Computers and Mathematics with Applications 66 (2013) 737–745

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Correlation structure of fractional Pearson diffusions

Nikolai N. Leonenko [a], Mark M. Meerschaert [b,*], Alla Sikorskii [b]

[a] Cardiff School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF24 4YH, UK

[b] Department of Statistics and Probability, Michigan State University, East Lansing, MI 48824, USA

ARTICLE INFO

Keywords:
Pearson diffusion
Fractional derivative
Correlation function
Mittag-Leffler function

ABSTRACT

The stochastic solution to a diffusion equations with polynomial coefficients is called a Pearson diffusion. If the first time derivative is replaced by a Caputo fractional derivative of order less than one, the stochastic solution is called a fractional Pearson diffusion. This paper develops an explicit formula for the covariance function of a fractional Pearson diffusion in steady state, in terms of Mittag-Leffler functions. That formula shows that fractional Pearson diffusions are long-range dependent, with a correlation that falls off like a power law, whose exponent equals the order of the fractional derivative.

1. Introduction

Fractional differential equations are an important and useful tool in many areas of science and engineering [1–4]. The literature includes important applications to physics [5–7], hydrology [8,9], and finance [10,11], among others. Fractional calculus began with a letter from Leibniz to L'Hôpital in 1695, but recent applications have intensified interest in both analytical and numerical methods for solving fractional differential equations [12–15].

There are some interesting and fundamental connections between fractional calculus and probability [16]. It is well known that the diffusion equation with constant coefficients governs Brownian motion, the long-time scaling limit of a simple random walk [17]. If the first time derivative is replaced by a Caputo fractional derivative of order $0 < \alpha < 1$, the result is a fractional diffusion equation that governs the scaling limit of a continuous-time random walk [18–21]: the nth particle jump X_n is preceded by a waiting time W_n with a power law probability distribution $P(W_n > t) \approx t^{-\alpha}$ having the same index α. This random walk model converges to a Brownian motion with the deterministic time variable replaced by an inverse α-stable subordinator E_t [22], resulting in a subdiffusive process that spreads at a slower rate $t^{\alpha/2}$ than the usual rate $t^{1/2}$ for a traditional Brownian motion. The resulting solutions are useful to model diffusive phenomena, in which particles rest for long periods between movements, in a homogeneous environment.

In a heterogeneous environment, the coefficients of the diffusion equation will naturally vary in space. Pearson diffusions form a tractable class of variable coefficient diffusion models with polynomial coefficients. They govern a class of Markov processes whose steady-state distributions belong to the class of Pearson distributions [23]. In a fractional Pearson diffusion, the time variable is replaced by an inverse α-stable subordinator. The resulting stochastic process is non-Markovian, but its one-dimensional distributions are governed by the fractional Pearson diffusion equation, obtained by replacing the first time derivative in the Pearson diffusion equation with a Caputo fractional derivative of the same order $0 < \alpha < 1$. The purpose of this paper is study the correlation structure of fractional Pearson diffusions in steady state. The correlation function is explicitly computed in Theorem 3.1, and from this it follows that fractional Pearson diffusions exhibit long-range dependence.

* Corresponding author.
E-mail address: mcubed@stt.msu.edu (M.M. Meerschaert).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2013.01.009

Consider the stochastic differential equation

$$dX_1(t) = \mu(X_1(t))dt + \sigma(X_1(t))dW(t), \tag{1}$$

where $W(t)$ is a standard Brownian motion. When

$$\mu(x) = a_0 + a_1 x \quad \text{and} \quad D(x) = \frac{\sigma^2(x)}{2} = d_0 + d_1 x + d_2 x^2, \tag{2}$$

the process $X_1(t)$ is called a *Pearson diffusion*. If $\sigma(x)$ is a positive constant, this is the Ornstein–Uhlenbeck process [24]. If $d_2 = 0$, this is the Cox–Ingersoll–Ross (CIR) process, which is used in finance [25]. The study of Pearson diffusions began with Kolmogorov [26] and Wong [27]; see also [28–33]. Let $p_1(x, t; y)$ denote the conditional probability density of $x = X_1(t)$ given $y = X_1(0)$, i.e., the transition density of this time-homogeneous Markov process. This transition density solves the Kolmogorov forward equation (Fokker–Planck equation)

$$\frac{\partial p_1(x,t;y)}{\partial t} = -\frac{\partial}{\partial x}\left[\mu(x)p_1(x,t;y)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)p_1(x,t;y)\right] \tag{3}$$

and the backward equation

$$\frac{\partial p_1(x,t;y)}{\partial t} = \mu(y)\frac{\partial p_1(x,t;y)}{\partial y} + \frac{\sigma^2(y)}{2}\frac{\partial^2 p_1(x,t;y)}{\partial y^2} \tag{4}$$

with the same initial condition $p_1(x, 0; y) = \delta(x-y)$. Then we say that $X_1(t)$ is the *stochastic solution* to the forward equation (3) and the backward equation (4).

