Journal of Computational Physics 293 (2015) 14–28

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Tempered fractional calculus ☆

Farzad Sabzikar [a], Mark M. Meerschaert [a,*], Jinghua Chen [b]

[a] Department of Statistics and Probability, Michigan State University, East Lansing, MI 48823, United States
[b] School of Sciences, Jimei University, Xiamen, Fujian, 361021, China

ARTICLE INFO

Article history:
Received 17 February 2014
Received in revised form 3 April 2014
Accepted 7 April 2014
Available online 16 April 2014

Keywords:
Fractional calculus
Anomalous diffusion
Random walk

ABSTRACT

Fractional derivatives and integrals are convolutions with a power law. Multiplying by an exponential factor leads to tempered fractional derivatives and integrals. Tempered fractional diffusion equations, where the usual second derivative in space is replaced by a tempered fractional derivative, govern the limits of random walk models with an exponentially tempered power law jump distribution. The limiting tempered stable probability densities exhibit semi-heavy tails, which are commonly observed in finance. Tempered power law waiting times lead to tempered fractional time derivatives, which have proven useful in geophysics. The tempered fractional derivative or integral of a Brownian motion, called a tempered fractional Brownian motion, can exhibit semi-long range dependence. The increments of this process, called tempered fractional Gaussian noise, provide a useful new stochastic model for wind speed data. A tempered fractional difference forms the basis for numerical methods to solve tempered fractional diffusion equations, and it also provides a useful new correlation model in time series.

1. Introduction

Fractional derivatives were invented by Leibnitz soon after the more familiar integer order derivatives [38,52], but have only recently become popular in applications. They are now used to model a wide variety of problems in physics [24,35,38,50,51,55,65], finance [23,27,37,42,60,61], biology [2,4,22,26,36], and hydrology [1,7,8,15,17,62]. Fractional derivatives can be most easily understood in terms of their connection to probability [45,46]. Einstein [20] explained the connection between random walks, Brownian motion, and the diffusion equation $\partial_t p(x,t) = \partial_x^2 p(x,t)$. Sokolov and Klafter [63] review the modern theory of anomalous diffusion, where the integer order derivatives in the diffusion equation are replaced by their fractional analogues: $\partial_t^\beta p(x,t) = \partial_x^\alpha p(x,t)$. A fractional space derivative of order $\alpha < 2$ corresponds to heavy tailed power law particle jumps $\mathbb{P}[J > x] \approx x^{-\alpha}$ (the famous Lévy flight), while a fractional time derivative of order $\beta < 1$ models heavy tailed power law waiting times $\mathbb{P}[W > t] \approx t^{-\beta}$ between jumps. Hence fractional space derivatives model anomalous super-diffusion, where a plume of particles spreads faster than the traditional diffusion equation predicts, and fractional time derivatives model anomalous sub-diffusion.

The goal of this paper is to describe a new variation on the fractional calculus, where power laws are tempered by an exponential factor. This exponential tempering has both mathematical and practical advantages. Mantegna and Stanley

☆ This work was partially supported by NSF grant DMS-1025486, a Fujian Governmental Scholarship, and the National Natural Science Foundation of China (Grant No. 11101344).

* Corresponding author.
E-mail addresses: sabzika2@stt.msu.edu (F. Sabzikar), mcubed@stt.msu.edu (M.M. Meerschaert), cjhdzdz@163.com (J. Chen).
URL: http://www.stt.msu.edu/users/mcubed/ (M.M. Meerschaert).

