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Asymptotic Behavior of Semistable Lévy Exponents and Applications to Fractal Path Properties

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Abstract This paper proves sharp bounds on the tails of the Lévy exponent of an operator semistable law on $\mathbb{R}^d$. These bounds are then applied to explicitly compute the Hausdorff and packing dimensions of the range, graph, and other random sets describing the sample paths of the corresponding operator semi-selfsimilar Lévy processes. The proofs are elementary, using only the properties of the Lévy exponent, and certain index formulae.

Keywords Lévy exponent · Operator semistable process · Semi-selfsimilarity · Hausdorff dimension · Packing dimension · Range · Graph · Multiple points · Recurrence · Transience

Mathematics Subject Classification (2010) Primary 60E10 · 60G51 · Secondary 28A78 · 28A80 · 60E07 · 60G17 · 60G18 · 60G52

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1 Introduction

Let $X = \{X(t)\}_{t\geq 0}$ be a *Lévy process* in $\mathbb{R}^d$, i.e., a stochastically continuous process with càdlàg paths that has stationary and independent increments and starts at the origin, i.e., $X(0) = 0$ almost surely. The distribution of X on the path space is uniquely determined by the distribution of $X(1)$ which can be an arbitrary infinitely divisible distribution in $\mathbb{R}^d$. The Lévy process X is called *operator semistable* if the (infinitely divisible) distribution μ of $X(1)$ is *full*, i.e., not supported on any lower dimensional hyperplane, and fulfills

$$\mu^c = c^E \mu * \varepsilon_u \tag{1.1}$$

for some fixed $c > 1$, $u \in \mathbb{R}^d$ and some linear operator E on $\mathbb{R}^d$, where μ^c denotes the c-fold convolution power of μ, $c^E \mu(dx) = \mu(c^{-E}dx)$ is the image measure of μ under the exponential operator $c^E = \sum_{n=0}^{\infty} \frac{(\log c)^n}{n!} E^n$, and ε_u denotes the Dirac measure at the point $u \in \mathbb{R}^d$. Operator semistable distributions were introduced by Jajte [11]; further early investigations can be found in [7,19,20]. In case $u = 0$ the distribution μ, respectively, the Lévy process X generated by μ, is called *strictly* operator semistable. Any *exponent* E is invertible, and any eigenvalue λ of E further fulfills $\text{Re}(\lambda) \geq \frac{1}{2}$, where $\text{Re}(\lambda) = \frac{1}{2}$ indicates a Gaussian component [25, Theorem 7.1.10]. We refer to the monograph [25] for a comprehensive overview on operator semistable distributions and their connection to limit theorems. As an easy consequence of (1.1), a strictly operator semistable Lévy process X is also *strictly operator semi-selfsimilar*, i.e.,

$$\{X(ct)\}_{t\geq 0} \stackrel{\text{fd}}{=} \{c^E X(t)\}_{t\geq 0}, \tag{1.2}$$

where $\stackrel{\text{fd}}{=}$ denotes equality of all finite-dimensional marginal distributions of the processes. The class of operator semi-selfsimilar processes is much larger than that of the semi-selfsimilar processes in the literature, see Maejima and Sato [22] and the references therein for more information. By induction, we clearly have $\{X(c^k t)\}_{t\geq 0} \stackrel{\text{fd}}{=} \{c^{kE} X(t)\}_{t\geq 0}$ for all $k \in \mathbb{Z}$. If (1.2) even holds for all $c > 0$, the Lévy process X is called *strictly operator selfsimilar* and the distribution of $X(t)$ is *strictly operator stable* [25, Definition 3.3.24]. If E is a scalar multiple of the identity, then an operator (semi-)stable law is called (semi-)stable, and an operator (semi-)selfsimilar process is called (semi-)selfsimilar. The operator scaling allows the tail behavior to vary with the coordinate, in an arbitrary coordinate system [25, Theorem 7.1.18]. This is important for many applications, including portfolio modeling in finance [26], pollution plumes in heterogeneous porous media [35], and diffusion tensor imaging [24]. Hence, operator semi-selfsimilarity generalizes the space-time scaling of selfsimilarity to a discrete scale and allows spatial scaling by linear operators, which gives more flexibility in modeling. We refer to [31] for several concrete applications of discrete scale-invariant phenomena from physics.

We remark that operator semistable Lévy processes are special cases of *group self-similar processes* introduced by Kolodyński and Rosiński [18]. To recall their definition, let G be a group of transformations of a set T and, for each $(g, t) \in G \times T$, let $C(g,t):\mathbb{R}^d\to\mathbb{R}^d$ be a bijection such that

$$C(g_1g_2,t)=C(g_1,g_2(t))\circ C(g_2,t),\quad \text{for all } g_1,g_2\in G \text{ and } t\in T,$$

and $C(e,t)=I$. Here e is the unit element of G and I is the identity operator on $\mathbb{R}^d$. In other words, C is a cocycle for the group action $(g,t)\mapsto g(t)$ of G on T. According to Kolodyński and Rosiński [18], a stochastic process $\{Y(t), t\in T\}$ taking values in $\mathbb{R}^d$ is called *G-self-similar with cocycle C* if

$$\{Y\big(g(t)\big),\ t\in T\}\overset{\text{fd}}{=}\{C(g,t)Y(t),\ t\in T\}. \tag{1.3}$$

In the setting of this paper, we take $T=[0,\infty)$ and $G=\{g_k : k\in\mathbb{Z}\}$, where g_k is the transformation on T defined by $g_k(t)=c^k t$. Thus, G is a subgroup of linear transformations on T and we may identify g_k with c^k. It is clear that a strictly operator semi-selfsimilar process $X=\{X(t)\}_{t\geq 0}$ is G-self-similar with cocycle C, where for each $g_k\in G$ and $t\geq 0$ the cocycle $C(g_k,t):\mathbb{R}^d\to\mathbb{R}^d$ is defined by $C(g_k,t)(x)=c^{kE}$. Note that $C(g_k,t)$ is a bijection since X is proper and $c>1$. Thus, operator semistable Lévy processes can also be studied by using the general framework of Kolodyński and Rosiński [18] and methods from ergodic theory, but this goes beyond the scope of the present paper.

We will need the following spectral decomposition of the exponent E as laid out in [25]. Factor the minimal polynomial of E into $f_1(x)\cdot\ldots\cdot f_p(x)$ such that every root of f_j has real part a_j, where $a_1<\cdots<a_p$ are the distinct real parts of the eigenvalues of E and $a_1\geq\frac{1}{2}$ by Theorem 7.1.10 in [25]. According to Theorem 2.1.14 in [25] we can decompose $\mathbb{R}^d$ into a direct sum $\mathbb{R}^d=V_1\oplus\ldots\oplus V_p$, where $V_j=\operatorname{Ker}(f_j(E))$ are E-invariant subspaces. Now, in an appropriate basis, E can be represented as a block-diagonal matrix $E=E_1\oplus\ldots\oplus E_p$, where $E_j:V_j\to V_j$ and every eigenvalue of E_j has real part a_j. In particular, every V_j is an E_j-invariant subspace of dimension $d_j=\dim V_j$. Now we can write $x\in\mathbb{R}^d$ as $x=x_1+\cdots+x_p$ and $t^E x=t^{E_1}x_1+\cdots+t^{E_p}x_p$ with respect to this direct sum decomposition, where $x_j\in V_j$ and $t>0$. Moreover, for the strictly operator semistable Lévy process we have $X(t)=X^{(1)}(t)+\ldots+X^{(p)}(t)$ with respect to this direct sum decomposition, where $\{X^{(j)}(t)\}_{t\geq 0}$ is an operator semistable Lévy process on $V_j\cong\mathbb{R}^{d_j}$ with exponent E_j by Lemma 7.1.17 in [25]. We can further choose an inner product on $\mathbb{R}^d$ such that the subspaces V_j, $1\leq j\leq p$, are mutually orthogonal and throughout this paper for $x\in\mathbb{R}^d$ we may choose $\|x\|=\langle x,x\rangle^{1/2}$ as the associated Euclidean norm on $\mathbb{R}^d$. With this choice, in particular, we have

$$\|X(t)\|^2=\|X^{(1)}(t)\|^2+\ldots+\|X^{(p)}(t)\|^2 \tag{1.4}$$

for all $t\geq 0$. Exponents need not be unique, because of possible symmetries [25, Remark 7.1.22]. However, since the real parts of the eigenvalues determine the tail behavior of μ [25, Theorem 7.1.18], the spectral decomposition is the same for any exponent. In case $d=1$, a spectral decomposition is superfluous and we simply say that X is a strictly *α-semistable Lévy process* with $\alpha=\alpha_1=a_1^{-1}=E^{-1}\in(0,2]$.

Since densities of operator semistable distributions exist but are in general not explicitly known, to show properties of operator semistable processes it is often natural to use Fourier transforms which are given in terms of the Lévy-Khintchine representation.

