\alpha_2 < \alpha$ and equals (see, e.g., [44]):

$$\mathbb{E}|X_2|^{\alpha_2} = \frac{\eta^{\alpha_2} (1 - \alpha_2) \Gamma\left(1 - \frac{\alpha_2}{\alpha}\right)}{(2 - \alpha_2) \cos \frac{\pi \alpha_2}{2}}. \quad (\text{A.34})$$

The moments of exponential and normal distributions are straightforward to compute and well known:

$$\mathbb{E}Z^{\frac{\alpha_1 + \alpha_2}{2}} = \Gamma\left(\frac{\alpha_1}{2} + \frac{\alpha_2}{\alpha} + 1\right), \quad (\text{A.35})$$

$$\mathbb{E}|X_1|^{\alpha_1} = \frac{1}{\sqrt{\pi}} 2^{\alpha_1} \sigma^{\alpha_1} \Gamma\left(\frac{\alpha_1}{2} + \frac{1}{2}\right). \quad (\text{A.36})$$

The result follows.

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