http://dx.doi.org/10.1016/j.jcp.2014.04.024
0021-9991/

[40] proposed a truncated Lévy flight to capture the natural cutoff in real physical systems. Koponen [29] introduced the tempered Lévy flight as a smoother alternative, without a sharp cutoff. Cartea and del-Castillo-Negrete [12] developed the tempered fractional diffusion equation that governs the probability densities of the tempered Lévy flight. Unlike the truncated model, tempered Lévy flights offer a complete set of statistical physics and numerical analysis tools. Random walks with exponentially tempered power law jumps converge to a tempered stable motion [13]. Probability densities of the tempered stable motion solve a tempered fractional diffusion equation that describes the particle plume shape [3], just like the original Einstein model for traditional diffusion. Tempered fractional derivatives are approximated by tempered fractional difference quotients, and this facilitates finite difference schemes for solving tempered fractional diffusion equations [3]. The tempered diffusion model has already proven useful in applications to geophysics [44,70,71] and finance [10,11]. In finance, the tempered stable process models price fluctuations with *semi-heavy tails*, resembling a pure power law at moderate time scales, but converging to a Gaussian at long time scales [5]. Since the anomalous diffusion eventually relaxes into a traditional diffusion profile at late time, this model is also called *transient anomalous diffusion* [71].

Kolmogorov [28] invented a new stochastic model for turbulence in the inertial range. Mandelbrot and Van Ness [39] pointed out that this stochastic process is the fractional derivative of a Brownian motion, and coined the name *fractional Brownian motion*. Fractional Brownian motion can exhibit a very useful property called *long range dependence*, where correlations fall off like a power law with time lag. A new variation on this model, called *tempered fractional Brownian motion*, is the tempered fractional derivative of a Brownian motion [47]. Tempered fractional Brownian motion can exhibit *semi-long range dependence*, with correlations that fall off like a power law at moderate time scales, but then eventually become short-range dependent at long time scales. This extends the Kolmogorov model for turbulence to also include low frequencies, and in fact tempered fractional Brownian motion provides a time-domain stochastic process model for the famous Davenport spectrum of wind speed [6,16,25,53], which is used to design electric power generation facilities.

2. Tempered fractional diffusion

We begin by recalling the connection between random walks, Brownian motion, and the diffusion equation (see [46] for complete details). Given a random walk of mean zero particle jumps $S(n) = X_1 + \cdots + X_n$, the Central Limit Theorem implies that $n^{-1/2}S(nt) \Rightarrow B(t)$ in distribution. The probability density $p(x,t)$ of the Brownian motion limit $B(t)$ solves the diffusion equation $\partial_t p(x,t) = D\partial_x^2 p(x,t)$. This useful connection between Brownian motion, random walks, and the diffusion equation assumes finite variance particle jumps. Power law jumps with density $f(x) = C\alpha x^{-\alpha-1}\mathbf{1}_{[C^{1/\alpha},\infty)}(t)$ for $1 < \alpha < 2$ have a finite mean but an infinite variance. Subtract the mean, and apply the extended central limit theorem [46, Theorem 3.37] to get $n^{-1/\alpha}S(nt) \Rightarrow A(t)$. Now the probability density $p(x,t)$ of the α-stable Lévy motion $A(t)$ solves the fractional diffusion equation $\partial_t p(x,t) = D\partial_x^\alpha p(x,t)$.

Tempered fractional diffusion applies an exponential tempering factor to the particle jump density. Consider a random walk $S^\varepsilon(n)$ with particle jump density

$$f_\varepsilon(x) = C_\varepsilon^{-1} x^{-\alpha-1} e^{-\lambda x}\mathbf{1}_{[\varepsilon,\infty)}(x) \quad \text{where } C_\varepsilon = \int_\varepsilon^\infty x^{-\alpha-1}e^{-\lambda x}dx \tag{1}$$

using the incomplete gamma function, and define the Poisson jump rate

$$\lambda_\varepsilon = D\frac{\alpha}{\Gamma(1-\alpha)}C_\varepsilon \tag{2}$$

for any $\varepsilon > 0$. To ease notation, we begin with the case of positive jumps.