The Caputo fractional derivative of order $0 < \alpha < 1$, defined by

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t f'(\tau)(t-\tau)^{-\alpha}\,d\tau, \tag{5}$$

has Laplace transform $s^\alpha \tilde{f}(s) - s^{\alpha-1}f(0)$, where $\tilde{f}(s) = \int_0^\infty e^{-st}f(t)\,dt$ [34,16]. The stochastic solution of the time-fractional forward equation

$$\frac{\partial^\alpha p_\alpha(x,t;y)}{\partial t^\alpha} = -\frac{\partial}{\partial x}\left[\mu(x)p_\alpha(x,t;y)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)p_\alpha(x,t;y)\right] \tag{6}$$

and the time-fractional backward equation

$$\frac{\partial^\alpha p_\alpha(x,t;y)}{\partial t^\alpha} = \mu(y)\frac{\partial p_\alpha(x,t;y)}{\partial y} + \frac{\sigma^2(y)}{2}\frac{\partial^2 p_\alpha(x,t;y)}{\partial y^2} \tag{7}$$

with point source initial condition $p_\alpha(x, 0; y) = \delta(x - y)$ is called a fractional Pearson diffusion [35], denoted by $X_\alpha(t)$. The fractional time derivative models particle sticking and trapping [36,20]. Because particle resting times are distributed like a power law, $X_\alpha(t)$ is no longer a Markov process. Hence the conditional probability density $p_\alpha(x, t; y)$ of $x = X_1(t)$ given $y = X_1(0)$ is not enough to determine the process. In this paper, we will study the correlation structure of fractional Pearson diffusions, and derive an explicit formula (17) for the correlation between $X_\alpha(t)$ and $X_\alpha(s)$ in terms of Mittag-Leffler functions. Then, an asymptotic expansion (29) will be obtained, to show that the correlation falls off like $t^{-\alpha}$ for t large, thus demonstrating that fractional Pearson diffusions exhibit long-range dependence.

2. Fractional Pearson diffusion

Let $\mathbf{m}(x)$ be the steady-state distribution of $X_1(t)$. The generator associated with the backward equation (7)

$$\mathcal{G}g(y) = \left[\mu(y)\frac{\partial}{\partial y} + \frac{\sigma^2(y)}{2}\frac{\partial^2}{\partial y^2}\right]g(y) \tag{8}$$

has a set of eigenfunctions $\mathcal{G}Q_n(y) = -\lambda_n Q_n(y)$ with eigenvalues $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ that form an orthonormal basis for $L^2(\mathbf{m}(y)\,dy)$. If $d_1 = d_2 = 0$ and $d_0 > 0$, then $\mathbf{m}(y)$ is a normal density, and Q_n are Hermite polynomials. When $d_2 = 0$, $\mathbf{m}(y)$ is a gamma density, and Q_n are Laguerre polynomials. For $D''(y) < 0$ with two positive real roots, $\mathbf{m}(y)$ is a beta density, and Q_n are Jacobi polynomials. In the remaining cases, the spectrum of $\mathcal{G}$ has a continuous part, and some moments of $X_1(t)$ do not exist. In every case, $\mathbf{m}(y)$ is one of the Pearson distributions [23]. For the remainder of this paper, we will assume one of the three cases (Hermite, Laguerre, Jacobi), so that all moments exist.

Assume a solution $p_1(x, t; y) = f(t)\phi(y)$ to the backward equation (4) and separate variables to see that

$$\frac{df(t)}{dt}\varphi(y) = f(t)\mathcal{G}\varphi(y) \quad \text{or} \quad \frac{1}{f(t)}\frac{df(t)}{dt} = \frac{\mathcal{G}\varphi(y)}{\varphi(y)} := -\lambda$$

so that $f(t)\phi(y) = e^{-\lambda_n t}Q_n(y)$ solves (4) for any $n \geq 0$. Then any linear combination $\sum_n b_n e^{-\lambda_n t}Q_n(y)$ is also a solution, with initial condition $g(x) = \sum_n b_n Q_n(x)$. Since

$$b_n = \langle g, Q_n\rangle_{L^2(\mathbf{m}(x)\,dx)} := \int g(x)Q_n(x)\mathbf{m}(x)\,dx,$$

it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} b_n e^{-\lambda_n t}Q_n(y) &= \sum_{n=0}^{\infty}\left(\int g(x)Q_n(x)\mathbf{m}(x)\,dx\right)e^{-\lambda_n t}Q_n(y)\\ &= \int\left(\mathbf{m}(x)\sum_{n=0}^{\infty} e^{-\lambda_n t}Q_n(x)Q_n(y)\right)g(x)\,dx,\end{aligned} \tag{9}$$

and hence

$$p_1(x, t; y) = \mathbf{m}(x)\sum_{n=0}^{\infty} e^{-\lambda_n t}Q_n(x)Q_n(y) \tag{10}$$

is the transition density of $X_1(t)$, i.e., the point source solution to the backward equation (4) and the forward equation (3).

