



Tempered fractional Brownian motion



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ABSTRACT

Tempered fractional Brownian motion (TFBM) modifies the power law kernel in the moving average representation of a fractional Brownian motion, adding an exponential tempering. Tempered fractional Gaussian noise (TFGN), the increments of TFBM, form a stationary time series that can exhibit semi-long range dependence. This paper develops the basic theory of TFBM, including moving average and spectral representations, sample path properties, and an application to modeling wind speed.

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1. Introduction

This paper defines a new stochastic process, which we call tempered fractional Brownian motion (TFBM), defined by exponentially tempering the power law kernel in the moving average representation of a fractional Brownian motion (FBM). The stationary increments of TFBM are called tempered fractional Gaussian noise (TFGN). When FGN is long range dependent, the corresponding TFGN exhibits semi-long range dependence: Its autocovariance function closely resembles that of FGN on an intermediate scale, but eventually falls off more rapidly. The spectral density of TFGN resembles a negative power law for low frequencies, but remains bounded at very low frequencies.

2. Moving average representation

Let $\{B(t)\}_{t \in \mathbb{R}}$ be a real-valued Brownian motion on the real line, a process with stationary independent increments such that $B(t)$ has a Gaussian distribution with mean zero and variance $\sigma^2|t|$ for all $t \in \mathbb{R}$, for some $\sigma > 0$. Define an independently scattered Gaussian random measure $B(dx)$ with control measure $m(dx) = \sigma^2 dx$ by setting $B[a, b] = B(b) - B(a)$ for any real numbers $a < b$, and then extending to all Borel sets. Then the stochastic integrals $I(f) := \int f(x)B(dx)$ are defined for all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $\int f(x)^2 dx < \infty$, as Gaussian random variables with mean zero and covariance $\mathbb{E}[I(f)I(g)] = \sigma^2 \int f(x)g(x)dx$, see for example Chapter 3 in Samorodnitsky and Taqqu (1994).

Definition 2.1. Given an independently scattered Gaussian random measure $B(dx)$ on \mathbb{R} with control measure $\sigma^2 dx$, for any $\alpha < \frac{1}{2}$ and $\lambda \geq 0$, the stochastic integral

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$$B_{\alpha,\lambda}(t) := \int_{-\infty}^{+\infty} [e^{-\lambda(t-x)_+} (t-x)_+^{-\alpha} - e^{-\lambda(-x)_+} (-x)_+^{-\alpha}] B(dx) \tag{2.1}$$

where $(x)_+ = xI(x > 0)$, and $0^0 = 0$, will be called a *tempered fractional Brownian motion* (TFBM).

It is easy to check that the function

$$g_{\alpha,\lambda,t}(x) := e^{-\lambda(t-x)_+} (t-x)_+^{-\alpha} - e^{-\lambda(-x)_+} (-x)_+^{-\alpha} \tag{2.2}$$

is square integrable over the entire real line for any $\alpha < \frac{1}{2}$, so that TFBM is well-defined. When $-1/2 < \alpha < 1/2$, FBM is a special case of TFBM with $\lambda = 0$. Note also that

$$g_{\alpha,\lambda,ct}(cx) = c^{-\alpha} g_{\alpha,c\lambda,t}(x) \tag{2.3}$$

for all $t, x \in \mathbb{R}$ and all $c > 0$. The next results shows that TFBM has a nice scaling property, involving both the time scale and the tempering. Here the symbol \triangleq indicates equality of finite dimensional distributions.

Proposition 2.2. *TFBM (2.1) is a Gaussian stochastic process with stationary increments, such that*

$$\{B_{\alpha,\lambda}(ct)\}_{t \in \mathbb{R}} \triangleq \{c^H B_{\alpha,c\lambda}(t)\}_{t \in \mathbb{R}} \tag{2.4}$$

for any scale factor $c > 0$, where the Hurst index $H = 1/2 - \alpha$.