Theorem 2.1. *Suppose* $0 < \alpha < 1$*. Given a random walk* $S^\varepsilon(n) = X_1^\varepsilon + \cdots + X_n^\varepsilon$ *with independent jumps, each having probability density function* (1)*, and an independent Poisson process* N_t^ε *with rate* (2)*, as* $\varepsilon \to 0$ *we have the convergence*

$$S^\varepsilon\big(N_t^\varepsilon\big) \quad \Rightarrow \quad A(t) \tag{3}$$

where the limit is a tempered stable process whose probability density function $p(x,t)$ *has Fourier transform*

$$\hat{p}(k,t) = e^{-tD[(\lambda+ik)^\alpha - \lambda^\alpha]} \tag{4}$$

for any $t \geq 0$*.*

Proof. The Poisson random variable satisfies $\mathbb{P}(N_t^\varepsilon = n) = e^{-\lambda_\varepsilon t}(\lambda_\varepsilon t)^n/n!$ for $n \geq 0$, and then

$$P_\varepsilon(x,t) = \mathbb{P}\big(S^\varepsilon\big(N_t^\varepsilon\big) \leq x\big) = \sum_{n=0}^\infty \mathbb{P}\big(S^\varepsilon(n) \leq x \big| N_t^\varepsilon = n\big)\mathbb{P}\big(N_t^\varepsilon = n\big)$$

by the law of total probability. Apply the Fourier–Stieltjes transform $\hat{f}(k) = \int e^{-ikx}F(dx)$ where $f(x) = F'(x)$, noting that $\hat{f}_\varepsilon(k)^n$ is the Fourier transform of the probability distribution of $S^\varepsilon(n)$, to get

$$\hat{p}_\varepsilon(k,t)=\sum_{n=0}^{\infty}\hat{f}_\varepsilon(k)^n e^{-\lambda_\varepsilon t}\frac{(\lambda_\varepsilon t)^n}{n!}=e^{-\lambda_\varepsilon t(1-\hat{f}_\varepsilon(k))} \tag{5}$$

using the Taylor series for the complex exponential function. Rewrite this in the form

$$\hat{p}_\varepsilon(k,t)=\exp\left[t\lambda_\varepsilon\int_\varepsilon^\infty \left(e^{-ikx}-1\right)f_\varepsilon(x)\,dx\right]=\exp\left[tD\frac{\alpha}{\Gamma(1-\alpha)}\int_\varepsilon^\infty \left(e^{-ikx}-1\right)x^{-\alpha-1}e^{-\lambda x}\,dx\right].$$

A straightforward computation using integration by parts and the formula for the gamma probability density [46, Eq. (3.14)] shows that

$$(ik)^\alpha=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \left(1-e^{-ikx}\right)x^{-\alpha-1}dx \quad \text{for } 0<\alpha<1. \tag{6}$$

Use this formula twice to compute

$$\begin{aligned}&\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \left(e^{-ikx}-1\right)e^{-\lambda x}x^{-\alpha-1}dx\\ &\quad=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \left(e^{-(ik+\lambda)x}-1\right)x^{-\alpha-1}dx-\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \left(e^{-\lambda x}-1\right)x^{-\alpha-1}dx\\ &\quad=-\left[(\lambda+ik)^\alpha-\lambda^\alpha\right]\end{aligned} \tag{7}$$

when $0<\alpha<1$. Then as $\varepsilon\to 0$, we get

$$\hat{p}_\varepsilon(k,t)\to\hat{p}(k,t)=e^{-tD[(\lambda+ik)^\alpha-\lambda^\alpha]}$$

for any $t\ge 0$. $\square$

It follows from Theorem 2.1 that $\hat{p}(k,t)=e^{-tD[(\lambda+ik)^\alpha-\lambda^\alpha]}$ is the Fourier transform of the *tempered Lévy flight* $A(t)$, a stochastic process with tempered power law jumps. We define the *tempered fractional derivative* $\partial_x^{\alpha,\lambda}f(x)$ as the function with Fourier transform $[(\lambda+ik)^\alpha-\lambda^\alpha]\hat{f}(k)$ when $0<\alpha<1$. Note that