Our interest is focused on the asymptotic behavior of the *Lévy exponent* $\psi : \mathbb{R}^d \to \mathbb{C}$ which is the unique continuous function with $\psi(0) = 0$ and $\mathbb{E}[\exp(i\langle\xi, X(t)\rangle)] = \exp(-t\psi(\xi))$ given by the Lévy-Khintchine formula

$$\psi(\xi) = i\langle\xi, b\rangle + \frac{1}{2}\langle\xi, \Sigma\xi\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{i\langle\xi,x\rangle} + \frac{i\langle\xi, x\rangle}{1+\|x\|^2}\right)\phi(dx)$$

for some unique $b \in \mathbb{R}^d$, a symmetric and nonnegative definite $\Sigma \in \mathbb{R}^{d\times d}$, and a Lévy measure ϕ. The latter is a σ-finite Borel measure on $\mathbb{R}^d \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^d\setminus\{0\}} \min\{1, \|x\|^2\}\phi(dx) < \infty.$$

Our aim is to derive upper and lower bounds for the real and imaginary part of the Lévy exponent ψ in terms of the spectral decomposition. These results are presented in Sect. 2 and enable us to prove upper and lower bounds for $\mathrm{Re}((1 + \psi(\xi))^{-1})$ in the operator semistable setup, generalizing the corresponding result for operator stable Lévy processes given in Proposition 4.1 of [27]. The quantity $\mathrm{Re}((1+\psi(\xi))^{-1})$ appears in various formulas for the Hausdorff and packing dimensions of certain random sets that describe the sample path behavior of a Lévy process. This enables us to give alternative analytic proofs for the Hausdorff and packing dimensions of the range and the graph of operator semistable Lévy processes in Sect. 3. We will further show a connection to recurrence properties of operator semistable Lévy processes and to the Hausdorff dimension of multiple points of their sample paths.

2 Tail Estimates for Lévy Exponents

Suppose that $X = \{X(t)\}_{t\geq 0}$ is operator semistable with exponent E. Recall from Sect. 1 that $a_1 < \cdots < a_p$ are the distinct real parts of the eigenvalues of E, with $a_1 \geq 1/2$, and define $\alpha_i = a_i^{-1}$ so that $2 \geq \alpha_1 > \cdots > \alpha_p$. Now we can state the main technical result of this paper.

Theorem 2.1 *Let* $X = \{X(t)\}_{t\geq 0}$ *be a strictly operator semistable Lévy process in* $\mathbb{R}^d$ *with Lévy exponent* ψ*. Then for every* $\varepsilon > 0$ *there exists* $\tau > 1$ *such that for some* $K_i = K_i(\varepsilon, \tau)$ *and* $\|\xi\| > \tau$ *we have*

(a) $K_2 \sum_{i=1}^{p} \|\xi_i\|^{\alpha_i} \leq \mathrm{Re}(\psi(\xi)) \leq K_1\|\xi\|^{\varepsilon/2} \sum_{i=1}^{p} \|\xi_i\|^{\alpha_i}$,

(b) $|\mathrm{Im}(\psi(\xi))| \leq K_3\|\xi\|^{\varepsilon/2} \sum_{i=1}^{p} \|\xi_i\|^{\alpha_i}$.

Proof We will need the following refinement of the spectral decomposition of the exponent $E = E_1 \oplus \ldots \oplus E_p$ with respect to $\mathbb{R}^d = V_1 \oplus \ldots \oplus V_p$ laid out in Sect. 1. Apply the Jordan decomposition to get further direct sums $V_i = U_{i1} \oplus \ldots \oplus U_{iq(i)}$ of E-invariant subspaces such that in an appropriate basis $E_i = E_{i1} \oplus \ldots \oplus E_{iq(i)}$ is block-diagonal and every $x \in U_{ij} \setminus \{0\}$ is of order j, i.e., if we write $E_{ij} = S_{ij} + N_{ij}$, where S_{ij} is semisimple and N_{ij} is nilpotent, then $N_{ij}^{j-1}x \neq 0$ and $N_{ij}^{j}x = 0$. This $S + N$ decomposition is unique, e.g., see Hoffman and Kunze [10]. If we write $x = \sum_{i=1}^{p}\sum_{j=1}^{q(i)} x_{ij}$ with respect to these direct sum decompositions, so that $x_{ij} \in U_{ij}$, by the proof of Theorem 2.2.4 in [25], we have in an associated Euclidean norm

$$\|t^{-E^*}x\|^2 = \sum_{i=1}^{p}\sum_{j=1}^{q(i)} \frac{t^{-2a_i}(\log t)^{2(j-1)}}{((j-1)!)^2}\,\|x_{ij}\|^2 + o_{ij}(t,x),$$

where E^* denotes the adjoint of the exponent E and $o_{ij}(t,x)$ is a linear combination of terms of the form $t^{-2a_i}(\log t)^m$ with $m < 2(j-1)$. Then for fixed $x \neq 0$ the function $t \mapsto R(t) = \|t^{E^*}x\|^{-1}$ is regularly varying with index $a = \min\{a_i : x_i \neq 0\}$. Now let $r \mapsto t(r)$ be an asymptotic inverse of $R(t)$, i.e., a regularly varying function with index $\alpha = a^{-1}$ such that $R(t(r))/r \to 1$ as $r \to \infty$. An explicit choice of

$$t(r) = \sum_{i=1}^{p}\sum_{j=1}^{q(i)} \left(\frac{\alpha_i^{j-1}}{(j-1)!}\right)^{\alpha_i} r^{\alpha_i}(\log r)^{\alpha_i(j-1)}\|x_{ij}\|^{\alpha_i} \tag{2.1}$$

shows that the convergence $R(t(r))/r \to 1$ holds uniformly on compact subsets of $\{x \neq 0\}$. For a more detailed derivation of (2.1), we refer to the proof of Theorem 4.2 in [27].

Write $t > 0$ as $t = c^{k(t)}m(t)$ with $k(t) \in \mathbb{Z}$ and $m(t) \in [1, c)$. By (1.2), we have that $X(t)$ and $c^{k(t)E}X(m(t))$ are identically distributed and hence

$$t\,\psi(\xi) = m(t)\,\psi(c^{k(t)E^*}\xi) \quad \text{for all } t > 0,\ \xi \in \mathbb{R}^d. \tag{2.2}$$

Let $F(\xi) = \mathrm{Re}(\psi(\xi))$, then by (2.2) we get

$$t\,F(\xi) = m(t)\,F(c^{k(t)E^*}\xi) \quad \text{for all } t > 0,\ \xi \in \mathbb{R}^d. \tag{2.3}$$

Since X is full, $F(\xi)$ is bounded away from zero and infinity on compact subsets of $\{\xi \neq 0\}$ by Corollary 7.1.12 in [25]. Given $x \neq 0$ and $r > 0$ define $\theta_{r,x} = t(r)^{-E^*}rx$ using (2.1). Then $\|\theta_{r,x}\| = r\|t(r)^{-E^*}x\| = r/R(t(r)) \to 1$ as $r \to \infty$ uniformly on compact subsets of $\{x \neq 0\}$. Hence, given $\eta \in (0,1)$ there exists $r_0 > 0$ such that

$$1 - \eta < \|\theta_{r,x}\| < 1 + \eta \quad \text{for all } r \geq r_0,\ x \in S_d. \tag{2.4}$$

For $\xi \neq 0$ let $r = \|\xi\| > 0$ and $x = \xi/r \in S_d$, then by (2.3) we have

$$F(\xi) = F(rx) = F(t(r)^{E^*}\theta_{r,x}) = F(c^{k(t(r))E^*}m(t(r))^{E^*}\theta_{r,x})$$

$$= m(t(r))^{-1}t(r)F(m(t(r))^{E^*}\theta_{r,x}) = c^{k(t(r))}F(m(t(r))^{E^*}\theta_{r,x}) \tag{2.5}$$

and, since $m(t(r)) \in [1, c)$ together with (2.4), we get that $F(m(t(r))^{E^*}\theta_{r,x})$ is uniformly bounded away from zero and infinity for all $r \geq r_0$ and $x \in S_d$.

Now let $\varepsilon > 0$ be given and choose a constant $\tau \geq \max\{r_0, e\}$ such that for all $r \geq \tau$ we have $1 \leq (\log r)^{\alpha_i(q(i)-1)} \leq r^{\varepsilon/2}$ for all $1 \leq i \leq p$. Then, it follows from (2.5) and (2.1) that for all $r \geq \tau$ we have

$$\begin{aligned} F(\xi) &= c^{k(t(r))}F(m(t(r))^{E^*}\theta_{r,x}) \leq K_1' c^{k(t(r))}m(t(r)) \\ &= K_1' t(r) = \tilde{K}_1 \sum_{i=1}^{p}\sum_{j=1}^{q(i)} r^{\alpha_i}(\log r)^{\alpha_i(j-1)}\|x_{ij}\|^{\alpha_i} \\ &\leq \tilde{K}_1 r^{\varepsilon/2}\sum_{i=1}^{p} r^{\alpha_i}\sum_{j=1}^{q(i)}\|x_{ij}\|^{\alpha_i} \\ &\leq K_1\|\xi\|^{\varepsilon/2}\sum_{i=1}^{p}(r\|x_i\|)^{\alpha_i} = K_1\|\xi\|^{\varepsilon/2}\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}, \end{aligned} \tag{2.6}$$

where the constant K_1 does not depend on ξ and the inequality in the last line follows from $\sum_{j=1}^{q(i)}\|x_{ij}\|^{\alpha_i} \leq q(i)\|x_i\|^{\alpha_i} \leq d\|x_i\|^{\alpha_i}$. This proves the upper bound in part (a).