Proof. Since $B(dx)$ has control measure $m(dx) = \sigma^2 dx$, the random measure $B(c dx)$ has control measure $c^{1/2} \sigma^2 dx$. Given $t_1 < t_2 < \dots < t_n$, a change of variable $x = cx'$ then yields

$$\begin{aligned} (B_{\alpha,\lambda}(ct_i) : i = 1, \dots, n) &= \left(\int g_{\alpha,\lambda,ct_i}(x) B(dx) : i = 1, \dots, n \right) \\ &\stackrel{d}{=} \left(\int c^{-\alpha} g_{\alpha,c\lambda,t_i}(x') c^{1/2} B(dx') : i = 1, \dots, n \right) \end{aligned}$$

so that (2.4) holds with $H = 1/2 - \alpha$. For any $s, t \in \mathbb{R}$, the integrand (2.2) satisfies $g_{\alpha,\lambda,s+t}(s+x) - g_{\alpha,\lambda,s}(s+x) = g_{\alpha,\lambda,t}(x)$, and hence a change of variable $x = s + x'$ in the moving average representation yields

$$(B_{\alpha,\lambda}(s+t_i) - B_{\alpha,\lambda}(s) : i = 1, \dots, n) \triangleq \left(\int g_{\alpha,\lambda,t_i}(x') B(dx') : i = 1, \dots, n \right)$$

which shows that TFBM has stationary increments. \square

Proposition 2.3. *TFBM (2.1) has the covariance function*

$$\text{Cov}[B_{\alpha,\lambda}(t), B_{\alpha,\lambda}(s)] = \frac{\sigma^2}{2} [C_t^2 |t|^{2H} + C_s^2 |s|^{2H} - C_{t-s}^2 |t-s|^{2H}] \tag{2.5}$$

for any $s, t \in \mathbb{R}$, where $H = 1/2 - \alpha$. Here

$$C_t^2 = \frac{2\Gamma(2H)}{(2\lambda|t|)^{2H}} - \frac{2\Gamma(H + \frac{1}{2})}{\sqrt{\pi}} \frac{1}{(2\lambda|t|)^H} K_H(\lambda|t|), \tag{2.6}$$

for $t \neq 0$, where $K_\nu(z)$ is the modified Bessel function of the second kind, and $C_0^2 = 0$.

Proof. Use the moving average representation (2.1) with $\sigma = 1$ to define

$$\begin{aligned} C_t^2 &:= \mathbb{E}[B_{\alpha,\lambda|t}(1)^2] = \int_{-\infty}^{+\infty} [e^{-\lambda t(1-x)_+} (1-x)_+^{-\alpha} - e^{-\lambda t(-x)_+} (-x)_+^{-\alpha}]^2 dx \\ &= \int_{-\infty}^{+\infty} e^{-2\lambda t(1-x)_+} (1-x)_+^{-2\alpha} dx + \int_{-\infty}^{+\infty} e^{-2\lambda t(-x)_+} (-x)_+^{-2\alpha} dx \\ &\quad - 2 \int_{-\infty}^{+\infty} e^{-\lambda t(1-x)_+} (1-x)_+^{-\alpha} e^{-\lambda t(-x)_+} (-x)_+^{-\alpha} dx. \end{aligned} \tag{2.7}$$

Apply the definition of the gamma function, along with a standard integral formula from p. 344 in Gradshteyn and Ryzhik (2000), to see that (2.6) holds. Since TFBM has stationary increments, it follows from (2.4) that $\mathbb{E}[B_{\alpha,\lambda}(t)^2] = |t|^{2H} C_t^2$ for all t real. Recall the elementary formula $ab = \frac{1}{2}[a^2 + b^2 - (a-b)^2]$, set $a = B_{\alpha,\lambda}(t)$ and $b = B_{\alpha,\lambda}(s)$, take expectations, and use the stationary increments property again, to see that (2.5) holds. \square