Since the time-fractional analogue (6) to the backward equation (4) is a *fractional Cauchy problem* of the form

$$\frac{\partial^\alpha p_\alpha(x, t; y)}{\partial t^\alpha} = \mathcal{G}_y p_\alpha(x, t; y),$$

a general semigroup result [37, Theorem 3.1] implies that

$$p_\alpha(x, t; y) = \int_0^\infty p_1(x, u; y)f_t(u)\,du, \tag{11}$$

where

$$f_t(x) = \frac{t}{\alpha}x^{-1-\frac{1}{\alpha}}g_\alpha\left(tx^{-\frac{1}{\alpha}}\right), \tag{12}$$

and $g_\alpha(t)$ is the probability density of a stable subordinator with index $0 < \alpha < 1$ and Laplace transform $\tilde{g}_\alpha(s) = \exp(-s^\alpha)$. If $D(u)$ is the standard stable subordinator, a strictly increasing stochastic process with stationary independent increments such that $D(1)$ has probability density g_α, then a simple calculation [20, Corollary 3.1] shows that the *inverse stable subordinator*

$$E_t = \inf\{u > 0 : D(u) > t\} \tag{13}$$

has density (12). Then it follows that (11) is the conditional probability density of $x = X_\alpha(t)$ given $y = X_\alpha(0)$, where $X_\alpha(t) := X_1(E_t)$ and the time change E_t is independent of the outer process $X_1(t)$. Since E_t has the same distribution as $t^\alpha E_1$ [20, Proposition 3.1], the fractional Pearson diffusion $X_\alpha(t)$ is a kind of subdiffusion, where particles move along the same trajectories, but more slowly than the Pearson diffusion $X_1(t)$. Bingham [38] and Bondesson, Kristiansen, and Steutel [39] show that

$$\int_0^\infty e^{-su}f_t(u)\,du = E_\alpha(-st^\alpha) := \sum_{j=0}^{\infty}\frac{(-st^\alpha)^j}{\Gamma(1+\alpha j)}, \tag{14}$$

using the Mittag-Leffler function. Then it follows from (10) and (11) that the transition density

$$\begin{aligned}p_\alpha(x, t; y) &= \sum_{n=0}^{\infty}\mathbf{m}(x)Q_n(x)Q_n(y)\int_0^\infty e^{-\lambda_n u}f_t(u)\,du\\ &= \mathbf{m}(x)\sum_{n=0}^{\infty}E_\alpha(-\lambda_n t^\alpha)Q_n(x)Q_n(y).\end{aligned} \tag{15}$$

An alternative proof [35, Theorem 3.2] uses separation of variables, and the fact that $E_\alpha(-\lambda t^\alpha)$ is an eigenfunction of the Caputo derivative with eigenvalue $-\lambda$ [40,41].

3. Correlation function

If the time-homogeneous Markov process $X_1(t)$ is in steady state, then its probability density $\mathbf{m}(x)$ stays the same over all time. We will say that fractional Pearson diffusion is in steady state if it starts with the distribution $\mathbf{m}(x)$. The fractional Pearson diffusion in steady state is first-order stationary, i.e., $X_\alpha(t)$ has the same probability density $\mathbf{m}(x)$ for all $t > 0$. Indeed, in view of (11),

$$\int_0^\infty \mathbf{m}(x) f_t(u)\, du = \mathbf{m}(x).$$

Thus the fractional Pearson diffusion in steady state has mean $\mathbb{E}[X_\alpha](t) = \mathbb{E}[X_1(t)] = m_1$ and variance $\mathrm{Var}[X_\alpha(t)] = \mathrm{Var}[X_1(t)] = m_2^2$ which do not vary over time. The stationary Pearson diffusion has correlation function

$$\mathrm{corr}[X_1(s), X_1(t)] = \exp(-\theta|t-s|), \tag{16}$$

where the correlation parameter $\theta = \lambda_1$ is the smallest positive eigenvalue of the backward generator (8) [35]. Thus the Pearson diffusion exhibits *short-range dependence*, with a correlation function that falls off exponentially. The next result gives an explicit formula for the correlation function of a fractional Pearson diffusion in steady state.

Theorem 3.1. *Suppose that $X_1(t)$ is a Pearson diffusion in steady state, so that its correlation function is given by (16). Then the correlation function of the corresponding fractional Pearson diffusion $X_\alpha(t) = X_1(E_t)$, where E_t is an independent standard inverse α-stable subordinator (13), is given by*

$$\mathrm{corr}[X_\alpha(t), X_\alpha(s)] = E_\alpha(-\theta t^\alpha) + \frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)} \int_0^{s/t} \frac{E_\alpha(-\theta t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz \tag{17}$$

for $t \ge s > 0$, where $E_\alpha(z)$ is the Mittag-Leffler function.

Proof. Write

$$\begin{aligned}\mathrm{corr}[X_\alpha(t), X_\alpha(s)] &= \mathrm{corr}[X_1(E_t), X_1(E_s)] \\ &= \int_0^\infty \int_0^\infty e^{-\theta|u-v|} H(du, dv),\end{aligned} \tag{18}$$

a Lebesgue–Stieltjes integral with respect to the bivariate distribution function $H(u, v) := \mathbb{P}[E_t \le u, E_s \le v]$ of the process E_t.