Similarly, it follows from (2.5) and (2.1) that for all $r \geq \tau$ we have

$$\begin{aligned} F(\xi) &= c^{k(t(r))}F(m(t(r))^{E^*}\theta_{r,x}) \geq K_2' c^{k(t(r))}m(t(r)) \\ &= K_2' t(r) = \tilde{K}_2 \sum_{i=1}^{p}\sum_{j=1}^{q(i)} r^{\alpha_i}(\log r)^{\alpha_i(j-1)}\|x_{ij}\|^{\alpha_i} \\ &\geq \tilde{K}_2 \sum_{i=1}^{p} r^{\alpha_i}\sum_{j=1}^{q(i)}\|x_{ij}\|^{\alpha_i} \\ &\geq K_2\sum_{i=1}^{p}(r\|x_i\|)^{\alpha_i} = K_2\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}, \end{aligned} \tag{2.7}$$

where the constant K_2 does not depend on ξ and the inequality in the last line follows from $\|x_i\|^{\alpha_i} = \|\sum_{j=1}^{q(i)} x_{ij}\|^{\alpha_i} \leq C_1(\sum_{j=1}^{q(i)}\|x_{ij}\|^2)^{\alpha_i/2} \leq C_1\sum_{j=1}^{q(i)}\|x_{ij}\|^{\alpha_i}$.

Now we consider $G(\xi) = \operatorname{Im}(\psi(\xi))$ for which by (2.2) we have

$$t \cdot G(\xi) = m(t)\cdot G(c^{k(t)E^*}\xi) \quad \text{for all } t > 0,\ \xi \in \mathbb{R}^d$$

and G is bounded on compact subsets of $\mathbb{R}^d \setminus \{0\}$ by continuity of ψ. Hence as above we get for all $\|\xi\| = r \geq \tau$

$$|G(\xi)| = c^{k(t(r))}|G(m(t(r))^{E^*}\theta_{r,x})| \leq K_3' t(r) \leq K_3\|\xi\|^{\varepsilon/2}\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}, \tag{2.8}$$

where the constant K_3 does not depend on ξ, proving part (c). □

Corollary 2.2 *Let X be a strictly operator semistable Lévy process in $\mathbb{R}^d$ with Lévy exponent ψ. Then for every $\varepsilon > 0$ there exists $\tau > 1$ such that for some $K = K(\varepsilon, \tau)$ we have*

$$\frac{K^{-1}\|\xi\|^{-\varepsilon}}{\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}} \leq \mathrm{Re}\left(\frac{1}{1+\psi(\xi)}\right) \leq \frac{K}{\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}} \quad \textit{for all } \|\xi\| > \tau. \tag{2.9}$$

Proof Using the obvious identity

$$\mathrm{Re}\left(\frac{1}{1+\psi(\xi)}\right) = \frac{1+\mathrm{Re}(\psi(\xi))}{(1+\mathrm{Re}(\psi(\xi)))^2 + (\mathrm{Im}(\psi(\xi))^2} = \frac{1+F(\xi)}{(1+F(\xi))^2+(G(\xi))^2},$$

by Theorem 2.1 we get for all $\xi \in \mathbb{R}^d$ with $\|\xi\| \geq \tau$

$$\mathrm{Re}\left(\frac{1}{1+\psi(\xi)}\right) \leq \frac{1}{1+F(\xi)} \leq \frac{1}{F(\xi)} \leq \frac{K_2^{-1}}{\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}}$$

and

$$\begin{aligned}\mathrm{Re}\left(\frac{1}{1+\psi(\xi)}\right) &\geq \frac{F(\xi)}{(1+F(\xi)))^2+(G(\xi))^2}\\ &\geq \frac{K_2\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}}{\left(1+K_1\|\xi\|^{\varepsilon/2}\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}\right)^2+\left(K_3\|\xi\|^{\varepsilon/2}\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}\right)^2}\\ &\geq K_{12}\frac{\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}}{\left(\|\xi\|^{\varepsilon/2}\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}\right)^2} = \frac{K_{12}\|\xi\|^{-\varepsilon}}{\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}},\end{aligned}$$

concluding the proof. □

3 Applications to Fractal Path Properties

3.1 Range and Graph

We will now apply the results of Sect. 2 to derive fractal properties of the range $X([0,1]) = \{X(t) : t \in [0,1]\}$ and the graph $G_X([0,1]) = \{(t, X(t)) : t \in [0,1]\}$ of a strictly operator semistable Lévy process X in terms of their Hausdorff and packing dimensions. We refer to [8] for a systemic account on fractal dimensions and their properties. With the help of the spectral decomposition of the exponent E, the Hausdorff dimension of the range of a strictly operator semistable Lévy process in $\mathbb{R}^d$ with $d \geq 2$ has been calculated in Theorem 3.1 of [12] as

$$\dim_{\mathrm{H}} X(B) = \begin{cases}\alpha_1 \dim_{\mathrm{H}} B & \text{if } \alpha_1 \dim_{\mathrm{H}} B \leq d_1\\ 1+\alpha_2(\dim_{\mathrm{H}} B - \alpha_1^{-1}) & \text{else}\end{cases} \tag{3.1}$$

almost surely, where $B \in \mathcal{B}(\mathbb{R}_+)$ is an arbitrary Borel set. In case $d = 1$ by Theorem 3.3 in [12] for a strictly α-semistable Lévy process we have

$$\dim_{\mathrm{H}} X(B) = \min\{\alpha \dim_{\mathrm{H}} B, 1\} \tag{3.2}$$

almost surely. In the special case of a strictly operator stable Lévy process, the formula (3.1) was established by Meerschaert and Xiao [27] generalizing an earlier partial result for $B = [0, 1]$ in [1]. Moreover, the Hausdorff dimension of the graph of a strictly operator semistable Lévy process in $\mathbb{R}^d$ with $d \geq 2$ was recently calculated in Theorem 3.1 of [33] as

$$\dim_{\mathrm{H}} G_X(B) = \begin{cases} \dim_{\mathrm{H}} B \cdot \max\{\alpha_1, 1\} & \text{if } \alpha_1 \dim_{\mathrm{H}} B \leq d_1 \\ 1 + \max\{\alpha_2, 1\}(\dim_{\mathrm{H}} B - \alpha_1^{-1}) & \text{else} \end{cases} \tag{3.3}$$

almost surely, and in case $d = 1$ by Theorem 3.2 in [33] for a strictly α-semistable Lévy process we have

$$\dim_{\mathrm{H}} G_X(B) = \begin{cases} \dim_{\mathrm{H}} B \cdot \max\{\alpha, 1\} & \text{if } \alpha \dim_{\mathrm{H}} B \leq 1 \\ 1 + \dim_{\mathrm{H}} B - \alpha^{-1} & \text{else} \end{cases} \tag{3.4}$$

almost surely. The derivation of (3.1)–(3.4) in [12,33] uses the standard method of showing that almost surely the right-hand side in (3.1)–(3.4) serves as an upper as well as a lower bound for the Hausdorff dimension on the corresponding left-hand side, following classical results for the range of one-dimensional stable Lévy processes in Blumenthal and Getoor [2–4] and Hendricks [9], and Lévy processes with independent stable components in Pruitt and Taylor [28,29,32]. The lower bound is shown by an application of Frostman's capacity theorem to prove that certain expected energy integrals are finite. The upper bound is shown by using the covering lemma of Pruitt and Taylor [29, Lemma 6.1] which needs sharp lower bounds for the expected sojourn time in small balls. For the latter in [12], uniform density bounds were derived in the semistable situation. For an overview, we refer to the survey article [34].

An alternative analytic approach for $B = [0, 1]$ uses an index formula proved in Corollary 1.8 of [17], valid for arbitrary Lévy processes X in $\mathbb{R}^d$, which states that almost surely

$$\dim_{\mathrm{H}} X([0, 1]) = \sup\left\{a < d : \int_{\{\|\xi\| \geq 1\}} \mathrm{Re}\left(\frac{1}{1 + \psi(\xi)}\right) \frac{d\xi}{\|\xi\|^{d-a}} < \infty\right\}. \tag{3.5}$$

Similarly, Khoshnevisan and Xiao [14] established the following formula for the packing dimension of $X([0, 1])$ in terms of the Lévy exponent $\psi(\xi)$:

$$\dim_{\mathrm{P}} X([0, 1]) = \sup\left\{\eta \geq 0 : \liminf_{r \to 0^+} \frac{W(r)}{r^\eta} = 0\right\} = \limsup_{r \to 0^+} \frac{\log W(r)}{\log r}, \tag{3.6}$$

almost surely, where $\sup\varnothing := 0$ and the function W is defined by

$$W(r) = \int_{\mathbb{R}^d} \operatorname{Re}\left(\frac{1}{1+\psi(\frac{x}{r})}\right) \frac{1}{\prod_{j=1}^d (1+x_j^2)}\,\mathrm{d}x. \tag{3.7}$$

In [14, Eq. (1.4)] they also provided a formula for $\dim_{\mathrm{H}} X([0,1])$ in terms of W. Notice that, when applied to the Lévy process $\{(t, X(t)) : t \geq 0\}$, (3.5) and (3.6) also provide analytic ways for computing the Hausdorff and packing dimensions of the graph of X.

Meerschaert and Xiao [27, Proposition 4.1] used (3.5) to give an alternative proof for (3.1) in case X is a full, strictly operator stable Lévy process and $B = [0,1]$ using bounds for $\operatorname{Re}((1+\psi(\xi))^{-1})$ as in (2.9). See also in Proposition 7.7 of [13]. Khoshnevisan and Xiao [14, Theorem 3.1] showed that, under condition (2.9), the packing dimension of $X([0,1])$ is also given by the right-hand side of (3.1) with $B = [0,1]$. Using Corollary 2.2, we immediately obtain the following special case of (3.3).