Remark 2.4. The integral representation (2.1) is causal, i.e., $B_{\alpha,\lambda}(t)$ depends only on the values of $B(s)$ for $s \leq t$. For applications to spatial statistics, consider

$$\begin{aligned}
 B_{\alpha,\lambda}^{p,q}(t) &= p \int_{-\infty}^{+\infty} \left[e^{-\lambda(t-x)_+} (t-x)_+^{-\alpha} - e^{-\lambda(-x)_+} (-x)_+^{-\alpha} \right] B(dx) \\
 &\quad + q \int_{-\infty}^{+\infty} \left[e^{-\lambda(x-t)_+} (x-t)_+^{-\alpha} - e^{-\lambda(x)_+} (x)_+^{-\alpha} \right] B(dx)
 \end{aligned}
 \tag{2.8}$$

for $p, q \geq 0$. It is not hard to check, by mimicking the proof of Proposition 2.2, that this process also has stationary increments, and satisfies the scaling property

$$\{B_{\alpha,\lambda}^{p,q}(ct)\}_{t \in \mathbb{R}} \triangleq \{c^H B_{\alpha,c\lambda}^{p,q}(t)\}_{t \in \mathbb{R}}
 \tag{2.9}$$

for any scale factor $c > 0$, where the Hurst index $H = 1/2 - \alpha$. When $p = q > 1$, (2.8) is a well-balanced TFBM.

3. Harmonizable representation

Let \hat{B}_1 and \hat{B}_2 be independent Gaussian random measures with $\hat{B}_1(A) = \hat{B}_1(-A)$, $\hat{B}_2(A) = -\hat{B}_2(-A)$ and $\mathbb{E}[(\hat{B}_i(A))^2] = m(A)/2$, where $m(dx) = \sigma^2 dx$, and define the complex-valued Gaussian random measure $\hat{B} = \hat{B}_1 + i\hat{B}_2$. If $f(x)$ is a complex-valued function of x real such that its Fourier transform $\hat{f}(k) := (2\pi)^{-1/2} \int e^{-ikx} f(x) dx$ exists and $\int |\hat{f}(k)|^2 dk < \infty$, we define the stochastic integral $\hat{I}(\hat{f}) = \int \hat{f}(k) \hat{B}(dk) := \int \hat{f}_1(k) \hat{B}_1(dk) - \int \hat{f}_2(k) \hat{B}_2(dk)$, where $\hat{f} = \hat{f}_1 + i\hat{f}_2$ is separated into real and imaginary parts. Then $\hat{I}(\hat{f})$ is a Gaussian random variable with mean zero, such that $\mathbb{E}[\hat{I}(\hat{f})\hat{I}(\hat{g})] = \int \hat{f}(k)\overline{\hat{g}(k)} dk$ for all such functions, and the Parseval identity $\int f(x)g(x) dx = \int \hat{f}(k)\overline{\hat{g}(k)} dk$ implies that $(\int \hat{f}(x)B(dx), \int g(x)B(dx)) \stackrel{d}{=} (\int \hat{f}(k)\hat{B}(dk), \int \hat{g}(k)\hat{B}(dk))$, see Proposition 7.2.7 in Samorodnitsky and Taqqu (1994).

Proposition 3.1. *The TFBM (2.1) has the harmonizable representation*

$$B_{\alpha,\lambda}(t) = \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-itk} - 1}{(\lambda - ik)^{1-\alpha}} \hat{B}(dk).
 \tag{3.1}$$

Proof. To show that the stochastic integral (3.1) exists, note that

$$\int_{-\infty}^{+\infty} \left| \frac{e^{-itx} - 1}{(\lambda - ix)^{1-\alpha}} \right|^2 dx \leq \int_{-\infty}^{+\infty} \frac{4}{(\lambda^2 + x^2)^{1-\alpha}} dx < \infty,$$

since the last integrand is bounded and $O(x^{2\alpha-2})$ as $|x| \rightarrow \infty$, with $2\alpha - 2 < -1$. Observe that the function $g_{\alpha,\lambda,t}$, given by (2.2), has the Fourier transform

$$\begin{aligned}
 \widehat{g_{\alpha,\lambda,t}}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} \left[e^{-\lambda(t-x)_+} (t-x)_+^{-\alpha} - e^{-\lambda(-x)_+} (-x)_+^{-\alpha} \right] dx \\
 &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^t e^{-ikx} e^{-\lambda(t-x)} (t-x)^{-\alpha} dx - \int_{-\infty}^0 e^{-ikx} e^{\lambda x} (-x)^{-\alpha} dx \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[e^{-ikt} \int_0^{+\infty} e^{-u(\lambda-ik)} u^{-\alpha} du - \int_0^{+\infty} e^{-u(\lambda-ik)} u^{-\alpha} du \right] \\
 &= \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}} \frac{e^{-ikt} - 1}{(\lambda - ik)^{1-\alpha}}
 \end{aligned}$$