$$\partial_t\hat{p}(k,t)=-D\left[(\lambda+ik)^\alpha-\lambda^\alpha\right]\hat{p}(k,t)$$

and invert the Fourier transform to see that the probability density $p(x,t)$ of the *tempered Lévy flight* $A(t)$ is the point source solution to the *tempered fractional diffusion equation*

$$\partial_t p(x,t)=-D\partial_x^{\alpha,\lambda}p(x,t).$$

We will also define the negative tempered fractional derivative $\partial_{-x}^{\alpha,\lambda}f(x)$ as the function with Fourier transform $[(\lambda-ik)^\alpha-\lambda^\alpha]\hat{f}(k)$ when $0<\alpha<1$. A combination of positive and negative jumps with particle jump density

$$f_\varepsilon(x)=C_\varepsilon^{-1}\left[px^{-\alpha-1}e^{-\lambda x}\mathbf{1}_{[\varepsilon,\infty)}(x)+q|x|^{-\alpha-1}e^{-\lambda|x|}\mathbf{1}_{(-\infty,-\varepsilon]}(x)\right] \tag{8}$$

for $p,q\ge 0$ with $p+q=1$ leads to a tempered stable limit process $A(t)$ having both positive and negative jumps. A simple extension of Theorem 2.1 shows that (3) holds, and now the limit density $p(x,t)$ solves the tempered fractional diffusion equation

$$\partial_t p(x,t)=-pD\partial_x^{\alpha,\lambda}p(x,t)-qD\partial_{-x}^{\alpha,\lambda}p(x,t).$$

To write the tempered fractional derivative in real space, use the formula (7) to see that

$$\left[(\lambda+ik)^\alpha-\lambda^\alpha\right]\hat{f}(k)=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \left(\hat{f}(k)-e^{-iky}\hat{f}(k)\right)e^{-\lambda y}y^{-\alpha-1}dy$$

and then use the shift property $\int e^{-ikx}f(x-y)\,dx=e^{-iky}\hat{f}(k)$ of the Fourier transform to see that

$$\partial_x^{\alpha,\lambda}f(x)=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \left(f(x)-f(x-y)\right)e^{-\lambda y}y^{-\alpha-1}dy \tag{9}$$

for $0<\alpha<1$. A similar argument shows that

$$\partial_{-x}^{\alpha,\lambda} f(x)=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \big(f(x)-f(x+y)\big)e^{-\lambda y}y^{-\alpha-1}dy. \tag{10}$$

Theorem 2.2. *Suppose $1<\alpha<2$. Given the mean-centered random walk $S^\varepsilon(n)=X_1^\varepsilon+\cdots+X_n^\varepsilon$ with independent jumps, each having probability density function (1) and mean $\mu_\varepsilon=\alpha C_\varepsilon^{1/\alpha}/(\alpha-1)$, and an independent Poisson process N_t^ε with rate (2), as $\varepsilon\to 0$ we have the convergence*

$$S^\varepsilon\big(N_t^\varepsilon\big)-t\lambda_\varepsilon\mu_\varepsilon \quad\Rightarrow\quad A(t) \tag{11}$$

where the limit is a mean zero tempered stable Lévy motion whose probability density function $p(x,t)$ has Fourier transform

$$\hat p(k,t)=e^{-tD[(\lambda+ik)^\alpha-\lambda^\alpha-ik\alpha\lambda^{\alpha-1}]} \tag{12}$$

for any $t\ge 0$.