To compute the integral in (18), we use the bivariate integration by parts formula [42, Lemma 2.2]

$$\begin{aligned}\int_0^a \int_0^b F(u, v) H(du, dv) = &\int_0^a \int_0^b H([u, a] \times [v, b]) F(du, dv) + \int_0^a H([u, a] \times (0, b]) F(du, 0) \\ &+ \int_0^b H((0, a] \times [v, b]) F(0, dv) + F(0, 0) H((0, a] \times (0, b]),\end{aligned} \tag{19}$$

with $F(u, v) = e^{-\theta|u-v|}$, and the limits of integration a and b are infinite:

$$\begin{aligned}\int_0^\infty \int_0^\infty F(u, v) H(du, dv) &= \int_0^\infty \int_0^\infty H([u, \infty] \times [v, \infty]) F(du, dv) + \int_0^\infty H([u, \infty] \times (0, \infty]) F(du, 0) \\ &\quad + \int_0^\infty H((0, \infty] \times [v, \infty]) F(0, dv) + F(0, 0) H((0, \infty] \times (0, \infty]) \\ &= \int_0^\infty \int_0^\infty \mathbb{P}[E_t \ge u, E_s \ge v] F(du, dv) + \int_0^\infty \mathbb{P}[E_t \ge u] F(du, 0) + \int_0^\infty \mathbb{P}[E_s \ge v] F(0, dv) + 1,\end{aligned} \tag{20}$$

since $E_t > 0$ with probability 1 for all $t > 0$. Note that $F(du, v) = f_v(u) du$ for all $v \ge 0$, where

$$f_v(u) = -\theta e^{-\theta(u-v)} I\{u > v\} + \theta e^{-\theta(v-u)} I\{u \le v\}. \tag{21}$$

Integrate by parts to get

$$\begin{aligned}\int_0^\infty \mathbb{P}[E_t \ge u] F(du, 0) &= \int_0^\infty (1 - \mathbb{P}[E_t < u]) \left(-\theta e^{-\theta u}\right) du \\ &= \left[e^{-\theta u} \mathbb{P}[E_t \ge u]\right]_0^\infty + \int_0^\infty e^{-\theta u} f_t(u) du \\ &= E_\alpha(-\theta t^\alpha) - 1,\end{aligned} \tag{22}$$

in view of (14). Similarly,

$$\int_0^\infty \mathbb{P}[E_s \geq v]F(0, dv) = E_\alpha(-\theta s^\alpha) - 1,$$

and hence (20) reduces to

$$\int_0^\infty \int_0^\infty F(u, v)H(du, dv) = I + E_\alpha(-\theta t^\alpha) + E_\alpha(-\theta s^\alpha) - 1, \tag{23}$$

where

$$I = \int_0^\infty \int_0^\infty \mathbb{P}[E_t \geq u, E_s \geq v]F(du, dv).$$

Assume (without loss of generality) that $t \geq s$. Then $E_t \geq E_s$, so $\mathbb{P}[E_t \geq u, E_s \geq v] = \mathbb{P}[E_s \geq v]$ for $u \leq v$. Write $I = I_1+I_2+I_3$, where

$$\begin{aligned}
I_1 &:= \int_{u<v} \mathbb{P}[E_t \geq u, E_s \geq v]F(du, dv) = \int_{u<v} \mathbb{P}[E_s \geq v]F(du, dv) \\
I_2 &:= \int_{u=v} \mathbb{P}[E_t \geq u, E_s \geq v]F(du, dv) = \int_{u=v} \mathbb{P}[E_s \geq v]F(du, dv) \\
I_3 &:= \int_{u>v} \mathbb{P}[E_t \geq u, E_s \geq v]F(du, dv).
\end{aligned}$$

Since $F(du, dv) = -\theta^2 e^{-\theta(v-u)} du\, dv$ for $u < v$, we may write

$$\begin{aligned}
I_1 &= -\theta^2 \int_{v=0}^\infty \int_{u=0}^v \mathbb{P}[E_s \geq v]e^{\theta(u-v)} du dv \\
&= -\theta \int_{v=0}^\infty \mathbb{P}[E_s \geq v](1 - e^{-\theta v}) dv \\
&= -\theta\, \mathbb{E}[E_s] - \theta \int_{v=0}^\infty \mathbb{P}[E_s \geq v]e^{-\theta v} dv \\
&= -\frac{\theta s^\alpha}{\Gamma(1+\alpha)} - (E_\alpha(-\theta s^\alpha) - 1),
\end{aligned} \tag{24}$$

using the well-known formula $\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \geq x]dx$ for any positive random variable, the relation (22), and the formula $\mathbb{E}[E_t] = t^\alpha/\Gamma(1+\alpha)$ for the mean of the standard inverse α-stable subordinator [43, Eq. (9)].