Theorem 3.1 *Let X be a strictly operator semistable Lévy process in $\mathbb{R}^d$ with $d \geq 2$. Then*

$$\dim_{\mathrm{H}} X([0,1]) = \dim_{\mathrm{P}} X([0\,,1]) = \begin{cases} \alpha_1 & \text{if } \alpha_1 \leq d_1 \\ 1+\alpha_2(1-\alpha_1^{-1}) & \text{else} \end{cases} \tag{3.8}$$

almost surely, in accordance with (3.1).

Proof Use Corollary 2.2 and follow the arguments for [27, Proposition 4.1] and [14, Theorem 3.1]. □

We can also obtain a special case of (3.2) as follows.

Corollary 3.2 *Let X be a strictly α-semistable Lévy process in $\mathbb{R}$. Then*

$$\dim_{\mathrm{H}} X([0,1]) = \dim_{\mathrm{P}} X([0\,,1]) = \min\{\alpha, 1\}$$

almost surely.

Proof In case $d = 1$ the conclusion (2.9) of Corollary 2.2 reads as

$$K^{-1}|\xi|^{-\varepsilon-\alpha} \leq \operatorname{Re}\left(\frac{1}{1+\psi(\xi)}\right) \leq K|\xi|^{-\alpha} \quad \text{for all } |\xi| > \tau. \tag{3.9}$$

Note that for $d = 1$ we can strengthen (3.9) to

$$K^{-1}|\xi|^{-\alpha} \leq \operatorname{Re}\left(\frac{1}{1+\psi(\xi)}\right) \leq K|\xi|^{-\alpha} \quad \text{for all } |\xi| \geq 1, \tag{3.10}$$

since in this case $R(t) = t^{1/\alpha}|x|^{-1}$ and the asymptotic inverse can be chosen as $t(r) = (r|x|)^{\alpha}$ such that $R(t(r)) = r$ for all $r > 0$. Following the line of arguments given in the proof of Theorem 2.1, it is easy to arrive at (3.10) instead of (3.9). Using (3.10), it is obvious that

$$\int_{|\xi|\geq 1} \mathrm{Re}\left(\frac{1}{1+\psi(\xi)}\right)\frac{d\xi}{|\xi|^{1-a}} < \infty \quad\Longleftrightarrow\quad a < \alpha.$$

Hence by (3.5), we immediately get

$$\dim_{\mathrm{H}} X([0,1]) = \sup\left\{a < 1 : \int_{\{|\xi|\geq 1\}} \mathrm{Re}\left(\frac{1}{1+\psi(\xi)}\right)\frac{d\xi}{|\xi|^{1-a}} < \infty\right\} = \min\{\alpha, 1\}$$

almost surely. Since $\dim_{\mathrm{H}} X([0,1]) \leq \dim_{\mathrm{P}} X([0,1]) \leq 1$, we see that, in order to prove $\dim_{\mathrm{P}} X([0,1]) = \min\{\alpha, 1\}$ a.s., it is sufficient to consider the case $\alpha < 1$ and verify $\dim_{\mathrm{P}} X([0,1]) \leq \alpha$ a.s. It follows from (3.7) and (3.10) that for $r \in (0,1)$,

$$W(r) \geq K^{-1} r^{\alpha} \int_{\mathbb{R}} \frac{1}{|x|^{\alpha}(1+x^2)} = K r^{\alpha}, \tag{3.11}$$

which implies that $\lim_{r\to 0} r^{-\eta} W(r) = \infty$ for all $\eta > \alpha$. By (3.6), we obtain $\dim_{\mathrm{P}} X([0,1]) \leq \alpha$ a.s. This concludes the proof. □

We now turn to the graph process $\{(t, X(t))\}_{t\geq 0}$ which is a Lévy process in $\mathbb{R}^{d+1}$ such that (1.1) holds with block diagonal exponent $1 \oplus E$. Since the Lévy measure of the graph process is concentrated on a d-dimensional subspace of $\mathbb{R}^{d+1}$, it is not full [25, Proposition 3.1.20], and hence it is not operator semistable. However, neither (3.5) nor (3.6) assume fullness of the Lévy process.

Write $\tilde{\xi} = (\xi_0, \xi) \in \mathbb{R}^{d+1}$ with $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ and observe that the Lévy exponent $\tilde{\psi}$ of the graph process is given by $\tilde{\psi}(\tilde{\xi}) = \psi(\xi) - i\xi_0$, which leads to

$$\mathrm{Re}\left(\frac{1}{1+\tilde{\psi}(\tilde{\xi})}\right) = \frac{1+F(\xi)}{(1+F(\xi))^2 + (G(\xi)-\xi_0)^2} =: H(\xi_0, \xi), \tag{3.12}$$

where $F = \mathrm{Re}\psi$ and $G = \mathrm{Im}\psi$ are as in the proof of Theorem 2.1. Next we prove a special case of (3.3). The proof is elementary, using only the sharp bounds of Theorem 2.1 along with the index formulae (3.5) and (3.6).

Theorem 3.3 *Let X be a strictly operator semistable Lévy process in $\mathbb{R}^d$ with $d \geq 2$. Then*

$$\dim_{\mathrm{H}} G_X([0,1]) = \dim_{\mathrm{P}} G_X([0,1]) = \begin{cases} \max\{\alpha_1, 1\} & \text{if } \alpha_1 \leq d_1 \\ 1 + \max\{\alpha_2, 1\}(1-\alpha_1^{-1}) & \text{else} \end{cases}$$

almost surely.

To clarify the proof of Theorem 3.3, it will be helpful to derive the corresponding statement for $d = 1$ first. This one-dimensional result is a special case of (3.4).

Proposition 3.4 *Let X be a strictly α-semistable Lévy process in $\mathbb{R}$. Then*

$$\dim_{\mathrm{H}} G_X([0,1]) = \dim_{\mathrm{P}} G_X([0,1]) = \max\{1, 2-\alpha^{-1}\} = \begin{cases} 1 & \text{if } \alpha \le 1 \\ 2-\alpha^{-1} & \text{else} \end{cases}$$

almost surely.

In the next two proofs, K denotes an unspecified positive constant whose value may vary at each occurrence.

Proof We will first establish lower bounds. In case $\alpha \le 1$ clearly $\dim_{\mathrm{H}} G_X([0,1]) \ge 1$ by projecting the graph $\{(t, X(t))\}_{t\ge 0}$ onto the first (deterministic) component. In case $\alpha > 1$ let $\gamma \in (0, 2-\alpha^{-1})$ and then note that in view of (3.5) and (3.12) we need to show that

$$\begin{aligned} I_\gamma &:= \int_{\{|\xi_0|\ge 2K_3'', |\xi|\ge\tau\}} \mathrm{Re}\left(\frac{1}{1+\tilde{\psi}(\tilde{\xi})}\right) \frac{d\tilde{\xi}}{\|\tilde{\xi}\|^{2-\gamma}} \\ &= \left(\int_{A_1} + \int_{A_2} + \int_{A_3}\right) \frac{H(\xi_0,\xi)}{(\xi_0^2+\xi^2)^{1-\gamma/2}}\, d\tilde{\xi} < \infty, \end{aligned}$$

where we use a similar decomposition of the domain of integration as in the proof of Theorem 2.1 in Manstavičius [23]; cf. also Figure 1 in [23]. Namely we set

$$\begin{aligned} A_1 &= \left\{|\xi_0| \ge 2K_3'' \max\left\{1, (|\xi|/\tau)^q\right\}\right\}, \\ A_2 &= \left\{|\xi| \ge \tau, |\xi_0| \le 2K_3''|\xi|/\tau\right\}, \\ A_3 &= \left\{|\xi| \ge \tau, 2K_3''|\xi|/\tau < |\xi_0| < 2K_3''(|\xi|/\tau)^q\right\}, \end{aligned}$$

where $q = \alpha + \varepsilon/2 > 1$, $K_3'' = K_3\tau^q$ and $\tau > 1$ is chosen such that Theorem 2.1 holds for $\varepsilon > 0$ with the following constraints. Since we always have $2-\alpha^{-1} \le \alpha$, we know that $\gamma < \alpha$ and can choose $\varepsilon > 0$ such that $\gamma < \frac{2\alpha-1+\varepsilon/2}{\alpha+\varepsilon/2}$.

On A_1 we use $(\xi_0^2+\xi^2)^{1-\gamma/2} \ge |\xi_0|^{2-\gamma}$ and by Theorem 2.1(b) we have

$$(G(\xi)-\xi_0)^2 \ge (|\xi_0| - |G(\xi)|)^2 \ge \left(|\xi_0| - K_3|\xi|^q\right)^2 \ge (|\xi_0|/2)^2$$

so that, using also part (a) of Theorem 2.1, we get by (3.12)

$$H(\xi_0,\xi) \le 4\,\frac{1+F(\xi)}{\xi_0^2} \le K\,\frac{|\xi|^{\alpha+\varepsilon/2}}{\xi_0^2}.$$

Hence, using symmetry with respect to ξ_0, we get

$$\int_{A_1} \frac{H(\xi_0,\xi)}{(\xi_0^2+\xi^2)^{1-\gamma/2}}\, d\tilde{\xi} \le K \int_{A_1} \frac{|\xi|^{\alpha+\varepsilon/2}}{|\xi_0|^{4-\gamma}}\, d\tilde{\xi}$$

$$
\begin{aligned}
&= K \int_{2K_3''}^{\infty} \frac{1}{\xi_0^{4-\gamma}} \int_{\{|\xi|^q \le \tau^q/(2K_3'')\,\xi_0\}} |\xi|^{\alpha+\varepsilon/2}\, \mathrm{d}\xi\, \mathrm{d}\xi_0 \\
&\le K \int_{2K_3''}^{\infty} \frac{1}{\xi_0^{4-\gamma}} \int_{\{|\xi| \le \tau/(2K_3'')^{1/q}\,\xi_0^{1/q}\}} |\xi_0|\, \mathrm{d}\xi\, d\xi_0 \\
&\le K \int_{2K_3''}^{\infty} \frac{\xi_0^{1+\frac{1}{q}}}{\xi_0^{4-\gamma}}\, \mathrm{d}\xi_0 = K \int_{2K_3''}^{\infty} \frac{1}{\xi_0^{3-\gamma-\frac{1}{q}}}\, \mathrm{d}\xi_0 < \infty,
\end{aligned}
$$

since $\gamma < 2-\alpha^{-1} < 2-(\alpha+\varepsilon/2)^{-1} = 2-q^{-1}$.

