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Regular variation and generalized domains of attraction in $\mathbb{R}^k$

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Abstract: William Feller used regular variation to give an elegant description of domains of attraction in $\mathbb{R}^1$. In this paper we extend Feller's results to the case of generalized domains of attraction in $\mathbb{R}^k$.

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1. Introduction

Let μ, ν be probability distributions on $\mathbb{R}^k$ and suppose that ν is full, i.e. that it cannot be supported on any $(k-1)$-dimensional hyperplane. Let X, X_1, X_2, $X_3, \ldots$ denote independent random vectors with common distribution μ and let Y denote a random vector with distribution ν. We say that μ belongs to the generalized domain of attraction (GDOA) of ν if there exist linear operators A_n and constants b_n such that

$$A_n(X_1 + \cdots + X_n) - b_n \Rightarrow Y. \tag{1.1}$$

We say that ν is operator stable if it has a nonempty GDOA. Operator stable laws and generalized domains of attraction are the natural multidimensional analogues to the stable laws and domains of attraction in one variable. In the one variable case Feller (1971) gives an elegant treatment of the topic using regular variation. This paper extends the results of Feller to the multidimensional case, using a multivariable analogue to regular variation.

Early research on multidimensional regular variation by Stam (1977), de Haan and Resnick (1979), and Jakimiv (1981) lead to the results in Meerschaert (1986b) in which we used a simple version of regular variation in $\mathbb{R}^k$ to treat (1.1) in the special case of norming by scalars $A_n = a_n I$. A slightly more general version of regular variation was used by de Haan, Omey and Resnick (1984) and Meerschaert (1991b) to treat the case of 'vector norming' where every A_n is assumed to be diagonal. In order to treat the general case we will use a slightly extended version of the general theory of regular variation in $\mathbb{R}^k$ laid out in Meerschaert (1988).

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2. Results

Suppose $F(x) > 0$ is defined for nonzero x in $\mathbb{R}^k$ such that $F(rx)$ is a monotone function of $r > 0$ for all x. We will say that F is RV(a) provided that there exist nonsingular linear operators A_n tending in norm to zero such that $nF(A_n x_n) \to g(x)$ whenever $x_n \to x$, where $g > 0$ satisfies $g(tx) = t^a g(x)$ for all x and all $t > 0$. We will say that a σ-finite Borel measure μ defined on $\mathbb{R}^k - \{0\}$ is RV(B) provided that there exist linear operators A_n tending in norm to zero such that $n\mu\{A_n^{-1}\,\mathrm{d}x\} \to \phi\{\mathrm{d}x\}$ in the weak topology for some ϕ which cannot be supported on any proper subspace of $\mathbb{R}^k$ and which satisfies $t^{-1}\{\mathrm{d}x\} = \phi\{t^B\,\mathrm{d}x\}$ for all $t > 0$.

The case where the operator stable limit law ν has both a normal and a nonnormal component will be dealt with via a reduction argument. In this case Sharpe (1969) shows that we can write $\mathbb{R}^k$ as the direct sum of proper subspaces L_1 and L_2 with associated projections P_1 and P_2 so that $P_1\nu$ is normal on L_1 and $P_2\nu$ is nonnormal operator-stable on L_2. Here $P\nu\{\mathrm{d}x\} = \nu\{P^{-1}\mathrm{d}x\}$.

Theorem. *Suppose X is a random vector on $\mathbb{R}^k$ with distribution μ.*

(i) *Suppose $EX = 0$. Then μ is in the GDOA of some normal law ν if and only if the truncated second moment function*

$$F(x) = E\langle X, x\rangle^2 I(|\langle X, x\rangle| < 1). \tag{2.1}$$

is RV(2).

(ii) *μ is in the GDOA of some operator stable law ν having no normal component if and only if μ is* RV(B) *where every eigenvalue of B exceeds $\frac{1}{2}$.*

(iii) *If $P_i\mu$ belongs to the GDOA of $P_i\nu$ for $i = 1, 2$ then μ is in the GDOA of ν.*

Corollary. *Suppose X is a random vector on $\mathbb{R}^k$ whose distribution μ belongs to the GDOA of some operator stable law ν.*

(i) *If ν is normal then $m_b(x) = E|\langle X, x\rangle|^b$ is finite for all x and $b < 2$. In particular μ has an expectation, and we may choose b_n to zero expectation.*

