1. Introduction

Given \{X_n\} independent and identically distributed random vectors on \(\mathbb{R}^k\) with distribution \(\mu\), let \(S_n = X_1 + \cdots + X_n\). We say that \(\mu\) belongs to the vector-normed domain of attraction of \(\nu\) if there exist \(a_n, b_n \in \mathbb{R}^k\) with \(a^{(i)}_n > 0\) for all \(i = 1, 2, \ldots, k\) such that

\[
\left(\frac{S_n^{(i)}}{a^{(i)}_n}, \ldots, \frac{S_n^{(k)}}{a^{(k)}_n}\right) - b_n \Rightarrow Y,
\]

where \(Y\) is a nondegenerate random vector on \(\mathbb{R}^k\) with distribution \(\nu\). Vector-normed domains of attraction were first considered by Resnick and Greenwood (1979), who obtained a complete characterization in the case \(k = 2\). Some connections with regular variation in \(\mathbb{R}^k\) were examined in de Haan, Omey and Resnick (1984). In this paper we use regular variation in \(\mathbb{R}^k\) to obtain a new characterization of vector-normed domains of attraction, thereby extending the results of Feller (1971) in \(\mathbb{R}^1\).

2. Results

Regular variation in \(\mathbb{R}^k\) was defined in Meerschaert (1988). If \(x, y \in \mathbb{R}^k\) we denote by \(xy\) the componentwise product \((x_1y_1, \ldots, x_ky_k)\). Let \(\mathbb{R}^+_k = \{(x_1, \ldots, x_k) : \text{all } x_i > 0\}\) and \(\mathbb{R}^-_k = -\mathbb{R}^+_k\). If \(\lambda > 0\) and \(\alpha \in \mathbb{R}^k\) let \(\lambda^\alpha = (\lambda^{\alpha_1}, \ldots, \lambda^{\alpha_k})\).

A function \(f: [a, \infty) \to \mathbb{R}^k\) will be said to vary regularly with index \(\alpha\) if it is Borel measurable and if for all \(\lambda > 0\) we have

\[
\lim_{r \to \infty} f(\lambda r) f(r)^{-1} = \lambda^\alpha. \tag{2.1}
\]

Suppose now that \(F: \mathbb{R}^k \to \mathbb{R}^+\) is Borel measurable. We will say that \(F\) is regularly varying at infinity (respectively, zero) if there exists \(f: \mathbb{R}^+ \to \mathbb{R}^k\) regularly varying with index \(\alpha\) in \(\mathbb{R}^k_+\) (respectively, \(\mathbb{R}^k_-\)) and \(e \neq 0\) such that whenever \(x_r \to x\neq 0\) we have

\[
\lim_{r \to \infty} F(f(r)x_r)/F(f(r)e) = \gamma(x) \tag{2.2}
\]

for some \(\gamma: \mathbb{R}^k - \{0\} \to \mathbb{R}^+\). In this case the choice of \(e \neq 0\) is arbitrary and affects the limit \(\gamma\) only in terms of a multiplicative constant. It follows from (2.2) that \(R(r) = F(f(r)e)\) varies regularly with some index \(\beta \in \mathbb{R}\) and that for all \(\lambda > 0\), all \(x \neq 0\) we have

\[
\lambda^\beta \gamma(x) = \gamma(\lambda^\alpha x). \tag{2.3}
\]
While $\alpha$, $\beta$ are not uniquely determined by $F$, their ratio $\rho = \beta \alpha^{-1}$ is uniquely determined, and we call $\rho$ the index of regular variation of $F$.

Let $\{ X_n \}$ be as above and define the truncated second moment function

$$ F(y) = E(X_n, y)^2 I\{|X_n, y| < 1\} \quad (2.3) $$

for $y \neq 0$.

**Theorem 2.1.** $\mu$ is in the vector-normed domain of attraction of a nondegenerate normal law if and only if the function $F(y)$ defined by (2.3) varies regularly at zero with index $(2, 2, \ldots, 2)$.