On A_2 we use $(\xi_0^2+\xi^2)^{1-\gamma/2} \ge |\xi|^{2-\gamma}$ and by Theorem 2.1(a) we get

$$
H(\xi_0,\xi) \le \frac{1}{1+F(\xi)} \le K\, \frac{1}{|\xi|^{\alpha}}.
$$

Hence, using symmetry with respect to ξ, we get

$$
\begin{aligned}
\int_{A_2} \frac{H(\xi_0,\xi)}{(\xi_0^2+\xi^2)^{1-\gamma/2}}\, \mathrm{d}\tilde{\xi} &\le K \int_{A_2} \frac{1}{|\xi|^{\alpha}}\, \frac{1}{|\xi|^{2-\gamma}}\, \mathrm{d}\tilde{\xi} \\
&= K \int_{\tau}^{\infty} \frac{1}{\xi^{2-\gamma+\alpha}} \int_{\{|\xi_0| \le 2K_3''|\xi|/\tau\}} \mathrm{d}\xi_0\, \mathrm{d}\xi \\
&= K \int_{\tau}^{\infty} \frac{1}{\xi^{1-\gamma+\alpha}}\, \mathrm{d}\xi < \infty,
\end{aligned}
$$

since $\gamma < \alpha$.

On A_3 we use $(\xi_0^2+\xi^2)^{1-\gamma/2} \ge |\xi_0|^{2-\gamma}$ as on A_1 and by Theorem 2.1(a) we get

$$
H(\xi_0,\xi) \le \frac{1}{1+F(\xi)} \le K\, \frac{1}{|\xi|^{\alpha}}
$$

as on A_2. Hence, using symmetry with respect to ξ_0 and ξ, we get

$$
\begin{aligned}
\int_{A_3} \frac{H(\xi_0,\xi)}{(\xi_0^2+\xi^2)^{1-\gamma/2}}\, \mathrm{d}\tilde{\xi} &\le K \int_{A_3} \frac{1}{|\xi|^{\alpha}}\, \frac{1}{|\xi_0|^{2-\gamma}}\, \mathrm{d}\tilde{\xi} \\
&= K \int_{\tau}^{\infty} \frac{1}{\xi^{\alpha}} \int_{2K_3''|\xi|/\tau}^{2K_3''(|\xi|/\tau)^q} \frac{1}{|\xi_0|^{2-\gamma}}\, \mathrm{d}\xi_0\, \mathrm{d}\xi \\
&\le K \int_{\tau}^{\infty} \frac{\xi^{q(\gamma-1)}}{\xi^{\alpha}}\, \mathrm{d}\xi = K \int_{\tau}^{\infty} \frac{1}{\xi^{2\alpha+\varepsilon/2-\gamma(\alpha+\varepsilon/2)}}\, \mathrm{d}\xi < \infty,
\end{aligned}
$$

since $\gamma < \frac{2\alpha-1+\varepsilon/2}{\alpha+\varepsilon/2}$ by our choice of $\varepsilon > 0$.

Altogether we have shown that $I_\gamma < \infty$ for every $0 < \gamma < 2-\alpha^{-1}$ so that $\dim_{\mathrm{H}} G_X([0\,,1]) \ge 2-\alpha^{-1}$ almost surely for $\alpha > 1$.

Since $\dim_{\mathrm{H}} G_X([0\,,1]) \le \dim_{\mathrm{P}} G_X([0\,,1])$, it remains to prove the upper bound for the packing dimension. In the following, we obtain the upper bound in a similar manner as in the proof of Theorem 3.1 in [14]. In case $\alpha \in (0,1)$ let $\eta > 1$ be arbitrary and choose $\varepsilon > 0$ such that $\alpha - 1 + \varepsilon < 0$ and $\eta > 1+\varepsilon$. Recall that $\tau > 1$ in Theorem 2.1. Then it follows from (3.7), (3.12) and Theorem 2.1 that for $r \in (0,1)$ and hence $r < r^{1-\alpha-\varepsilon}$ we have

$$
\begin{aligned}
W(r) &= \int_{\mathbb{R}^2} \mathrm{Re}\left(\frac{1}{1+\tilde{\psi}(\tilde{\xi}/r)}\right) \frac{\mathrm{d}\tilde{\xi}}{(1+\xi_0^2)(1+\xi^2)} \\
&\geq K \int_{\tau r}^{\infty}\int_{\tau r}^{\infty} \frac{(\xi/r)^{\alpha}}{(\xi/r)^{2\alpha+\varepsilon} + \big((\xi/r)^{\alpha+\varepsilon/2} + \xi_0/r\big)^2} \frac{\mathrm{d}\xi}{1+\xi^2}\frac{\mathrm{d}\xi_0}{1+\xi_0^2} \\
&\geq K r^{\alpha+\varepsilon} \int_{\tau r}^{\infty}\int_{\tau}^{\infty} \frac{\xi^{\alpha}}{\xi^{2\alpha+\varepsilon} + \big(\xi^{\alpha+\varepsilon/2} + r^{\alpha-1+\varepsilon/2}\xi_0\big)^2} \frac{\mathrm{d}\xi}{1+\xi^2}\frac{\mathrm{d}\xi_0}{1+\xi_0^2} \\
&\geq K r^{\alpha+\varepsilon} \int_{\tau r}^{\infty} \frac{1}{1+(1+r^{\alpha-1+\varepsilon/2}\xi_0)^2} \int_{\tau}^{\infty} \frac{\mathrm{d}\xi}{\xi^{\alpha+\varepsilon}(1+\xi)^2}\frac{\mathrm{d}\xi_0}{1+\xi_0^2} \\
&\geq K r^{\alpha+\varepsilon} \int_{\tau r^{1-\alpha-\varepsilon}}^{1} \frac{\mathrm{d}\xi_0}{r^{2\alpha-2+\varepsilon}\xi_0^2(1+\xi_0^2)} \\
&\geq K r^{2-\alpha} \int_{\tau r^{1-\alpha-\varepsilon}}^{1} \frac{\mathrm{d}\xi_0}{\xi_0^2} = K r^{1+\varepsilon}.
\end{aligned}
$$

This implies $\lim_{r\to 0^+} r^{-\eta} W(r) = \infty$, since $\eta > 1+\varepsilon$ by our choice of $\varepsilon > 0$. Hence, by (3.6) we obtain $\dim_{\mathrm{P}} G_X([0,1]) \leq 1$ almost surely, since $\eta > 1$ is arbitrary.

In case $\alpha \geq 1$ let $\eta > 2-\alpha^{-1}$ be arbitrary and choose $\varepsilon > 0$ such that $\alpha > 1+\varepsilon/2$ and $\eta > 2-\alpha^{-1}+2\varepsilon$. Note that

$$
\beta := \frac{2-\alpha-\alpha^{-1}+\varepsilon}{1-\alpha-\varepsilon} = 1 + \frac{1-\alpha^{-1}+2\varepsilon}{1-\alpha-\varepsilon} < 1
$$

and observe that $1-(\alpha+\varepsilon/2)(1-\beta) < 0$ by elementary calculations. Then it follows from (3.7), (3.12) and Theorem 2.1 that for $r \in (0,1)$ and hence $r < r^{\beta}$ we have

$$
\begin{aligned}
W(r) &= \int_{\mathbb{R}^2} \mathrm{Re}\left(\frac{1}{1+\tilde{\psi}(\tilde{\xi}/r)}\right) \frac{\mathrm{d}\tilde{\xi}}{(1+\xi_0^2)(1+\xi^2)} \\
&\geq K \int_{\tau r}^{\infty}\int_{\tau r}^{\infty} \frac{(\xi/r)^{\alpha}}{(\xi/r)^{2\alpha+\varepsilon} + \big((\xi/r)^{\alpha+\varepsilon/2} + \xi_0/r\big)^2} \frac{\mathrm{d}\xi_0}{1+\xi_0^2}\frac{\mathrm{d}\xi}{1+\xi^2} \\
&\geq K r^{2-\alpha} \int_{\tau r}^{\infty}\int_{\tau}^{\infty} \frac{\xi^{\alpha}}{\xi^{2\alpha+\varepsilon}r^{2-2\alpha-\varepsilon} + \big(\xi^{\alpha+\varepsilon/2}r^{1-\alpha-\varepsilon/2} + \xi_0\big)^2} \frac{\mathrm{d}\xi_0}{1+\xi_0^2}\frac{\mathrm{d}\xi}{1+\xi^2} \\
&\geq K r^{2-\alpha} \int_{\tau r^{\beta}}^{1} \frac{\xi^{\alpha}}{\xi^{2\alpha+\varepsilon}r^{2-2\alpha-\varepsilon} + \big(\xi^{\alpha+\varepsilon/2}r^{1-\alpha-\varepsilon/2} + 1\big)^2} \int_{\tau}^{\infty} \\
&\qquad \frac{\mathrm{d}\xi_0}{\xi_0^2(1+\xi_0)^2}\frac{\mathrm{d}\xi}{1+\xi^2}.
\end{aligned}
$$