Proof. An easy computation shows that the jump variable X^ε with probability density (1) has mean $\mu_\varepsilon=\int xf_\varepsilon(x)\,dx=\alpha C_\varepsilon^{1/\alpha}/(\alpha-1)$. Then it follows from (5) that the density of $S^\varepsilon(N_t^\varepsilon)-t\lambda_\varepsilon\mu_\varepsilon$ has Fourier transform

$$\hat p_\varepsilon(k,t)=e^{-\lambda_\varepsilon t(1-\hat f_\varepsilon(k))}e^{-ikt\lambda_\varepsilon\mu_\varepsilon}$$

for any $t\ge 0$. Rewrite this in the form

$$\hat p_\varepsilon(k,t)=\exp\left[-t\lambda_\varepsilon\int_\varepsilon^\infty \big(1-e^{-ikx}+ikx\big)f_\varepsilon(x)\,dx\right]=\exp\left[tD\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\int_\varepsilon^\infty (e^{-ikx}-1-ikx)x^{-\alpha-1}e^{-\lambda x}\,dx\right]$$

using the fact that $\Gamma(2-\alpha)=(1-\alpha)\Gamma(1-\alpha)$. Use (6) and another integration by parts [46, p. 58] to see that

$$(ik)^\alpha=\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\int_0^\infty (e^{-ikx}-1+ikx)x^{-\alpha-1}dx \quad \text{for } 1<\alpha<2. \tag{13}$$

Let $C=\alpha(\alpha-1)/\Gamma(2-\alpha)$ and compute

$$\begin{aligned} C\int_0^\infty (e^{-iky}-1+iky)e^{-\lambda y}y^{-\alpha-1}dy &= C\int_0^\infty \big(e^{-(\lambda+ik)y}-1+(\lambda+ik)y\big)y^{-\alpha-1}dy \\ &\quad -C\int_0^\infty \big(e^{-\lambda y}-1+\lambda y\big)y^{-\alpha-1}dy-C\int_0^\infty \big(e^{-\lambda y}-1\big)iky\,y^{-\alpha-1}dy \\ &=(\lambda+ik)^\alpha-\lambda^\alpha-ik\alpha\lambda^{\alpha-1} \end{aligned} \tag{14}$$

using (13). Then as $\varepsilon\to 0$, we get

$$\hat p_\varepsilon(k,t)\to\hat p(k,t)=e^{-tD[(\lambda+ik)^\alpha-\lambda^\alpha-ik\alpha\lambda^{\alpha-1}]}$$

for any $t\ge 0$. $\square$

It follows from Theorem 2.2 that $\hat p(k,t)=e^{-tD[(\lambda+ik)^\alpha-\lambda^\alpha-ik\alpha\lambda^{\alpha-1}]}$ is the Fourier transform of the tempered Lévy flight $A(t)$. We define the *tempered fractional derivative* $\partial_x^{\alpha,\lambda}f(x)$ as the function with Fourier transform $[(\lambda+ik)^\alpha-\lambda^\alpha-ik\alpha\lambda^{\alpha-1}]\hat f(k)$ when $1<\alpha<2$. Since $\partial_t\hat p(k,t)=D[(\lambda+ik)^\alpha-\lambda^\alpha-ik\alpha\lambda^{\alpha-1}]\hat p(k,t)$, we can invert the Fourier transform to see that the probability density $p(x,t)$ of $A(t)$ is the point source solution to the tempered fractional diffusion equation

$$\partial_t p(x,t)=D\partial_x^{\alpha,\lambda}p(x,t). \tag{15}$$

Fig. 1 shows a typical particle path, illustrating the effect of tempering to cool the long particle jumps.

To write this tempered fractional derivative in real space, use the formula (14) to see that

$$\big[(\lambda+ik)^\alpha-\lambda^\alpha-ik\alpha\lambda^{\alpha-1}\big]\hat f(k)=\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\hat f(k)\int_0^\infty (e^{-iky}-1+iky)e^{-\lambda y}y^{-\alpha-1}dy$$