Since $F(du, v) = f_v(u)du$, where the function (21) has a jump of size 2θ at the point $u = v$, we also have

$$I_2 = 2\theta \int_0^\infty \mathbb{P}[E_s \geq v]dv = 2\theta\, \mathbb{E}[E_s] = \frac{2\theta s^\alpha}{\Gamma(1+\alpha)}.$$

Since $F(du, dv) = -\theta^2 e^{-\theta(u-v)} du\, dv$ for $u > v$ as well, we have

$$I_3 = -\theta^2 \int_{v=0}^\infty \mathbb{P}[E_t \geq u,\ E_s \geq v] \int_{u=v}^\infty e^{-\theta(u-v)} du\, dv. \tag{25}$$

Next, we obtain an expression for $\mathbb{P}[E_t \geq u,\ E_s \geq v]$. Since the process E_t is inverse to the stable subordinator D_u, we have $\{E_t > u\} = \{D_u < t\}$ [20, Eq. (3.2)], and since E_t has a density, it follows that $\mathbb{P}[E_t \geq u,\ E_s \geq v] = \mathbb{P}[D_u < t, D_v < s]$. Since $D(u)$ has the same distribution as $u^{1/\alpha}D(1)$, the random variable $D(u)$ has the density function $g_\alpha(x, u) = u^{-1/\alpha} g_\alpha(xu^{-1/\alpha})$. Then, a comparison with (12) reveals that

$$\frac{x}{\alpha} g_\alpha(x, u) = u f_x(u),$$

where $f_t(u)$ is the probability density (12) of $u = E_t$. Since D_u has stationary independent increments, it follows that

$$\begin{aligned}
\mathbb{P}[E_t \geq u, E_s \geq v] &= \mathbb{P}[D_u < t, D_v < s] \\
&= \mathbb{P}[(D_u - D_v) + D_v < t, D_v < s] \\
&= \int_{y=0}^s g_\alpha(y, v) \int_{x=0}^{t-y} g_\alpha(x, u-v) dx dy \\
&= \int_{y=0}^s \frac{\alpha}{y} v f_y(v) \int_{x=0}^{t-y} \frac{\alpha}{x}(u-v) f_x(u-v) dx dy.
\end{aligned}$$

Substituting the expression above back into (25) and using the Fubini theorem, it follows that

$$\begin{aligned} I_3 &= -\theta^2 \int_{y=0}^{s} \frac{\alpha}{y} \int_{x=0}^{t-y} \frac{\alpha}{x} \int_{v=0}^{\infty} v f_y(v) \int_{u=v}^{\infty} (u-v) f_x(u-v) e^{-\theta(u-v)} du\, dv\, dx\, dy \\ &= -\theta^2 \int_{y=0}^{s} \frac{\alpha}{y} \int_{x=0}^{t-y} \frac{\alpha}{x} \int_{v=0}^{\infty} v f_y(v) dv \int_{z=0}^{\infty} z f_x(z) e^{-\theta z} dz\, dx\, dy, \end{aligned}$$

where

$$\int_{v=0}^{\infty} v f_y(v) dv = \mathbb{E}[E_y] = \frac{y^{\alpha}}{\Gamma(1+\alpha)}. \tag{26}$$

Next, we claim that

$$\int_0^{\infty} z f_x(z) e^{-\theta z} dz = -\frac{x}{\alpha\theta} \frac{d}{dx} E_\alpha(-\theta x^\alpha). \tag{27}$$

To see that (27) holds, first differentiate the power series expansion for the Mittag-Leffler function to obtain

$$\begin{aligned} \frac{d}{dx} E_\alpha(-\theta x^\alpha) &= \sum_{j=1}^{\infty} \frac{(-\theta x^\alpha)^{j-1} j}{\Gamma(1+\alpha j)} (-\theta \alpha x^{\alpha-1}) \\ &= \frac{\alpha}{x} \sum_{j=1}^{\infty} \frac{(-\theta x^\alpha)^j j}{\Gamma(1+\alpha j)}. \end{aligned} \tag{28}$$

Then expand $e^{-\theta z}$ in a Taylor series expansion, and integrate term by term:

$$\begin{aligned} \int_0^{\infty} z f_x(z) e^{-\theta z} dz &= \sum_{k=0}^{\infty} \frac{(-\theta)^k}{k!} \int_0^{\infty} z^{k+1} f_x(z) dz \\ &= \sum_{k=0}^{\infty} \frac{(-\theta)^k}{k!} E[E_x^{k+1}] = \sum_{k=0}^{\infty} \frac{(-\theta)^k}{k!} x^{\alpha(k+1)} \frac{(k+1)!}{\Gamma(1+\alpha(k+1))} \\ &= -\frac{1}{\theta} \sum_{k=0}^{\infty} \frac{(-\theta x^\alpha)^{k+1}(k+1)}{\Gamma(1+\alpha(k+1))} = -\frac{1}{\theta} \sum_{j=0}^{\infty} \frac{(-\theta x^\alpha)^j j}{\Gamma(1+\alpha j)}. \end{aligned}$$

Then, apply (28) to see that (27) holds.