Since for $\xi \geq \tau r^{\beta}$ we have $\xi^{\alpha+\varepsilon/2} r^{1-\alpha-\varepsilon/2} \geq K\, r^{1-(\alpha+\varepsilon/2)(1-\beta)} \to \infty$ as $r \to 0^+$ by our choice of β, for sufficiently small $r \in (0,1)$ we get

$$W(r) \geq K r^{2-\alpha} \int_{\tau r^{\beta}}^{1} \frac{\xi^{\alpha}}{\xi^{2\alpha+\varepsilon} r^{2-2\alpha-\varepsilon}} \frac{d\xi}{1+\xi^2}$$
$$\geq K r^{\alpha+\varepsilon} \int_{\tau r^{\beta}}^{1} \frac{d\xi}{\xi^{\alpha+\varepsilon}} \geq K r^{\alpha+\varepsilon+\beta(1-\alpha-\varepsilon)} = K r^{2-\alpha^{-1}+2\varepsilon}.$$

This implies $\lim_{r\to 0^+} r^{-\eta} W(r) = \infty$, since $\eta > 2 - \alpha^{-1} + 2\varepsilon$ by our choice of $\varepsilon > 0$. Hence, by (3.6) we obtain $\dim_P G_X([0,1]) \leq 2 - \alpha^{-1}$ almost surely, since $\eta > 2 - \alpha^{-1}$ is arbitrary, concluding the proof of Proposition 3.4. □

Proof of Theorem 3.3 We will first prove the lower bounds. In case $\alpha_1 \leq d_1$ clearly $\dim_H G_X([0,1]) \geq 1$ by projecting the graph $\{(t, X(t))\}_{t\geq 0}$ onto the first (deterministic) component and by projection of the graph onto the range we get $\dim_H G_X([0,1]) \geq \alpha_1$ almost surely by (3.8). In case $\alpha_1 > d_1$ we have $d_1 = 1$, hence by projecting the graph $\{(t, X(t))\}_{t\geq 0}$ onto the subgraph $\{(t, X^{(1)}(t))\}_{t\geq 0}$ we get $\dim_H G_X([0,1]) \geq \dim_H G_{X^{(1)}}([0,1]) = 2 - \alpha_1^{-1}$ almost surely by Proposition 3.4 and a projection of the graph onto the range yields $\dim_H G_X([0,1]) \geq 1+\alpha_2(1+\alpha_1^{-1})$ almost surely by (3.8).

Since $\dim_H G_X([0,1]) \leq \dim_P G_X([0,1])$, again it remains to prove the upper bound for the packing dimension. To prove this upper bound, we rewrite the tail indices $\alpha_0 = 1$ and $\alpha_1 > \cdots > \alpha_p$ for each of the $d+1$ coordinates so that $\tilde{\alpha}_0 \geq \cdots \geq \tilde{\alpha}_d$. In principle, we now have to distinguish four cases:

1. $\alpha_1 \leq 1 = d_1$, then we have $\tilde{\alpha}_0 = \alpha_0 = 1$, $\tilde{\alpha}_1 = \alpha_1$ and we need to show that $r^{-\eta} W(r) \to \infty$ as $r \to 0^+$ for all $\eta > 1$.
2. $1 < \alpha_1 < 2 \leq d_1$, then $\tilde{\alpha}_0 = \alpha_1 = \tilde{\alpha}_1$ and we have to show that $r^{-\eta} W(r) \to \infty$ as $r \to 0^+$ for all $\eta > \alpha_1$.
3. $\alpha_1 > 1 = d_1 \geq \alpha_2$, then $\tilde{\alpha}_0 = \alpha_1$, $\tilde{\alpha}_1 = \alpha_0 = 1$ and we have to show that $r^{-\eta} W(r) \to \infty$ as $r \to 0^+$ for all $\eta > 2 - \alpha_1^{-1}$.
4. $\alpha_1 > \alpha_2 > 1 = d_1$, then $\tilde{\alpha}_0 = \alpha_1$, $\tilde{\alpha}_1 = \alpha_2$ and we have to show that $r^{-\eta} W(r) \to \infty$ as $r \to 0^+$ for all $\eta > 1 + \alpha_2(1 - \alpha_1^{-1})$.

Note that these four cases can be summarized in the sense that we have to show $r^{-\eta} W(r) \to \infty$ as $r \to 0^+$ for all $\eta > 1 + \tilde{\alpha}_1(1 - \tilde{\alpha}_0^{-1}) \geq 1$, where $2 \geq \tilde{\alpha}_0 \geq \tilde{\alpha}_1$ and $\tilde{\alpha}_0 \geq 1$. We write $\tilde{\xi} = (\tilde{\xi}_0, \ldots, \tilde{\xi}_d) \in \mathbb{R}^{d+1}$ and define

$$i^* := \min\{i = 0, \ldots, d : \tilde{\alpha}_i = 1 = \alpha_0\}.$$

Then it follows from (3.7), (3.12) and Theorem 2.1 that for $r \in (0,1)$ we have

$$W(r) = \int_{\mathbb{R}^{d+1}} \mathrm{Re}\left(\frac{1}{1+\tilde{\psi}(\tilde{\xi}/r)}\right) \frac{d\tilde{\xi}}{\prod_{i=0}^{d}(1+\tilde{\xi}_i^2)}$$
$$\geq K \int_{\{|\tilde{\xi}_i|\geq \tau r,\, 0\leq i\leq d\}} \frac{\sum_{i\neq i^*} |\tilde{\xi}_i/r|^{\tilde{\alpha}_i}}{\left(\|\tilde{\xi}/r\|^{\varepsilon/2} \sum_{i\neq i^*} |\tilde{\xi}_i/r|^{\tilde{\alpha}_i}\right)^2 + \left(\|\tilde{\xi}/r\|^{\varepsilon/2} \sum_{i\neq i^*} |\tilde{\xi}_i/r|^{\tilde{\alpha}_i} + |\tilde{\xi}_{i^*}/r|\right)^2}$$

$$\frac{\mathrm{d}\tilde{\xi}}{\prod_{i=0}^{d}(1+\tilde{\xi}_i^2)}$$

$$\geq K \int_{\left\{\substack{\tilde{\xi}_i \geq \tau r,\, 0\leq i\leq d \\ \tilde{\xi}_i\leq 1,\, 2\leq i\leq d}\right\}} \frac{\sum_{i\neq i^*}(\tilde{\xi}_i/r)^{\tilde{\alpha}_i}}{\left(\sum_{i=0}^{d}(\tilde{\xi}_i/r)^{\tilde{\alpha}_i}\right)^2}\,\frac{\mathrm{d}\tilde{\xi}}{\|\tilde{\xi}/r\|^{\varepsilon}(1+\tilde{\xi}_0^2)(1+\tilde{\xi}_1^2)}$$

$$\geq K \int_{\{\tau r\leq\tilde{\xi}_i\leq 1,\, 2\leq i\leq d\}} \mathrm{d}\tilde{\xi}_2\cdots d\tilde{\xi}_d$$

$$\cdot \int_{\tau r}^{\infty}\int_{\tau r}^{\infty} \frac{C+\sum_{i\in\{0,1\}\setminus\{i^*\}}(\tilde{\xi}_i/r)^{\tilde{\alpha}_i}}{(A+(\tilde{\xi}_1/r)^{\tilde{\alpha}_1}+(\tilde{\xi}_0/r)^{\tilde{\alpha}_0})^2(B+(\tilde{\xi}_1/r)^2+(\tilde{\xi}_0/r)^2)^{\varepsilon/2}}$$

$$\frac{\mathrm{d}\tilde{\xi}_0}{1+\tilde{\xi}_0^2}\,\frac{\mathrm{d}\tilde{\xi}_1}{1+\tilde{\xi}_1^2},$$

where

$$A=\sum_{i=2}^{d}(\tilde{\xi}_i/r)^{\tilde{\alpha}_i},\quad B=\sum_{i=2}^{d}(\tilde{\xi}_i/r)^2,\quad C=\sum_{i\in\{2,\ldots,d\}\setminus\{i^*\}}(\tilde{\xi}_i/r)^{\tilde{\alpha}_i}.$$

For fixed $(\tilde{\xi}_2,\ldots,\tilde{\xi}_d)\in[\tau r,1]^{d-1}$ and thus fixed A, B, C we consider the inner integrals

$$I(r):=\int_{\tau r}^{\infty}\int_{\tau r}^{\infty}\frac{C+\sum_{i\in\{0,1\}\setminus\{i^*\}}(\tilde{\xi}_i/r)^{\tilde{\alpha}_i}}{(A+(\tilde{\xi}_1/r)^{\tilde{\alpha}_1}+(\tilde{\xi}_0/r)^{\tilde{\alpha}_0})^2(B+(\tilde{\xi}_1/r)^2+(\tilde{\xi}_0/r)^2)^{\varepsilon/2}}$$

$$\frac{\mathrm{d}\tilde{\xi}_0}{1+\tilde{\xi}_0^2}\,\frac{\mathrm{d}\tilde{\xi}_1}{1+\tilde{\xi}_1^2}.$$

