Statistics & Probability Letters 4 (1986) 43–45
North-Holland
January 1986

REGULAR VARIATION AND DOMAINS OF ATTRACTION IN $\mathbb{R}^k$

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Received December 1984
Revised May 1985

Abstract: The results of W. Feller characterizing domains of attraction in $\mathbb{R}^1$ in terms of regular variation are extended to the multivariable case.

Keywords: domains of attraction, regular variation, stable laws, infinitely divisible laws, Lévy measures.

1. Introduction

Suppose that μ and ν are probability distributions on $\mathbb{R}^k$ and that ν is full, i.e., cannot be supported on any $(k-1)$ dimensional affine subspace of $\mathbb{R}^k$. We say that μ belongs to the domain of attraction of ν if for a sequence of independent random vectors $\{X_n\}$ with common distribution μ there exists $a_n > 0$ and $b_n \in \mathbb{R}^k$ such that

$$a_n^{-1}(X_1 + \cdots + X_n) - b_n \Rightarrow Y \qquad (*)$$

where Y is a random vector with distribution ν. We say that ν is stable if it has a nonempty domain of attraction.

Multivariable domains of attraction were first characterized by E.L. Rvačeva, who generalized the work of Gnedenko and Kolmogorov in $\mathbb{R}^1$ (see [3,8]). William Feller later presented an elegant and intuitive treatment of the one variable problem using the theory of regular variation. In this paper we apply a multivariable theory of regular variation developed recently by A. Jakimiv [4] to obtain an extension of Feller's results to the case of random vectors. As in the one variable case the theory of regular variation allows a more concise statement of criteria for attraction to a stable law, along with a considerably simpler proof. Another advantage of this approach is that it can be extended to solve the more general domains of attraction problem in which the scalars a_n in $(*)$ are replaced by linear transformations (see [6,7]).

2. Results

Let $\Gamma = \mathbb{R}^k - \{0\}$. A Borel measurable function $F: \Gamma \to \mathbb{R}^+$ varies regularly at infinity if there exists $e \in \Gamma$ and $\psi: \Gamma \to \mathbb{R}^+$ such that for all $x \in \Gamma$

$$\lim_{t \to \infty} F(tx)/F(te) = \psi(x). \qquad (1)$$

In this case there exists some $\rho \in \mathbb{R}$ called the index of F such that

$$\psi(\lambda x) = \lambda^\rho \psi(x) \qquad (2)$$

for all $\lambda > 0$ and all $x \in \Gamma$ (see [4]). In the case $\rho = 0$ we say that F varies slowly.

Theorem 1. *μ is in the domain of attraction of a full normal law on $\mathbb{R}^k$ if and only if the truncated second moment function*

$$M(x) = \int \langle y, (x/\|x\|)\rangle I(|\langle y, (x/\|x\|)\rangle| \leqslant \|x\|)\mu\{dy\} \qquad (3)$$

is slowly varying.

Now let Π denote the set of σ-finite Borel measures on Γ which assign finite measure to sets bounded away from the origin. Write $\nu_n \to \nu$ if ν_n, $\nu \in \Pi$ and if $\nu_n(A) \to \nu(A)$ for all Borel sets A bounded away from the origin such that $\nu(\partial A) = 0$. Here ∂A denotes the topological boundary of A. We say that $\mu \in \Pi$ varies regularly at infinity if there exists a Borel subset E of Γ and a measure $\phi \in \Pi$ which cannot be supported on any proper subspace of $\mathbb{R}^k$ such that, as $t \to \infty$,

$$\mu\{t\,\mathrm{d}x\}/\mu(tE) \to \phi\{\mathrm{d}x\}. \tag{4}$$

In this case there exists some $\rho < 0$ called the index of μ such that, for all $\lambda > 0$,

$$\mu\{\lambda\,\mathrm{d}x\} = \lambda^\rho \mu\{\mathrm{d}x\}. \tag{5}$$

Theorem 2. *μ belongs to the domain of attraction of a full nonnormal stable law on $\mathbb{R}^k$ if and only if μ varies regularly with index $\rho \in (-2, 0)$.*

If F varies regularly then (1) holds for any nonzero vector e, except that the limit Ψ changes by a multiplicative constant. The properties of regularly varying functions needed below are obtained using (1) and the fact that if F varies regularly then $F(te)$ is a (one variable) regularly varying function of $t > 0$ with the same index. Similarly if μ varies regularly then (4) holds for any Borel set E bounded away from the origin such that $\phi(\partial E) = 0$, with the same effect on the limit. Then $u(tE)$ is a regularly varying function of t having the same index as μ.