Now, it follows using (26) and (27) and then a substitution $z = y/t$ that

$$\begin{aligned} I_3 &= -\theta^2 \int_{y=0}^{s} \frac{\alpha}{y} \int_{x=0}^{t-y} \frac{\alpha}{x} \left[\frac{y^\alpha}{\Gamma(1+\alpha)} \right] \left[-\frac{x}{\alpha\theta} \frac{d}{dx} E_\alpha(-\theta x^\alpha) \right] dx\, dy \\ &= \frac{\theta\alpha}{\Gamma(1+\alpha)} \int_{y=0}^{s} \frac{1}{y^{1-\alpha}} \int_{x=0}^{t-y} \frac{d}{dx} E_\alpha(-\theta x^\alpha) dx\, dy \\ &= \frac{\theta\alpha}{\Gamma(1+\alpha)} \int_{y=0}^{s} \frac{1}{y^{1-\alpha}} (E_\alpha(-\theta(t-y)^\alpha) - 1) dy \\ &= \frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)} \int_0^{s/t} \frac{E_\alpha(-\theta t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz - \frac{\theta s^\alpha}{\Gamma(1+\alpha)}. \end{aligned}$$

Then, it follows from (18) and (23) that

$$\begin{aligned} \mathrm{corr}[X_\alpha(t), X_\alpha(s)] &= \int_0^{\infty} \int_0^{\infty} F(u,v) H(du, dv) \\ &= I_1 + I_2 + I_3 + E_\alpha(-\theta t^\alpha) + E_\alpha(-\theta s^\alpha) - 1 \\ &= \left[-\frac{\theta s^\alpha}{\Gamma(1+\alpha)} - E_\alpha(-\theta s^\alpha) + 1 \right] + \frac{2\theta s^\alpha}{\Gamma(1+\alpha)} + \frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)} \\ &\quad \times \int_0^{s/t} \frac{E_\alpha(-\theta t^\alpha(1-z)^\alpha)}{z^{1-\alpha}} dz - \frac{\theta s^\alpha}{\Gamma(1+\alpha)} + E_\alpha(-\theta t^\alpha) + E_\alpha(-\theta s^\alpha) - 1 \\ &= \frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)} \int_0^{s/t} \frac{E_\alpha(-\theta t^\alpha(1-z)^\alpha)}{z^{1-\alpha}} dz + E_\alpha(-\theta t^\alpha), \end{aligned}$$

which agrees with (17). $\square$

Remark 3.2. When $t = s$, it must be true that $\mathrm{corr}[X_\alpha(t), X_\alpha(s)] = 1$. To see that this follows from (17), recall the formula for the beta density:

$$\int_0^x y^{a-1}(x-y)^{b-1}\,dy = B(a,b)x^{a+b-1} \quad \text{where } B(a,b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

for $a > 0$ and $b > 0$, and write

$$\begin{aligned}
\frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)}\int_0^1 \frac{E_\alpha(-\theta t^\alpha(1-z)^\alpha)}{z^{1-\alpha}}dz &= \frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)}\int_0^1 \sum_{j=0}^\infty \frac{(-\theta t^\alpha(1-z)^\alpha)^j}{\Gamma(1+\alpha j)} z^{1-\alpha}dz \\
&= \frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)}\sum_{j=0}^\infty \frac{(-\theta t^\alpha)^j}{\Gamma(1+\alpha j)}\int_0^1 (1-z)^{\alpha j} z^{\alpha-1}dz \\
&= \frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)}\sum_{j=0}^\infty \frac{(-\theta t^\alpha)^j}{\Gamma(1+\alpha j)} B(\alpha j+1,\alpha) \\
&= \frac{\theta t^\alpha}{\Gamma(1+\alpha)}\sum_{j=0}^\infty \frac{\alpha\Gamma(\alpha)(-\theta t^\alpha)^j}{\Gamma(1+\alpha(j+1))} \\
&= -\sum_{j=0}^\infty \frac{(-\theta t^\alpha)^{j+1}}{\Gamma(1+\alpha(j+1))} = 1 - E_\alpha(-\theta t^\alpha).
\end{aligned}$$

Then, it follows from (17) that $\mathrm{corr}[X_\alpha(t), X_\alpha(s)] = 1$.

Remark 3.3. Stationary Pearson diffusions exhibit short-range dependence, since their correlation function (16) falls off exponentially fast. However, the correlation function of a fractional Pearson diffusion falls off like a power law with exponent $\alpha \in (0, 1)$, and so this process exhibits *long-range dependence*. To see this, fix $s > 0$ and recall [44, Eq. (2.14)] that

$$E_\alpha(-\theta t^\alpha) \sim \frac{1}{\Gamma(1-\alpha)\theta t^\alpha} \quad \text{as } t \to \infty.$$

Then

$$E_\alpha(-\theta t^\alpha(1-sy/t)^\alpha) \sim \frac{1}{\Gamma(1-\alpha)\theta t^\alpha(1-sy/t)^\alpha}$$

as $t \to \infty$ for any $y \in [0, 1]$. In addition, from [45],

$$|E_\alpha(-\theta t^\alpha(1-sy/t)^\alpha)| \le \frac{c}{1+\theta t^\alpha(1-sy/t)^\alpha}$$

for all $t > 0$, and using the dominated convergence theorem we get

$$\begin{aligned}
\frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)}\int_0^{s/t} \frac{E_\alpha(-\theta t^\alpha(1-z)^\alpha)}{z^{1-\alpha}}dz &= \left(\frac{s}{t}\right)^\alpha \frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)}\int_0^1 y^{\alpha-1}E_\alpha\left(-\theta t^\alpha(1-sy/t)^\alpha\right)dy \\
&\sim \left(\frac{s}{t}\right)^\alpha \frac{\alpha}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\int_0^1 y^{\alpha-1}dy = \left(\frac{s}{t}\right)^\alpha \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)}
\end{aligned}$$

as $t \to \infty$. It follows from (17) that for any fixed $s > 0$ we have

$$\mathrm{corr}(X_\alpha(t), X_\alpha(s)) \sim \frac{1}{t^\alpha\Gamma(1-\alpha)}\left(\frac{1}{\theta} + \frac{s^\alpha}{\Gamma(\alpha+1)}\right) \quad \text{as } t \to \infty. \tag{29}$$