(ii) *Suppose μ is in the GDOA of ν nonnormal so that μ is* RV(B) *and define $m = \min\{\mathrm{Re}(\lambda)\}$, $M = \max\{\mathrm{Re}(\lambda)\}$ where λ ranges over the eigenvalues of B. Then $m_b(x)$ is finite whenever $bM < 1$ and $mB(x)$ is infinite for all nonzero x whenever $bm > 1$. If $m > 1$ then we may take all $b_n = 0$ and if $M < 1$ we may center to zero expectation.*

Generalized domains of attraction in the case of a normal limit were characterized by Hahn and Klass (1980a) using a version of uniform regular variation, i.e. they apply one-dimensional regular variation uniformly in all radial directions. The statement of our result (i) differs in that the condition for attraction to a normal law is inherently multidimensional. Although uniformity over radial directions is not transparent in our statement, it is a fundamental component of our proof of (i) via the uniformity on compact sets of the regular variation of $r \to F(r^{-1}x)$. Since any law in the GDOA of a normal law must have an expectation, the assumption $EX = 0$ entails no loss of generality.

Part (ii) of the theorem is essentially just a restatement of the result in Meerschaert (1986a) in terms of regular variation. Part (iii) is a new result which is related to the spectral decomposition of Meerschaert (1991a). See the remarks at the end of this paper for more details.

In the case of a normal limit Klosowska (1980) shows that $\langle X, \theta\rangle$ belongs to the one-dimensional domain of attraction of a normal law for every unit vector $\theta \in \mathbb{R}^k$. Then the portion of the corollary (i) concerning the existence of radial moments may be obtained by reference to one-variable results. The protion concerning the centering constants was originally proven by Hahn and Klass (1980b). We include these results here for the sake of completeness. We also wish to emphasize that all of these results follow immediately from the proof of theorem above. The Corollary's results on the existence and nonexistence of radial moments extend prior results of Hudson, Veeh and Weiner (1988).

Generalized domains of attraction were characterized by Hahn and Klass (1985) using classical methods. The theorem above provides an alternative characterization of GDOA in terms of regular variation. The use of regular variation allows us to treat the problem in a simple and unified manner. It also produces easily obtainable information about moments, centering, and tail behavior.

3. Proof

Let X be an arbitrary indexing set and suppose that $R:(0,\infty)\times X\to(0,\infty)$ is defined such that $R(r,x)$ is a Borel measurable function of r for each x. We will say that R is uniformly regularly varying with index $a\in\mathbb{R}$ if for all $t>0$,

$$\lim_{r\to\infty} R(tr,x)/R(r,x)=t^a, \tag{3.1}$$

and this convergence is uniform in x. In the applications to come the indexing set X is the unit sphere, in which case the definition (3.1) imposes a uniform growth rate in all directions.

Lemma 1. *If F is* RV(a) *then $R(r,x)=F(r^{-1}x)$ is uniformly regularly varying with index $-a$ on compact subsets of* $\{\|x\|>0\}$.

Proof. Suppose $x_r\to x$ and define $n(r)=\sup\{n:\|A_n^{-1}x_r\|\leqslant r\}$ so that $n=n(r)$ is well-defined for large r and $n\to\infty$ as $r\to\infty$. If we let $y_n=A_n^{-1}x_r/r$ then the set $\{y_n\}$ is relatively compact. If $y_n\to y$ along a subsequence then

$$\frac{R(tr,x_r)}{R(r,x_r)}=\frac{F(A_nt^{-1}y_n)}{F(A_ny_n)}\to t^{-a} \tag{3.2}$$

along a subsequence. Since any subsequence $r_n\to\infty$ has a further subsequence with this property, the lemma follows. □

This form of regular variation is adequate for the treatment of normal GDOA, but for the nonnormal case we require a generalization which allows the growth rate to vary with x. A Borel measurable function $R(r)$ is R-O varying if it is real-valued and positive for $r\geqslant A$ and if there exist positive constants $a>1$, $m<1$, $M>1$ such that $m\leqslant R(tr)/R(r)\leqslant M$ whenever $1\leqslant t\leqslant a$ and $r\geqslant A$. We will say that the function $R(r,x)$ is uniformly R-O varying if it is an R-O varying function of r for each x, and the constants A, a, m, M can be chosen independent of x. A necessary and sufficient condition for uniform R-O variation is that $mt^h\leqslant R(tr,x)/R(r,x)\leqslant Mt^H$ whenever $t\geqslant 1$ and $r\geqslant A$. Here we have let $h=\log m/\log a$ and $H=\log M/\log a$. The infimum of all such H is called the upper Matuszewska index and the supremum of all such h is called the lower Matuszewska index.