using the well-known formula for the characteristic function of the gamma density. Then (2.1) along with Proposition 7.2.7 in Samorodnitsky and Taqqu (1994) implies

$$\begin{aligned}
 B_{\alpha,\lambda}(t) &= \int_{-\infty}^{+\infty} g_{\alpha,\lambda,t}(x) B(dx) \\
 &\triangleq \int_{-\infty}^{+\infty} \widehat{g_{\alpha,\lambda,t}}(k) \hat{B}(dk) = \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-ikt} - 1}{(\lambda - ik)^{1-\alpha}} \hat{B}(dk)
 \end{aligned}$$

which is equivalent to (3.1). \square

Remark 3.2. The spectral representation (3.1) reduces to that of causal FBM in the special case $\lambda = 0$ and $-1/2 < \alpha < 1/2$, see for example Eq. (7.2.17) in Samorodnitsky and Taqqu (1994). The general TFBM (2.8) has spectral representation

$$B_{\alpha,\lambda}^{p,q}(t) = \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-itk} - 1}{ik} \left[\frac{p ik}{(\lambda - ik)^{1-\alpha}} - \frac{q ik}{(\lambda + ik)^{1-\alpha}} \right] \hat{B}(dk).
 \tag{3.2}$$

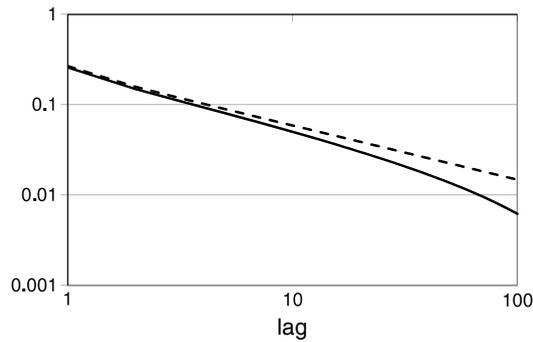


Fig. 1. The autocovariance function (4.4) for TFGN with $\sigma = 1$, $\lambda = 0.001$ and $H = 0.7$ (solid line) and for the corresponding FGN with $\sigma = 1$, $\lambda = 0$ and $H = 0.7$ (dotted line).

4. Tempered fractional Gaussian noise

Given a TFBM (2.1), we define tempered fractional Gaussian noise (TFGN)

$$X_j = B_{\alpha,\lambda}(j + 1) - B_{\alpha,\lambda}(j) \quad \text{for integers } -\infty < j < \infty. \tag{4.1}$$

It follows easily from (2.1) that TFGN has the moving average representation

$$X_j = \int_{-\infty}^{+\infty} [e^{-\lambda(j+1-x)+} (j + 1 - x)_+^{-\alpha} - e^{-\lambda(j-x)+} (j - x)_+^{-\alpha}] B(dx). \tag{4.2}$$

Using (3.1), it also follows that the harmonizable representation of TFGN is

$$X_j = \frac{\Gamma(1 - \alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikj} \frac{e^{-ik} - 1}{(\lambda - ik)^{1-\alpha}} \hat{B}(dk). \tag{4.3}$$

It follows from (2.5) that TFGN is a stationary Gaussian time series with mean zero and covariance function

$$r(j) := \mathbb{E}[X_0 X_j] = \frac{\sigma^2}{2} [|j + 1|^{2H} C_{j+1}^2 - 2 |j|^{2H} C_j^2 + |j - 1|^{2H} C_{j-1}^2], \tag{4.4}$$

where $H = 1/2 - \alpha$, and C_j is given by (2.6).