In case $i^*=0$, i.e., $1=\tilde{\alpha}_0\geq\tilde{\alpha}_1$, let $\eta>1$ be arbitrary and choose $\varepsilon>0$ such that $\eta>1+2\varepsilon$. Then we have

$$I(r)=\int_{\tau r}^{\infty}\int_{\tau r}^{\infty}\frac{C+(\tilde{\xi}_1/r)^{\tilde{\alpha}_1}}{(A+(\tilde{\xi}_1/r)^{\tilde{\alpha}_1}+(\tilde{\xi}_0/r)^{\tilde{\alpha}_0})^2}\,\frac{\mathrm{d}\tilde{\xi}_1}{(B+(\tilde{\xi}_1/r)^2+(\tilde{\xi}_0/r)^2)^{\varepsilon/2}(1+\tilde{\xi}_1^2)}$$

$$\frac{\mathrm{d}\tilde{\xi}_0}{1+\tilde{\xi}_0^2}$$

$$\geq K\,r^{\tilde{\alpha}_1+\varepsilon}\int_{\tau r}^{\infty}\int_{\tau}^{\infty}\frac{\tilde{\xi}_1^{\tilde{\alpha}_1}}{(r^{\tilde{\alpha}_1}A+\tilde{\xi}_1^{\tilde{\alpha}_1}+r^{\tilde{\alpha}_1-\tilde{\alpha}_0}\tilde{\xi}_0^{\tilde{\alpha}_0})^2}\,\frac{\mathrm{d}\tilde{\xi}_1}{(r^2B+\tilde{\xi}_1^2+\tilde{\xi}_0^2)^{\varepsilon/2}(1+\tilde{\xi}_1^2)}$$

$$\frac{\mathrm{d}\tilde{\xi}_0}{1+\tilde{\xi}_0^2}$$

$$\geq K\,r^{\tilde{\alpha}_1+\varepsilon}\int_{\tau r^{1-\tilde{\alpha}_1-\varepsilon}}^{1}\frac{1}{(1+r^{\tilde{\alpha}_1-\tilde{\alpha}_0}\tilde{\xi}_0^{\tilde{\alpha}_0})^2}\int_{\tau}^{\infty}\frac{\mathrm{d}\tilde{\xi}_1}{(r^2B+\tilde{\xi}_1^2+1)^{\varepsilon/2}\tilde{\xi}_1^{\tilde{\alpha}_1}(1+\tilde{\xi}_1^2)}$$

$$\frac{\mathrm{d}\tilde{\xi}_0}{1+\tilde{\xi}_0^2},$$

where in the last step we used $r^{\tilde{\alpha}_1}A \le d-1$, $\tilde{\xi}_1^{-\tilde{\alpha}_1} \le \tau^{-\tilde{\alpha}_1}$, $r^2 B \le d-1$ and $\tilde{\xi}_0 \le 1$. Since the inner integral is positive and finite, $\tilde{\alpha}_0 = 1$ and for $\tilde{\xi}_0 \ge \tau r^{1-\tilde{\alpha}_1-\varepsilon}$ we have $r^{\tilde{\alpha}_1-\tilde{\alpha}_0}\tilde{\xi}_0^{\tilde{\alpha}_0} \ge K\, r^{-\varepsilon} \to \infty$ as $r \to 0^+$, for sufficiently small $r \in (0,1)$ we further get

$$\begin{aligned} I(r) &\ge K\, r^{\tilde{\alpha}_1+\varepsilon} \int_{\tau r^{1-\tilde{\alpha}_1-\varepsilon}}^{1} \frac{\mathrm{d}\tilde{\xi}_0}{r^{2\tilde{\alpha}_1-2\tilde{\alpha}_0}\tilde{\xi}_0^{2\tilde{\alpha}_0}(1+\tilde{\xi}_0^2)} \\ &\ge K\, r^{2-\tilde{\alpha}_1+\varepsilon} \int_{\tau r^{1-\tilde{\alpha}_1-\varepsilon}}^{1} \frac{\mathrm{d}\tilde{\xi}_0}{\tilde{\xi}_0^2} \ge K\, r^{1+2\varepsilon}. \end{aligned}$$

This implies $\lim_{r\to 0^+} r^{-\eta} W(r) = \infty$, since $\eta > 1+2\varepsilon$ by our choice of $\varepsilon > 0$. Hence, by (3.6) we obtain $\dim_{\mathrm{P}} G_X([0,1]) \le 1 = 1 + \tilde{\alpha}_1(1-\tilde{\alpha}_0^{-1})$ almost surely, since $\eta > 1$ is arbitrary.

In case $i^* \ge 1$, i.e., $\tilde{\alpha}_0 > 1 \ge \tilde{\alpha}_1$, let $\eta > 1 + \tilde{\alpha}_1(1-\tilde{\alpha}_0^{-1})$ be arbitrary and choose $\varepsilon > 0$ such that $\eta > 1 + \tilde{\alpha}_1(1-\tilde{\alpha}_0^{-1}) + 2\varepsilon$. Note that

$$\beta := \frac{1+\tilde{\alpha}_1(1-\tilde{\alpha}_0^{-1}) - \tilde{\alpha}_0 + \varepsilon}{1-2\tilde{\alpha}_0} = 1 + \frac{\tilde{\alpha}_1(1-\tilde{\alpha}_0^{-1}) + \varepsilon}{1-2\tilde{\alpha}_0} < 1$$

and observe that $\tilde{\alpha}_1 - \tilde{\alpha}_0(1-\beta) = \varepsilon\tilde{\alpha}_0/(1-2\tilde{\alpha}_0) < 0$ by elementary calculations. Then we have

$$\begin{aligned} I(r) &\ge \int_{\tau r}^{\infty}\int_{\tau r}^{\infty} \frac{C + (\tilde{\xi}_0/r)^{\tilde{\alpha}_0}}{(A + (\tilde{\xi}_1/r)^{\tilde{\alpha}_1} + (\tilde{\xi}_0/r)^{\tilde{\alpha}_0})^2} \frac{\mathrm{d}\tilde{\xi}_1}{(B + (\tilde{\xi}_1/r)^2 + (\tilde{\xi}_0/r)^2)^{\varepsilon/2}(1+\tilde{\xi}_1^2)} \\ &\quad \frac{\mathrm{d}\tilde{\xi}_0}{1+\tilde{\xi}_0^2} \\ &\ge K\, r^{2\tilde{\alpha}_1-\tilde{\alpha}_0+\varepsilon} \int_{\tau r}^{\infty}\int_{\tau}^{\infty} \frac{\tilde{\xi}_0^{\tilde{\alpha}_0}}{(r^{\tilde{\alpha}_1}A + \tilde{\xi}_1^{\tilde{\alpha}_1} + r^{\tilde{\alpha}_1-\tilde{\alpha}_0}\tilde{\xi}_0^{\tilde{\alpha}_0})^2} \frac{\mathrm{d}\tilde{\xi}_1}{(r^2 B + \tilde{\xi}_1^2 + \tilde{\xi}_0^2)^{\varepsilon/2}(1+\tilde{\xi}_1^2)} \\ &\quad \frac{\mathrm{d}\tilde{\xi}_0}{1+\tilde{\xi}_0^2} \\ &\ge K\, r^{2\tilde{\alpha}_1-\tilde{\alpha}_0+\varepsilon} \int_{\tau r^{\beta}}^{1} \frac{\tilde{\xi}_0^{\tilde{\alpha}_0}}{(1 + r^{\tilde{\alpha}_1-\tilde{\alpha}_0}\tilde{\xi}_0^{\tilde{\alpha}_0})^2} \int_{\tau}^{\infty} \frac{\mathrm{d}\tilde{\xi}_1}{(r^2 B + \tilde{\xi}_1^2 + 1)^{\varepsilon/2}\tilde{\xi}_1^{2\tilde{\alpha}_1}(1+\tilde{\xi}_1^2)} \\ &\quad \frac{\mathrm{d}\tilde{\xi}_0}{1+\tilde{\xi}_0^2}, \end{aligned}$$

where in the last step we used $r^{\tilde{\alpha}_1}A \le d-1$, $\tilde{\xi}_1^{-\tilde{\alpha}_1} \le \tau^{-\tilde{\alpha}_1}$, $r^2 B \le d-1$ and $\tilde{\xi}_0 \le 1$. Since the inner integral is positive and finite and for $\tilde{\xi}_0 \ge \tau r^{\beta}$, we have $r^{\tilde{\alpha}_1-\tilde{\alpha}_0}\tilde{\xi}_0^{\tilde{\alpha}_0} \ge K\, r^{\tilde{\alpha}_1-\tilde{\alpha}_0(1-\beta)} \to \infty$ as $r \to 0^+$ by our choice of β, and for sufficiently small $r \in (0,1)$, we further get

$$I(r) \geq K\, r^{2\tilde{\alpha}_1-\tilde{\alpha}_0+\varepsilon} \int_{\tau r^\beta}^1 \frac{\mathrm{d}\tilde{\xi}_0}{r^{2\tilde{\alpha}_1-2\tilde{\alpha}_0}\tilde{\xi}_0^{2\tilde{\alpha}_0}(1+\tilde{\xi}_0^2)}$$
$$\geq K\, r^{\tilde{\alpha}_0+\varepsilon} \int_{\tau r^\beta}^1 \frac{\mathrm{d}\tilde{\xi}_0}{\tilde{\xi}_0^{2\tilde{\alpha}_0}} \geq K\, r^{\tilde{\alpha}_0+\varepsilon+\beta(1-2\tilde{\alpha}_0)} = K\, r^{1+\tilde{\alpha}_1(1-\tilde{\alpha}_0^{-1})+2\varepsilon}.$$