Remark 3.4. A Pearson diffusion in steady state is a stationary stochastic process, i.e., the joint distribution of $X_1(t_1), \ldots, X_1(t_n)$ is the same as that of $X_1(s+t_1), \ldots, X_1(s+t_n)$ for any $s > 0$. A fractional Pearson diffusion is not stationary, since the inverse stable subordinator is not stationary. The joint distribution of the inverse stable subordinator E_t at multiple times has recently been computed [46], and, in principle, this can be used to give a different proof of (17). However, the resulting integrals do not seem tractable.

Remark 3.5. Since fractional Pearson diffusions are not Markovian, the transition density (15) is not sufficient to determine the finite-dimensional distributions of the process (since the Chapman–Kolmogorov formula does not apply). That is, neither the governing backward equation (6) nor the corresponding forward equation (7) uniquely determines the process. For diffusions with constant coefficients, there has been some work on identifying and solving the governing equations of the joint density for multiple times [47,48]. It would be interesting to extend this work, to obtain the governing equations for fractional Pearson diffusions at multiple times.

4. Summary

Fractional Pearson diffusions are governed by time-fractional diffusion equations with polynomial coefficients. Theorem 3.1 in this paper gives an explicit formula for the covariance function of a fractional Pearson diffusion in steady state, in terms of Mittag-Leffler functions, which also shows that fractional Pearson diffusions are long-range dependent. The correlation falls off like a power law, with exponent equal to the order of the fractional derivative in time.

Acknowledgements

NNL was partially supported by a grant from the European Commission PIRSES-GA-2008-230804 (Marie Curie). MMM was partially supported by NSF grant DMS-1025486 and NIH grant R01-EB012079. We would like to thank Dumitru Baleanu for organizing and editing this special issue on Fractional Differentiation and its Applications I. We also thank two anonymous reviewers for helpful suggestions.