Remark 4.1. Using the well-known fact that $K_\nu(x) \sim \sqrt{\pi}(2x)^{-1/2} e^{-x}$ as $x \rightarrow \infty$, it follows easily from (2.6) that

$$t^{2H} C_t^2 \rightarrow 2\Gamma(2H)(2\lambda)^{-2H} \quad \text{as } t \rightarrow \infty, \tag{4.5}$$

and hence $C_j \sim C_{j+1}$ as $j \rightarrow \infty$. Then (4.4) along with a Taylor series expansion shows that

$$r(j) \sim \sigma^2 C_j^2 H(2H - 1) |j|^{2H-2} \quad \text{as } j \rightarrow \infty,$$

compare Proposition 7.2.10 in Samorodnitsky and Taqqu (1994). For $\lambda > 0$ sufficiently small, the power law terms in (2.7) dominate, C_j^2 remains almost constant, and $r(j)$ falls off like $|j|^{2H-2}$ for moderate values of $j > 0$. For larger j , the exponential terms in (2.7) dominate, and (4.5) implies that $r(j) \sim j^{-2H}(2H - 1)\Gamma(2H)(2\lambda)^{-2H}$ as $j \rightarrow \infty$. Hence TFGN is short range dependent, but its covariance function is arbitrarily close to that of long range dependent FGN for small values of λ , and moderate lags, when $-1/2 < \alpha < 1/2$. We call this property *semi-long range dependence*, since it is analogous to the *semi-heavy tails* of Barndorff-Nielsen (1998). Fig. 1 shows a log-log plot of $r(j)$ in the case $H = 0.7$ and $\lambda = 0.001$, where FGN exhibits long range dependence.

Proposition 4.2. TFGN (4.1) has the spectral density

$$h(k) = \frac{\Gamma(1 - \alpha)^2}{2\pi} |e^{-ik} - 1|^2 \sum_{\ell=-\infty}^{+\infty} \frac{\sigma^2}{[\lambda^2 + (k + 2\pi\ell)^2]^{H+1/2}}. \tag{4.6}$$

Proof. Recall that the spectral density

$$h(k) = \frac{1}{2\pi} \sum_{j=-\infty}^{+\infty} e^{ikj} r(j) \quad \text{and} \quad r(j) = \int_{-\pi}^{\pi} e^{-ikj} h(k) dk. \tag{4.7}$$

Define $C = \sqrt{2\pi}/\Gamma(1 - \alpha)$ and apply (4.3) to write

$$\begin{aligned}
 r(j) &= \frac{\sigma^2}{C^2} \int_{-\infty}^{+\infty} e^{-ikj} \frac{|e^{-ik} - 1|^2}{(\lambda^2 + k^2)^{(1-\alpha)}} dk \\
 &= \frac{1}{C^2} \int_{-\pi}^{+\pi} e^{-ikj} |e^{-ik} - 1|^2 \sum_{\ell=-\infty}^{+\infty} \frac{\sigma^2}{[\lambda^2 + (k + 2\pi\ell)^2]^{(1-\alpha)}} dk
 \end{aligned}
 \tag{4.8}$$

and then it follows from (4.7) that the spectral density of TFGN is given by (4.6). \square

Remark 4.3. Extending definition (4.1) to all j real, we obtain the continuous parameter TFGN

$$X_t = B_{\alpha,\lambda}(t + 1) - B_{\alpha,\lambda}(t).$$

The harmonizable representation of this process is given by (4.3) with j replaced by t , and the proof of Proposition 4.2 implies that X_t has spectral density

$$h(\omega) = \frac{\Gamma(1 - \alpha)^2}{2\pi} |e^{-i\omega} - 1|^2 \frac{\sigma^2}{[\lambda^2 + \omega^2]^{H+1/2}} \tag{4.9}$$

for all real ω . The fact that $e^{-i\omega} - 1 \sim -i\omega$ as $\omega \rightarrow 0$ yields the low frequency approximation

$$h(\omega) \approx \frac{\sigma^2 \Gamma(1 - \alpha)^2}{2\pi} \frac{\omega^2}{(\lambda^2 + \omega^2)^{H+1/2}},$$

see Section 6 for an application to wind speed data.

5. Sample path properties

We say that the sample paths of a stochastic process $X(t)$ satisfy a uniform Hölder condition of order β on the compact set $K \subset \mathbb{R}$ if there exists a positive random variable A such that

$$|X(x) - X(y)| \leq A|x - y|^\beta$$

almost surely for all $x, y \in K$. We say that the process has Hölder critical exponent $\gamma \in (0, 1)$ if the process satisfies a uniform Hölder condition of any order $\beta \in (0, \gamma)$ on any compact set $K \subset \mathbb{R}$, and fails to satisfy this condition for $\beta \in (\gamma, 1)$.