Suppose that μ is RV(B). The definition imposes both upper and lower bounds on the rate at which the tails of μ diminish. Indeed the bounds are essentially the same as for ϕ, which is easily seen to be RV(b) by taking $A_n=n^{-A}$. In order to obtain specific upper and lower bounds we will apply uniform R-O variation. Let $R(r,x)=\mu\{y:|\langle x,y\rangle|\leqslant r\}$ and define the truncated moments U_b, V_c as in Feller (1971) i.e. let

$$U_b(r,x)=\int_0^r t^bR(\mathrm{d}t,x),\qquad V_c(r,x)=\int_r^\infty t^cR(\mathrm{d}t,x). \tag{3.3}$$

Since we are interested only in the tail behavior of μ, there is no loss in generality in assuming that μ assigns zero measure to some neighborhood of the origin, so that R, U_b are always well-defined. Define $m = \min\{\text{Re}(\lambda)\}$, $M = \max\{\text{Re}(\lambda)\}$, where λ ranges over the eigenvalues of B. Let $h = -1/m$ and $H = -1/M$. The following result states in part that the tails of μ diminish no faster than t^h and no slower than t^H.

Lemma 2. *Suppose that μ is* RV(B).
(i) *If $h + c < 0$ then V_c is uniformly* R-O *varying with lower Matuszewska index $h + c$ on compact subsets of $\{\|x\| > 0\}$.*
(ii) *If $H + b > 0$ then U_b is uniformly* R-O *varying with upper Matuszewska index $H + b$ on compact subsets of $\{\|x\| > 0\}$.*

Proof. The proof of both parts is similar. We will only prove part (ii). Define the set-valued function

$$H(x) = \{y : |\langle x, y\rangle| \leqslant 1\} \tag{3.4}$$

and let

$$F(x) = \int |\langle x, y\rangle|^b \mu\{dy\}, \qquad g(x) = \int |\langle x, y\rangle|^b \phi\{dy\}, \tag{3.5}$$

where both integrals are taken over $H(x)$. Using the fact that $H(B^*x) = B^{-1}H(x)$ we obtain $tg(x) = g(t^{B^*}x)$ for all x and all $t > 0$, while $nF(A_n^* x_n) \to g(x)$ whenever $x_n \to x$ provided that the boundary of $H(x)$ has ϕ-measure zero. (Actually ϕ is continuous, but we will not need this fact.) Note also that $U_b(r, x) = r^b F(x/r)$.

Suppose $x_r \to x$ and define $n(r) = \sup\{n : \|(A_n^*)^{-1} x_r\| \leqslant r\}$ so that $n = n(r)$ is well-defined and tends to infinity along with r. If we let $y_n = (A_n^*)^{-1}(x_r/r)$ then $\{y_n\}$ is relatively compact. If $y_n \to y$ along a subsequence then

$$\frac{U_b(tr, x_r)}{U_b(r, x_r)} = \frac{t^b F(A_n^* t^{-1} y_n)}{F(A_n^* y_n)} \to \frac{t^b g(t^{-1} y)}{g(y)} \tag{3.6}$$

along a subsequence. Since any subsequence $r_n \to \infty$ has a further subsequence with this property, we have that U_b is uniformly R-O varying. Furthermore it follows by a straightforward computation that g is uniformly R-O varying with upper Matuszewska index $b + H$ on compact subsets of $\{\|x\| > 0\}$, and so U_b has upper index $b + H$ as well. □

Recall that $X, X_1, X_2, X_3, \ldots$ are i.i.d. according to μ, and that μ is said to belong to the GDOA of a random vector Y whose distribution ν has Lévy representation (a, Q, ϕ) if there exist linear operators A_n and constants b_n such that (1.1) holds.

Proof of theorem. We begin with the proof of (i). Suppose that (1.1) holds. By the standard convergence criteria for triangular arrays we have

$$n\left[\int \langle y, x\rangle^2 \mu\{A_n^{-1}\,dy\} - \left(\int \langle y, x\rangle \mu\{A_n^{-1}\,dy\}\right)^2\right] \to Q(x) \tag{3.7}$$

for all x, where the domain of integration is the set $H(x)$ defined by (3.4). Write $A_n^* x = r_n \theta_n$ where $\|\theta_n\| = 1$ and $r_n > 0$. A change of variable in (3.1) yields

$$nr_n^2\left[\int \langle y, \theta_n\rangle^2 \mu\{dy\} - \left(\int \langle y, \theta_n\rangle \mu\{dy\}\right)^2\right] \to Q(x) \tag{3.8}$$

where we integrate over $H(A_n^* x)$. This set tends to $\mathbb{R}^k$ as $n \to \infty$ and it follows that the second integral in (3.8) tends to zero ($EX = 0$) while the first remains bounded away from zero as $n \to \infty$. Then (3.8) remains true with the second integral deleted. Rewrite in terms of F to obtain

$$nF(A_n^* x) \to Q(x) \tag{3.9}$$

and since both sides of (3.9) are quadratic forms it is easy to check that this convergence is uniform on compact sets. Then F is RV(2).