Now let $\Pi$ denote the class of $\alpha$-finite Borel measures on $\mathbb{R}^k - \{0\}$ which are finite on sets bounded away from the origin, and write $\nu_n \rightarrow \nu$ if $\nu_n, \nu \in \Pi$ and $\nu_n(A) \rightarrow \nu(A)$ for all Borel subsets bounded away from the origin such that $\nu(A) = 0$. We will say that $\mu \in \Pi$ is regularly varying at infinity (respectively, zero) if there exists $f: \mathbb{R}^k \rightarrow \mathbb{R}^k$ regularly varying with index $\alpha \in \mathbb{R}^k$ (respectively, $\mathbb{R}^k$) and a Borel set $E$ such that

$$ \mu\{f(r)\,dx\}/\mu\{f(r)\,E\} \rightarrow \phi(dx) \quad (2.4) $$

for some measure $\phi \in \Pi$ which cannot be supported on any proper subspace of $\mathbb{R}^k$. The set $E$ is arbitrary and affects the limit $\phi$ only in terms of a multiplicative constant. It follows from (2.3) that $\mu(f(r)\,E)$ is a regularly varying function of $r > 0$ with some index $\beta \in \mathbb{R}$, and that for all $\lambda > 0$,

$$ \lambda^\beta \phi(dx) = \phi(\lambda^\alpha \,dx). \quad (2.5) $$

One again we will call $\rho = \beta \alpha^{-1}$ the index of regular variation.

**Theorem 2.2.** $\mu$ belongs to the vector-normed domain of attraction of a nondegenerate limit law having no normal component if and only if $\mu$ varies regularly at infinity with index $\rho = (\rho_1, \ldots, \rho_k)$ where all $\rho_i \in (-2, 0)$.

If the limit distribution has both normal and nonnormal components, then according to Sharpe (1969) we can decompose $\nu$ into the product of two marginals, one normal and one strictly non-normal. Let $L_1 = \text{Span}\{ e_i : Y_i \text{ normal and } L_1 = L_1^+ \}$. Denote by $\sigma_i$ the projection map onto $L_i$.

Then $\sigma_1(Y)$ is normal, $\sigma_2(Y)$ has no normal component, and $\sigma_1(Y), \sigma_2(Y)$ are independent. The following result allows us to treat the general limit case by reduction to the two cases considered above.

**Theorem 2.3.** $\mu$ is in the vector-normed domain of attraction of $\nu$ if and only if for $i = 1, 2$,

$$ \sigma_i(a_n^{-1}S_n - b_n) \Rightarrow \nu_i(Y). \quad (2.6) $$

3. Proofs

In this section we prove the theorems stated in Section 2 characterizing vector-normed domains of attraction in terms of regular variation. We will proceed by extending the arguments Feller employed in $\mathbb{R}^k$. Using the notation introduced in the beginning of Section 2, we may rewrite (1.1) in the abbreviated form

$$ a_n^{-1}S_n - b_n \Rightarrow Y. \quad (3.1) $$

**Proof of Theorem 2.1.** Without loss of generality $EX_n = 0$. For all $y \in \mathbb{R}^k$ we have

$$ n\left[ \int_{\{|x,y|<\epsilon\}} \langle x, y \rangle^2 \mu(a_n \,dx) \right. $$

$$ \left. - \left( \int_{\{|x,y|<\epsilon\}} \langle x, y \rangle \mu(a_n \,dx) \right)^2 \right] \rightarrow Q(y). \quad (3.2) $$

for all $\epsilon > 0$. Since $Y$ is nondegenerate normal, $\phi$ is the zero measure and $Q$ is positive definite. As $n \rightarrow \infty$ the second integral in (3.2) tends to zero, and so (3.2) remains true with this term deleted. Taking $\epsilon = 1$ we have

$$ nF(a_n^{-1}y) \rightarrow Q(y). \quad (3.3) $$

It follows that $F$ varies regularly at zero, and since $Q(ry) = r^2Q(y)$ the index of $F$ is $(2, 2, \ldots, 2)$.