3. Proofs

It is straightforward to verify that Theorems 1 and 2 are equivalent to the characterization of domains of attraction appearing in [8]. However, it is instructive to provide a new proof of these results here, since by way of the theory of regular variation the original proofs can be greatly simplified.

In the following proofs we will use the standard criteria for convergence of an infinitesimal triangular array of random vectors due to E.L. Rvačeva (see [8, Theorem 2.3 and Theorem 2.4]) with a minor modification. It is easy to check that both theorems remain true if the domain of integration in the integral formula is changed from $|x| < \varepsilon$ to $|t'x| < \varepsilon |t|$. We will consistently use this modified form below without further comment. On this subject we should also comment that Rvačeva makes a slight error in the statement of Theorem 2.3, in that convergence of the integral formula both with the lim sup and the lim inf is necessary. The correct statement, along with a more modern treatment of the subject, can be found in [1] (see p. 67).

Proof of Theorem 1. Suppose that (*) holds and that Y is normal with Lévy representation $(a, Q, 0)$ where Q is positive definite. By [8, Theorem 2.4] we obtain

$$a_n^{n/2}\left[\int \langle y, x\rangle^2 \delta_n(y)\mu\{\mathrm{d}y\} - \left(\int \langle y, x\rangle \delta_n(y)\mu\{\mathrm{d}y\}\right)^2\right] \to Q(X) \tag{6}$$

for all $x \in \mathbb{R}^k$, where $\delta_n(y) = I(y\colon |\langle y, (x/\|x\|)\rangle < a_n\varepsilon)$. If the function M defined by (3) is bounded, then it is easy to check that M varies slowly. Otherwise we have that $M(tx) \to \infty$ as $t \to \infty$ for all $x \in \Gamma$. An application of the Schwartz Inequality shows that in this case the squared term in (6) is dominated as $n \to \infty$ by the first integral, and so we have

$$\lim_{n\to\infty} (n/a_n^2) M(a_n x) = Q(x)/\|x\|^2 \tag{7}$$

for all $x \in \Gamma$. Now an application of [2, VIII. 8, Lemma 3] along with (1) yields that M is slowly varying.

Conversely, suppose we are given that M varies slowly and (1) holds. If M is bounded then the Central Limit Theorem applies, and we are done. Otherwise we may define

$$a_n = \sup\{t > 0\colon M(te)/t^2 \geqslant 1/n\} \tag{8}$$

and then $(n/a_n^2)M(a_n e) \to 1$. From this and (1) we again obtain that (7) holds for all $x \in \Gamma$ with $Q(x) = \|x\|^2\psi(x)$, and again this is equivalent to (6). Now it only remains to show that $n\mu\{a_n\,\mathrm{d}x\} \to 0$, by [8, Theorem 2.4]. But this is an immediate consequence of (7) and [2, VIII.9, Theorem 2].

Proof of Theorem 2. Suppose that $(*)$ holds where Y is nondegenerate with Lévy representation $(a, 0, \phi)$. By [8, Theorem 2.3] we obtain $n\mu\{a_n \, dx\} \to \phi\{dx\}$. It is easy to see from $(*)$ that $(a_n + 1/a_n) \to 1$, and now it follows that (4) holds, where $E = \{x: \|x\| > c\}$ is chosen to make $\phi(\partial E) = 0$. Then (5) holds also, and from Lévy's classical result (see [5]) we obtain $-2 < \rho < 0$.

Conversely, suppose that μ varies regularly with index $\rho \in (-2, 0)$ and (4) holds. By (5) we see that ϕ is a Lévy measure. Choose E as above so that $\phi(\partial E) = 1$ and define, for all n sufficiently large,

$$a_n = \sup\{t > 0: n\mu(tE) \geqslant 1\}. \quad (9)$$

By (5) we have that $\phi(\partial E) = 0$ so that $\mu(tE)$ varies regularly (see remarks following Theorem 2 above). It follows that $nu(a_n E) \to 1$ and then by (4) we have that

$$n\mu\{a_n \, dx\} \to \phi\{dx\} \quad (10)$$

and now by virtue of [8, Theorem 2.3] and an application of the Schwartz Inequality it will suffice to show that

$$\overline{\lim_{\varepsilon \to 0}} \; \overline{\lim_{n \to \infty}} \; (a_n^{n/2}) \int \langle y, x \rangle^2 \delta_n(y) \mu\{dy\} = 0 \quad (11)$$

where δ_n is as defined above. But this follows directly from (10) and [2, VIII.9, Theorem 2].

Acknowledgement

This paper was drawn from my dissertation research under the guidance of Professor Fred Hoppe at the University of Michigan. I would also like to thank the referee for suggesting the reference [1] and for making several useful comments which improved the presentation of these results.

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