References

[1] S. Bhalekar, V. Daftardar-Gejji, D. Baleanu, R. Magin, Transient chaos in fractional Bloch equations, Comput. Math. Appl. 64 (12) (2012) 3367–3376.
[2] R.L. Magin, Fractional Calculus in Bioengineering, Begell House, Redding, 2006.
[3] F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics, Vol. 378, Springer Verlag, Berlin, 1997.
[4] J. Sabatier, O.P. Agrawal, J.A.T. Machado, Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer, New York, 2007.
[5] W. Chen, H. Sun, Multiscale statistical model of fully-developed turbulence particle accelerations, Modern Phys. Lett. B 23 (3) (2009) 449–452.
[6] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339 (1) (2000) 1–77.
[7] R. Metzler, J. Klafter, The random walk: recent developments in the description of anomalous transport by fractional dynamics, J. Phys. A: Math. Gen. 37 (31) (2004) R161–R208.
[8] D. Benson, S. Wheatcraft, M. Meerschaert, Application of a fractional advection-dispersion equation, Water Resour. Res. 36 (2000) 1403–1412.
[9] B. Berkowitz, A. Cortis, M. Dentz, H. Scher, Modeling non-Fickian transport in geological formations as a continuous time random walk, Rev. Geophys. 44 (2) (2006) 1–49.
[10] E. Scalas, Five years of continuous-time random walks in econophysics, in: A. Namatame, T. Kaizouji, Y. Aruka (Eds.), The Complex Networks of Economic Interactions, Springer, New York, 2006, pp. 3–16.
[11] J. Masoliver, M. Montero, J. Perelló, G.H. Weiss, J. Perello, The continuous time random walk formalism in financial markets, J. Econ. Behav. Organ. 61 (4) (2006) 577–598.
[12] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, Fractional Calculus Models and Numerical Methods, in: Series on Complexity, Nonlinearity and Chaos, World Scientific, Singapore, 2012.
[13] I. Podlubny, Fractional Differential Equations, Academic Press, Boston, 1999.
[14] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, London, 1993.
[15] V.V. Uchaikin, V.M. Zolotarev, Chance and Stability. Stable Distributions and Their Applications, VSP, Utrecht, 1999.
[16] M.M. Meerschaert, A. Sikorskii, Stochastic Models for Fractional Calculus, De Gruyter, Berlin, 2012.
[17] A. Einstein, On the movement of small particles suspended in a stationary liquid demanded by the molecular kinetic theory of heat, Ann. Phys. 17 (1905) 549–560.
[18] W. Chen, Time-space fabric underlying anomalous diffusion, Chaos Solitons Fractals 28 (4) (2006) 923–929.
[19] Y.J. Liang, Wen Chen, A survey on numerical evaluation of Lévy stable distributions and a new MATLAB toolbox, Signal Process. 93 (1) (2013) 242–251.
[20] M.M. Meerschaert, H.-P. Scheffler, Limit theorems for continuous time random walks with infinite mean waiting times, J. Appl. Probab. 41 (2004) 623–638.
[21] R.R. Nigmatullin, D. Baleanu, The derivation of the generalized functional equations describing self-similar processes, Fract. Calc. Appl. Anal. 15 (4) (2012) 718–740.
[22] M.M. Meerschaert, P. Straka, Inverse stable subordinators, Math. Model. Nat. Phenom. 2013 (in press). Preprint available at www.stt.msu.edu/users/mcubed/hittingTime.pdf.
[23] K. Pearson, Tables for Statisticians and Biometricians, Cambridge University Press, Cambridge, UK, 1914.
[24] G.E. Uhlenbeck, L.S. Ornstein, On the theory of Brownian motion, Phys. Rev. 36 (1930) 823–841.
[25] J.C. Cox, J.E. Ingersoll Jr., S.A. Ross, A theory of the term structure of interest rates, Econometrica 53 (1985) 385–407.
[26] A.N. Kolmogorov, Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung (on analytical methods in probability theory), Math. Ann. 104 (1931) 415–458.
[27] E. Wong, The construction of a class of stationary Markov processes, in: Proc. Sympos. Appl. Math. XVI, American Math. Soc, Providence RI, 1964, pp. 264–276.
[28] F. Avram, N.N. Leonenko, L. Rabehasaina, Series expansions for the first passage distribution of Wong–Pearson jump-diffusions, Stoch. Anal. Appl. 27 (4) (2009) 770–796.
[29] F. Avram, N.N. Leonenko, N. Šuvak, Parameter estimation for Fisher–Snedecor diffusion, Statistics 45 (1) (2011) 1–16.
[30] J.L. Forman, M. Sørensen, The Pearson diffusions: a class of statistically tractable diffusion processes, Scand. J. Stat. 35 (2008) 438–465.
[31] N.N. Leonenko, N. Šuvak, Statistical inference for reciprocal gamma diffusion processes, J. Statist. Plann. Inference 140 (2010) 30–51.
[32] N.N. Leonenko, N. Šuvak, Statistical inference for student diffusion process, Stoch. Anal. Appl. 28 (6) (2010) 972–1002.
[33] W.T. Shaw, A. Manir, Dependency without copulas and ellipticity, Eur. J. Finan. 15 (7–8) (2009) 661–674.
[34] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, part II, Geophys. J. R. Astron. Soc. 13 (1967) 529–539.
[35] N.N. Leonenko, M.M. Meerschaert, A. Sikorskii, Fractional Pearson diffusion, 2012 (submitted for publication). Preprint available at www.stt.msu.edu/users/mcubed/LMS.pdf.
[36] A.N. Kochubei, A Cauchy problem for evolution equations of fractional order, Differ. Equ. 25 (1989) 967–974.
[37] B. Baeumer, M.M. Meerschaert, Stochastic solutions for fractional Cauchy problems, Fract. Calc. Appl. Anal. 4 (2001) 481–500.
[38] N.H. Bingham, Limit theorems for occupation times of Markov processes, Z. Wahrscheinlichkeitstheor. Verwandte Geb. 17 (1971) 1–22.
[39] L. Bondesson, G. Kristiansen, F. Steutel, Infinite divisibility of random variables and their integer parts, Statist. Probab. Lett. 28 (1996) 271–278.
[40] F. Mainardi, R. Gorenflo, On Mittag-Leffler-type functions in fractional evolution processes, J. Comput. Appl. Math. 118 (1–2) (2000) 283–299.
[41] M.M. Meerschaert, E. Nane, P. Vellaisamy, Fractional Cauchy problems on bounded domains, Ann. Probab. 37 (3) (2009) 979–1007.

[42] R.D. Gill, M.J. van der Laan, J.A. Wellner, Inefficient estimators of the bivariate survival function for three models, Ann. Inst. Henri Poincaré 31 (3) (1995) 545–597.
[43] B. Baeumer, M.M. Meerschaert, Fractional diffusion with two time scales, Physica A 373 (2007) 237–251.
[44] A. Piryatinska, A.I. Saichev, W. Woyczynski, Models of anomalous diffusion: the subdiffusive case, Physica A 349 (2005) 375–420.
[45] A.M. Krägeloh, Two families of functions related to the fractional powers of generators of strongly continuous contraction semigroups, J. Math. Anal. Appl. 283 (2003) 459–467.
[46] M.M. Meerschaert, P. Straka, Semi-Markov approach to continuous time random walk limit processes, 2012 (submitted for publication). Preprint available at www.stt.msu.edu/users/mcubed/fddCTRW.pdf.
[47] A. Baule, R. Friedrich, A fractional diffusion equation for two-point probability distributions of a continuous-time random walk, Europhys. Lett. 77 (1) (2007) 10002.
[48] M.M. Meerschaert, P. Straka, Fractional dynamics at multiple times, J. Stat. Phys. 149 (5) (2012) 878–886.