Theorem 5.1. *The sample paths of the TFBM (2.1) have Hölder critical exponent $H = 1/2 - \alpha$ for any $\alpha \in (-1/2, 1/2)$ and any $\lambda \geq 0$.*

Proof. Since $B_{\alpha,\lambda}(0) = 0$, it follows from Proposition 4 in Bonami and Estrade (2003) that if

$$\gamma = \sup \{ \beta > 0 : \mathbb{E} [B_{\alpha,\lambda}(t)^2] = o(|t|^{2\beta}) \text{ as } |t| \rightarrow 0 \}, \tag{5.1}$$

then the TFBM $B_{\alpha,\lambda}(t)$ satisfies a uniform Hölder condition of order β on any compact set for any $\beta \in (0, \gamma)$, and moreover, if we also have

$$\gamma = \inf \{ \beta > 0 : |t|^{2\beta} = o(\mathbb{E} [B_{\alpha,\lambda}(t)^2]) \text{ as } |t| \rightarrow 0, \}, \tag{5.2}$$

then this TFBM has Hölder critical exponent γ . Use the harmonizable representation (3.1) to write

$$\begin{aligned}
 \mathbb{E} [B_{\alpha,\lambda}(t)^2] &= \frac{1}{C^2} \int_{-\infty}^{+\infty} \frac{e^{-itk} - 1}{(\lambda - ik)^{1-\alpha}} \overline{\left[\frac{e^{-itk} - 1}{(\lambda - ik)^{1-\alpha}} \right]} dk \\
 &= \frac{2}{C^2} \int_{-\infty}^{+\infty} [1 - \cos(tk)] (\lambda^2 + k^2)^{\alpha-1} dk
 \end{aligned}$$

where $C = \sqrt{2\pi}/\Gamma(1 - \alpha)$, and apply the Tauberian theorem for Fourier transforms, Theorem 1 in Pitman (1968), to see that $\mathbb{E} [B_{\alpha,\lambda}(t)^2] \sim H(1/t)$ as $t \rightarrow 0$, where

$$H(x) = \frac{2}{C^2} \int_{|k|>x} (\lambda^2 + k^2)^{\alpha-1} dk.$$

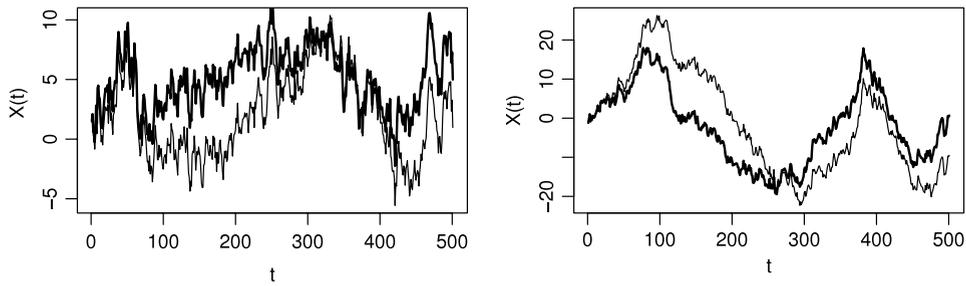


Fig. 2. Left panel: Sample paths of TFBM (thick black line) with $\lambda = 0.03$ and $H = 0.3$, and FBM (thin black line) with $H = 0.3$. Both graphs use the same noise realization $B(t)$. The right panel shows the same plots for $\lambda = 0.01$ and $H = 0.7$.

Since $\lambda^2 + k^2 \sim k^2$ as $k \rightarrow \infty$, for any $\varepsilon > 0$, there exists some $M > 0$ such that $(1 - \varepsilon)k^{2\alpha-2} < (\lambda^2 + k^2)^{\alpha-1} < (1 + \varepsilon)k^{2\alpha-2}$ for all $k > M$, and hence we have

$$\frac{4(1 - \varepsilon)}{(1 - 2\alpha)C^2} x^{2\alpha-1} < H(x) < \frac{4(1 + \varepsilon)}{(1 - 2\alpha)C^2} x^{2\alpha-1}$$

for all $x > M$. Substitute $t = 1/x$ to see that both (5.1) and (5.2) hold with $\gamma = 1 - 2\alpha = 2H$, which completes the proof. \square