Conversely suppose $F$ varies regularly at zero with index $(2, 2, \ldots, 2)$. Letting $a_n^{-1} = f(r_n)$ where $r_n = \sup\{ r > 0 : nF(f(r)e) \leq 1 \}$ we arrive at (3.3).

Since $EX_n = 0$ this is again equivalent to (3.2), and now we need only show that $n\mu(A_n \,dx) \rightarrow 0$.

This follows easily from (3.3) by a reduction to the one variable case: $nF(e_i/a_n^{(i)}) \rightarrow Q(e_i)$ and so the
truncated second moment of $X_n^{(i)}$ varies slowly at infinity. Hence $n\mu \{ a, A \} \to 0$ for sets of the form $A = \{ x : |x| \geq \epsilon \}$. But any subset of $\mathbb{R}^k$ which is bounded away from the origin is contained in the union of a finite number of these. □

**Proof of Theorem 2.2.** Suppose that (3.1) holds and $Y$ has no normal component. Then $Q \equiv 0$ and $\phi$ cannot be supported on any proper subspace of $\mathbb{R}^k$. From the standard convergence criteria for triangular arrays of random vectors we obtain immediately that $\mu$ varies regularly at infinity. Since $\phi$ is a Lévy measure, and in particular
\[
\int_{0 < |x| < 1} |x|^2 \phi \{ dx \} < \infty \tag{3.4}
\]
we must have $-2 < \rho_1 < 0$.

Conversely suppose $\mu$ varies regularly at infinity with index $\rho$, all $\rho_i \in (-2, 0)$. Let $a_n = f(r_n)$ where $r_n = \sup \{ r > 0 : n\mu \{ f(r)E \} \geq 1 \}$, so that $n\mu \{ a_n \ dx \} \to \phi \{ dx \}$. To show that (3.1) holds, by an application of Schwartz inequality it will suffice to show that for all $\varepsilon$,
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{|x| < \varepsilon} \langle x, y \rangle^2 \mu \{ a_n \ dx \} = 0. \tag{3.5}
\]
Clearly it suffices to show (3.5) for $y = e_i, i = 1, \ldots, k$. And once again, this follows directly by a reduction to the one variable case. □

**Proof of Theorem 2.3.** The direct half is obvious. As to the converse, suppose that (2.6) holds for $i = 1, 2$. Since $\phi \{ \pi^{-1}_{12} \ (d x) \}$ is the Lévy measure of $\pi Y$ we have for $i = 1, 2$,
\[
n\mu \{ a_n \pi^{-1}_{12} (d x) \} \to \phi \{ \pi^{-1}_{12} (d x) \}; \tag{3.6}
\]
and since the limit in (3.6) is the zero measure when $i = 1$, this implies that $n\mu \{ a_n \ dx \} \to \phi \{ dx \}$. We also have
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \left\{ n \left[ \int_{|x| < \varepsilon} \langle x, y \rangle^2 \mu \{ a_n \ dx \} \right] - \left( \int_{|x| < \varepsilon} \langle x, y \rangle \mu \{ a_n \ dx \} \right)^2 - Q(y) \right\} = 0
\]
(3.7)
for all $y \in L_1$ and $y \in L_2$. Suppose then that $y = y_1 + y_2$ where both $y_1 \in L_1$ and $y_2 \in L_2$ are nonzero. We need to show that
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{|x| < \varepsilon} \langle x, y_1 \rangle \langle x, y_2 \rangle \mu \{ a_n \ dx \} = 0,
\]
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{|x| < \varepsilon} \langle x, y_1 \rangle \mu \{ a_n \ dx \} \cdot \int_{|x| < \varepsilon} \langle x, y_2 \rangle \mu \{ a_n \ dx \} = 0. \tag{3.8}
\]
Both integral expressions are dominated by
\[
n\int_{|x| < \varepsilon} \langle x, y_1 \rangle^2 \mu \{ a_n \ dx \}
\]
\[
\cdot n\int_{|x| < \varepsilon} \langle x, y_2 \rangle^2 \mu \{ a_n \ dx \}. \tag{3.9}
\]

The proof of Theorem 2.1 shows that the first term is bounded. Apply (3.5). □

**References**