Suppose now that F is RV(2) and so (3.9) holds. By reversing the arguments in the direct half of the proof we arrive back at (3.7), and now we need only show that $n\mu\{A_n^{-1}\,\mathrm{d}y\} \to 0$.

We will argue using uniform regular variation. Let U_b, V_c denote the truncated moments of μ defined by (3.3) with $R(r, x) = \mu\{y: |\langle x, y\rangle| \leqslant r\}$. Lemma 1 implies that $F(x/r)$ is regularly varying with index (-2) uniformly on the unit sphere $S = \{x: \|x\| = 1\}$. Therefore $U_2(r, x) = r^2 F(x/r)$ is slowly varying uniformly on S. Given a nonzero vector x choose r_n, θ_n as before so that $A_n^* x = r_n \theta_n$. Define $H(x) = \{y: |\langle x, y\rangle| > 1\}$. Then we have $n\mu(A_n^{-1}H(x)) = nV_0(r_n^{-1}, \theta_n)$ where $r_n \to 0$. From (3.7) we have that $nr_n^2 U_2(r_n^{-1}, \theta_n) \to Q(x)$ and now by applying a uniform version of Feller (1971, VIII.9, Theorem 2) we obtain $nV_0(r_n^{-1}, \theta_n) \to 0$ which finishes the proof of (i).

Part (ii) is simply a restatement of the result in Meerschaert (1986) in terms of regular variation. We recount the proof here for the convenience of the reader. The direct half is an immediate consequence of the standard convergence criteria for triangular arrays. As to the converse, an application of the Schwarz Inequality shows that it is enough to verify

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} n \int_{|\langle y, x\rangle| < \varepsilon} \langle y, x\rangle^2 \mu\{A_n^{-1}\,\mathrm{d}y\} = 0 \tag{3.10}$$

for all x. As in the proof of (i) let U_b, V_c denote the truncated absolute moments of μ. Writing $A_n^* x = r_n \theta_n$ as before, the expression under the limit in (3.10) becomes $nr_n^2 U_2(\varepsilon/r_n, \theta_n)$. Apply uniform R-O variation to obtain

$$nr_n^2 U_2(\varepsilon/r_n, \theta_n) \leqslant c_1 \varepsilon^2 nV_0(\varepsilon/r_n, \theta_n) \leqslant c_1 \varepsilon^\delta nV_0(r_n^{-1}, \theta_n) = c_2 \varepsilon^\delta n\mu(A_n^{-1}H(x)). \tag{3.11}$$

The result follows from the fact that $n\mu\{A_n^{-1}\,\mathrm{d}y\} \to \phi\{\mathrm{d}y\}$ as $n \to \infty$.

Now we prove (iii). Without loss of generality we have $EP_1 X = 0$. By assumption $P_i[A_n(X_1 + \cdots + X_n) - b_n] \Rightarrow P_i Y$ for $i = 1, 2$ where A_n commutes with P_i. Let $[a, Q, \phi]$ denote the Lévy representation of ν so that we have $n\mu\{A_n^{-1}P_i^{-1}\,\mathrm{d}y\} \to \phi\{P_i^{-1}\,\mathrm{d}y\}$ where P_i^{-1} denotes the pre-image, and the limit is zero for $i = 1$. It follows that $n\mu\{A_n^{-1}\,\mathrm{d}y\} \to \phi\{\mathrm{d}y\}$. We have also that the second criterion for the convergence of a triangular array holds whenever $x \in L_1$ or $x \in L_2$. Now in order to complete the proof it will suffice to show that this remains true for all other x.

Suppose $x = x_1 + x_2$ where $x_i \in L_i$ is nonzero for $i = 1, 2$. A straightforward computation shows that it is enough for

(i) $$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} n \int_{\|y\| < \varepsilon} \langle y, x_1\rangle \langle y, x_2\rangle \mu\{A_n^{-1}\,dy\} = 0;$$

(ii) $$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left[n \int_{\|y\| < \varepsilon} \langle y, x_1\rangle \mu\{A_n^{-1}\,dy\} \cdot \int_{\|y\| < \varepsilon} \langle y, x_2\rangle \mu\{A_n^{-1}\,dy\} \right] = 0.$$

But both of these follow immediately from the Schwarz Inequality along with (3.10) and the fact that (3.7) holds with the second integral deleted. □

Proof of corollary. We begin with part (i). The statement about moments follows immediately from Feller (1971, VIII.9, Theorem 2). As for the centering constants, it suffices to show that

$$n \int \langle x, y\rangle \mu\{A_n^{-1}\,\mathrm{d}y\} \to 0 \tag{3.12}$$

for all x, where we integrate over the set $H(x/c)$ used in the proof of the theorem above. Using the same notation as in that proof the expression on the left becomes

$$nr_nV_1(c/r_n, \theta_n) = o(n)nr_n^2c^{-1}U_2(c/r_n, \theta_n) \sim o(n)c^{-1}Q(x)$$

which vanishes in the limit for any c.