This implies $\lim_{r\to 0+} r^{-\eta} W(r) = \infty$, since $\eta > 1+\tilde{\alpha}_1(1-\tilde{\alpha}_0^{-1})+2\varepsilon$ by our choice of $\varepsilon > 0$. Hence, by (3.6) we obtain $\dim_{\mathrm{P}} G_X([0,1]) \leq 1+\tilde{\alpha}_1(1-\tilde{\alpha}_0^{-1})$ almost surely, since $\eta > 1+\tilde{\alpha}_1(1-\tilde{\alpha}_0^{-1})$ is arbitrary, concluding the proof. □

3.2 Recurrence and Transience

A Lévy process $X = \{X(t)\}_{t\geq 0}$ in $\mathbb{R}^d$ is called *recurrent* if $\liminf_{t\to\infty} \|X(t)\| = 0$ almost surely, and it is called *transient* if $\lim_{t\to\infty} \|X(t)\| = \infty$. Due to dichotomy, every Lévy process is either recurrent or transient, e.g., see Theorem 35.4 in [30]. In case of a full, strictly operator semistable Lévy process recurrence and transience of X is fully characterized by the following results.

- $d \geq 3$: Every full Lévy process in $\mathbb{R}^d$ is transient by Theorem 37.8 in [30].
- $d = 2$: By Choi and Sato [6], the only recurrent strictly operator semistable Lévy processes in $\mathbb{R}^2$ are Gaussian with $\alpha_1 = 2$ and $d_1 = 2$, see also Theorem 37.18 in [30].
- $d = 1$: By Theorems 3.1 and 3.2 in Choi [5], a strictly α-semistable Lévy process in $\mathbb{R}$ is recurrent if $1 \leq \alpha \leq 2$ and it is transient if $0 < \alpha < 1$.

Hence, together with (3.1) and (3.2) we immediately get a characterization of recurrence and transience by the Hausdorff dimension of the range.

Corollary 3.5 *A full, strictly operator semistable Lévy process $X = \{X(t)\}_{t\geq 0}$ in $\mathbb{R}^d$ is recurrent if and only if $\dim_{\mathrm{H}} X([0,1]) = d$ almost surely.*

A possible interpretation of this result is that a strictly operator semistable Lévy process is recurrent if and only if its sample paths are almost surely "space filling." Note that Corollary 3.5 is not true for arbitrary Lévy processes as follows. By Theorem 37.5 in [30] a Lévy process X in $\mathbb{R}^d$ is recurrent if and only if

$$\lim_{q\downarrow 0} \int_{\{\|\xi\|<1\}} \mathrm{Re}\left(\frac{1}{q-\psi(\xi)}\right) \mathrm{d}\xi = \infty. \tag{3.13}$$

Hence, recurrence and transience are determined by the behavior of $\psi(\xi)$ near the origin $\xi = 0$, i.e., the tail behavior of the process, whereas by (3.5) the Hausdorff dimension of the range is determined by the behavior of $\psi(\xi)$ as $\|\xi\| \to \infty$, i.e., the local behavior of the process. To illustrate this, we give the following example.

Example 3.6 Let ϕ be the symmetric Lévy measure on $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ with Lebesgue density

$$g(x) = \begin{cases} |x|^{-(\beta+1)} & \text{if } 0 < |x| \leq 1, \\ |x|^{-(\alpha+1)} & \text{if } |x| > 1, \end{cases}$$

where $\alpha > 0$ and $\beta < 2$ due to $\int_{\mathbb{R}^*}(1 \wedge x^2)\, d\phi(x) < \infty$. Then, it can be easily shown using the criteria (3.13) and (3.5) that the Lévy process $X = \{X(t)\}_{t \geq 0}$ with Lévy exponent

$$\psi(\xi) = \int_{\mathbb{R}^*} \left(e^{i\xi x} - 1 - \frac{i\xi x}{1+x^2} \right) d\phi(x) = \int_{\mathbb{R}^*} (\cos(\xi x) - 1)\, g(x)\, \mathrm{d}x$$

is recurrent if and only if $\alpha \geq 1$ and we have $\dim_{\mathrm{H}} X([0,1]) = \max\{0, \min\{\beta, 1\}\}$ almost surely. For $\beta < 1 \leq \alpha$ or $\alpha < 1 \leq \beta$, we see that the statement of Corollary 3.5 fails to hold.

Remark 3.7 It is also possible to characterize recurrence and transience of a full, strictly operator semistable Lévy process X by the Hausdorff dimension of its graph, but we have to distinguish between the cases $d = 1$ and $d \geq 2$. A comparison of the above results of Choi and Sato with (3.3) and (3.4) easily gives that a full, strictly operator semistable Lévy process is recurrent if and only if

$$\dim_{\mathrm{H}} G_X([0,1]) = \begin{cases} 2 - \alpha^{-1} & \text{if } d = 1 \\ d & \text{if } d \geq 2 \end{cases}$$

almost surely.

3.3 Double Points of Operator Semistable Lévy Processes

A point $x \in \mathbb{R}^d$ is called a *k-multiple point* of the stochastic process X for some $k \in \mathbb{N} \setminus \{1\}$ if there exist $0 \leq t_1 < \cdots < t_k$ such that $X(t_1) = \cdots = X(t_k) = x$. By $M_X(k)$, we denote the set of all k-multiple points of X. Recently, Luks and Xiao [21] derived a general formula for the Hausdorff dimension of $M_X(k)$ for symmetric, absolutely continuous Lévy processes in terms of the Lévy exponent ψ. In Theorem 1 of [21], they proved that

$$\dim_{\mathrm{H}} M_X(k) = d - \inf \left\{ \beta \in (0,d] : \int_{\mathbb{R}^{kd}} \frac{1}{1 + \left\| \sum_{j=1}^k \xi^{(j)} \right\|^\beta} \prod_{j=1}^k \frac{1}{1+\psi(\xi^{(j)})}\, d\bar{\xi} < \infty \right\},$$

almost surely with the convention $\inf \emptyset = d$, where $\bar{\xi} = (\xi^{(1)}, \ldots, \xi^{(d)})$ for $\xi^{(j)} \in \mathbb{R}^d$. Moreover, in case of symmetric operator stable Lévy processes X with exponent E, Luks and Xiao [21] were able to calculate $M_X(2)$ explicitly, based only on the fact that for $\varepsilon > 0$ there exists $\tau > 1$ such that for some $K > 1$ it holds that

$$\frac{K^{-1}\|\xi\|^{-\varepsilon}}{\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}} \le \frac{1}{1+\psi(\xi)} \le \frac{K\|\xi\|^{\varepsilon}}{\sum_{i=1}^{p}\|\xi_i\|^{\alpha_i}} \quad \text{for all } \|\xi\| > \tau,$$

which is known from (4.2) of [27]. Since from Corollary 2.2 we know that the same bounds hold true also for symmetric operator semistable Lévy processes, the explicit dimension results of [21] also hold in this more general situation. In the following, we reformulate Theorem 2 of [21] for the semistable case, where we rearrange the distinct real parts $\alpha_1 > \cdots > \alpha_p$ of the eigenvalues of the exponent E as $\tilde{\alpha}_1 \geq \cdots \geq \tilde{\alpha}_d$ including their multiplicities.

Corollary 3.8 *Let $X = \{X(t)\}_{t\geq 0}$ be a symmetric operator semistable Lévy process in $\mathbb{R}^d$ with exponent E. Then for the double points of X we have almost surely:*

(a) *If $d = 2$ then* $\dim_{\mathrm{H}} M_X(2) = \min\left\{\tilde{\alpha}_1(2-\tilde{\alpha}_1^{-1}-\tilde{\alpha}_2^{-1}), 2\tilde{\alpha}_2(1-\tilde{\alpha}_1^{-1})\right\}$.
(b) *If $d = 3$ then* $\dim_{\mathrm{H}} M_X(2) = \tilde{\alpha}_1(2-\tilde{\alpha}_1^{-1}-\tilde{\alpha}_2^{-1}-\tilde{\alpha}_3^{-1})$.
(c) *If $d \geq 4$ then* $M_X(2) = \emptyset$.

Note that a negative Hausdorff dimension means that $M_X(2) = \emptyset$ almost surely.

3.4 Concluding Remarks

The results in Sect. 3 show that many sample path properties of a strictly operator semistable Lévy processes can be described by the real parts of the eigenvalues of the exponent E, and the upper and lower bounds in Theorem 2.1 play an important role in studying these and other properties. Several interesting questions remain open. For example, Corollary 3.8, as well as [21], only provides information for the set of double points; it would be interesting to solve the problems for $k = 3$ (for $k \geq 4$, the set of multiple points is almost surely empty). Moreover, Khoshnevisan and Xiao [15], Khoshnevisan, Shieh and Xiao [16] have studied the existence of intersections of independent Lévy processes and the Hausdorff dimensions of the sets of intersection times and intersection points, respectively. Their results are expressed in terms of the Lévy exponents of the processes. For strictly operator semistable Lévy processes, we believe that these results could be explicitly expressed in terms of the real parts of the eigenvalues of their exponents.

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