Remark 5.2. When $\alpha < -1/2$, TFBM has continuously differentiable sample paths. To see this, write $B_{\alpha,\lambda}(t) = Z_t - Z_0$ where the stationary Gaussian stochastic process

$$Z_t := \int_{-\infty}^{+\infty} e^{-\lambda(t-x)_+} (t-x)_+^{-\alpha} B(dx)$$

belongs to the Matérn class. Hence its sample paths are p times continuously differentiable for any $H > p$, see for example Handcock and Stein (1993, p. 406).

Remark 5.3. The harmonizable representation

$$X(t) = \int_{-\infty}^{+\infty} (e^{-itk} - 1) \hat{f}(k) \hat{B}(dk)$$

defines a mean zero Gaussian processes with stationary increments for any Fourier filter $\hat{f}(k)$ such that $\int [1 - \cos(tk)] |\hat{f}(k)|^2 dk < \infty$. If $|\hat{f}(k)|^2$ is regularly varying at infinity with index $2\alpha - 2$ for some $-1/2 < \alpha < 1/2$, the Karamata Theorem (e.g., see Lemma 5.3.8 (d) in Meerschaert and Scheffler, 2001) implies that $H(x)$ varies regularly at infinity with index $2\alpha - 1$, and then the proof of Theorem 5.1 extends to show that $X(t)$ has Hölder critical exponent $1 - 2\alpha$. Several examples of such processes are given in Bonami and Estrade (2003).

The sample paths of TFBM closely resemble that of FBM for small values of the tempering parameter $\lambda > 0$. The left panel in Fig. 2 compares a typical sample path of both processes, simulated using the same white noise $B(dx)$, in a case where FBM is negative dependent. The right panel shows the corresponding sample paths in a case where FBM is long range dependent. These simulations use a discretized version of the moving average representation (2.1). It would also be interesting to develop a simulation method based on the harmonizable representation (3.1).

6. Discussion

Wind speed data are important for electrical power generation and structural engineering. The most popular model for wind speed near the earth’s surface, due to Davenport (1961), see also Li and Kareem (1990), can be written in the form $s_t = \mu + X_t$ where $\mu = \mathbb{E}[s_t]$ is the average wind speed, and X_t has normalized spectral density

$$4800DV_{10} \frac{x^2}{(1 + x^2)^{\frac{4}{3}}} \tag{6.1}$$

where V_{10} is the mean velocity (m/s) at an altitude of 10 m, D is the corresponding drag coefficient, and $x = 1200\omega/V_{10}$. In view of Remark 4.3, it is not hard to check that (6.1) corresponds to the spectral density of a continuous parameter TFGN with $\lambda = V_{10}/1200$ and $H = 5/6$. Hence TFGN can provide a useful stochastic process model for wind speed data. Fig. 3 compares the spectral density of TFGN and FGN in the case where FGN is long range dependent. The spectral density of FGN blows up at the origin like a power law. The spectral density of TFGN follows the same power law at moderate frequencies, but remains bounded at very low frequencies, a behavior typically seen in wind speed data. See for example in Davenport (1961), Norton (1981), Jang and Jyh-Shinn (1999), and Pérez Beaupuits et al. (2004).

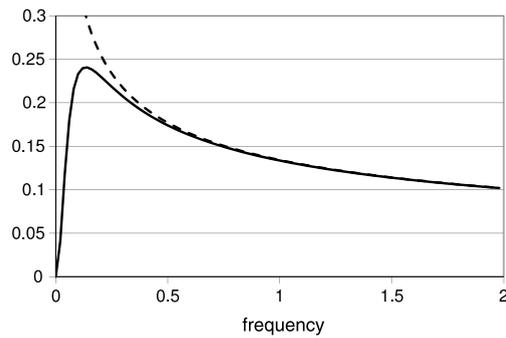


Fig. 3. The spectral density (4.9) for TFGN with $\sigma = 1$, $\lambda = 0.06$ and $H = 0.7$ (solid line) and FGN with $\sigma = 1$, $\lambda = 0$ and $H = 0.7$ (dotted line).

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