Now we prove part (ii) of the corollary. The statement about moments follows immediately from the proof of the theorem. To prove centering to zero expectation it suffices to show that the integral in (3.12) can be made arbitrarily small for large n by appropriate choice of c. Using the uniform R-O variation we obtain

$$nr_nV_1(c/r_n, \theta_n) \leqslant K_1nr_n^2c^{-1}U_2(c/r_n, \theta_n) \leqslant K_2nr_n^2c^{-\varepsilon}U_2(r_n^{-1}, \theta_n) \leqslant K_3c^{\varepsilon}\mu(H(x)).$$

Let $c \to 0$.

To prove we can set $b_n = 0$ when $m > 1$ we use

$$nr_nU_1(c/r_n, x_n) \leqslant K_1nr_nc^{-\varepsilon}U_1(r_n^{-1}, x_n) \leqslant K_2c^{\varepsilon}nV_0(r_n^{-1}, x_n) \leqslant K_3c^{\varepsilon}\mu(H(t)).$$

Let $c \to 0$. □

4. Remarks

As in the one-variable case, several properties of operator stable limits emerge naturally from the regular variation. The index of regular variation B which occurs in the theorem above is the exponent of the operator stable law, defined by Sharpe (1969). In general this index is not unique. Further information on exponents of operator stable laws can be found in Holmes, Hudson and Mason (1982), Hudson, Jurek and Veeh (1986), and Meerschaert and Veeh (1992). Many of the results which appear in those papers can be interpreted more broadly as pertaining to the index of a regularly varying measure on $\mathbb{R}^k$.

The spectral decomposition for generalized domains of attraction in Meerschaert (1991a) provides a partial converse to part (iii) of the theorem above. Suppose that μ belongs to the GDOA of ν operator stable. In general it is too much to hope that $P_i\mu$ will belong to the GDOA of $P_i\nu$ for $i = 1, 2$. However it is true that there always exists a nonsingular linear operator T on $\mathbb{R}^k$ such that μ also belongs to the GDOA of $\nu_0 = T\nu$ and that $P_i\mu$ belongs to the GDOA of $P_i\nu_0$ for $i = 1, 2$.

The spectral decomposition can also be used to sharpen the results of the corollary above. Suppose that μ belongs to the GDOA of some operator stable law ν having no normal component. Let $\{a_1, \dots, a_d\}$ be an enumeration of the real spectrum of the index B of the regularly varying measure μ. The spectral decomposition yields projection operators $P_1, \dots, P_d$ with the property that $P_i\mu$ is attracted to a spectrally simple operator stable law ν_i. This means that the exponent of ν_i (i.e. the index of regular variation of $P_i\mu$) has real spectrum consisting of the single element a_i. Then the radial moments $m_b(x)$ will be finite whenever $b < 1/a_i$ and infinite whenever $b > 1/a_i$. If $a_i > 1$ then we can center to zero expectation, and if $a_i < 1$ then no centering is necessary.

In the one-variable case every element of a nonnormal domain of attraction has regularly varying tails. In the case of generalized domains of attraction, the tail function

$$V_0(r, \theta) = \Pr\{|\langle X, \theta\rangle| > r\}$$

need not vary regularly. In fact Meerschaert (1990) gives an example to show that the operator stable laws themselves need not have regularly varying tails. However we can obtain a slightly weaker tail condition from the remarks in the preceding paragraph. In the one–variable case attraction to a stable law with index α requires that the tail function varies regularly with index $-\alpha$. In other words the tails tend to zero about as fast as $r^{-\alpha}$. When $\mu_i = P_i\mu$ is attracted to the spectrally simple operator stable law ν_i, the tails of μ_i are R-O varying with both upper and lower Matuszewska index equal to $\alpha_i = 1/a_i$. Hence they tend to zero faster than $r^{-\alpha_i+\varepsilon}$ and slower than $r^{-\alpha_i-\varepsilon}$ for any $\varepsilon > 0$.

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