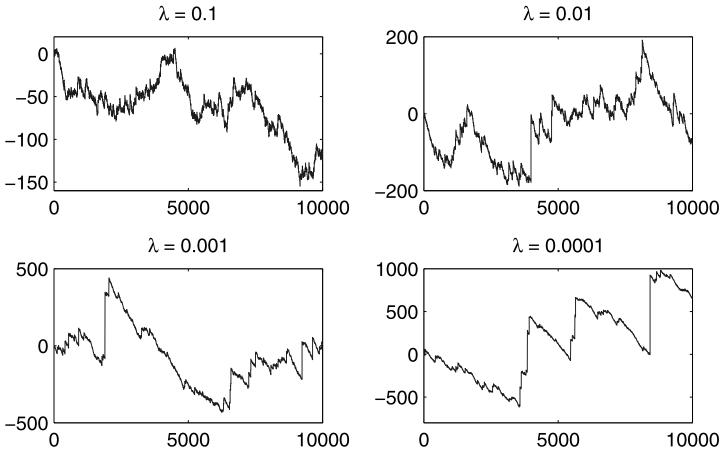

Fig. 1. Tempered fractional diffusion particle paths with $\alpha = 1.2$, from [3], showing the effect of tempering on long particle jumps.

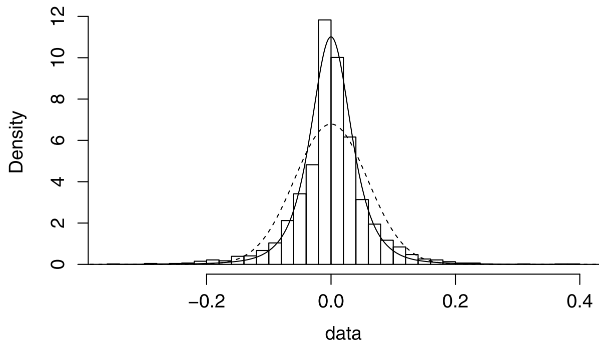

Fig. 2. Symmetric tempered stable model fit (solid line) with $\alpha = 1.1$, $\lambda = 12$, $D = 0.1$ to one-step bilinear ARMA forecast errors for annual inflation rates. The best fitting normal density (dashed line) fails to capture the sharp data peak.

and invert the Fourier transform to get

$$\partial_x^{\alpha,\lambda} f(x) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^\infty \big(f(x-y) - f(x) + yf'(x)\big)e^{-\lambda y} y^{-\alpha-1} dy \tag{16}$$

for $1 < \alpha < 2$.

The negative tempered fractional derivative $\partial_{-x}^{\alpha,\lambda} f(x)$ is the function with Fourier transform $[(\lambda - ik)^\alpha - \lambda^\alpha + ik\alpha\lambda^{\alpha-1}]\hat{f}(k)$ when $1 < \alpha < 2$. In real space, we can write

$$\partial_{-x}^{\alpha,\lambda} f(x) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^\infty \big(f(x+y) - f(x) - yf'(x)\big)e^{-\lambda y} y^{-\alpha-1} dy. \tag{17}$$

The particle jump density (8) leads to a tempered stable limit process $A(t)$ having both positive and negative jumps. A simple extension of Theorem 2.2 shows that (11) holds, and now the limit density $p(x,t)$ solves the tempered fractional diffusion equation

$$\partial_t p(x,t) = pD\partial_x^{\alpha,\lambda} p(x,t) + qD\partial_{-x}^{\alpha,\lambda} p(x,t).$$

If $p = q$ then this produces a symmetric profile, with *semi-heavy tails*. That is, the tails of the probability density $p(x,t)$ resemble a power law $p(x,t) \approx |x|^{-\alpha-1}$ for large $|x|$ and moderate $t > 0$, but as $t \to \infty$ the tails relax to a Gaussian profile. These semi-heavy tails are typical in applications to finance [10,11]. Fig. 2 fits a symmetric tempered stable density function to macroeconomic data on inflation rates. The data shows a sharper peak and a heavier tail than the best fitting Gaussian curve.

The tempered space-fractional diffusion equation models transient super-diffusion, where a particle plume initially spreads faster than the traditional diffusion equation predicts, but later relaxes to a typical diffusion profile. The corresponding model for transient sub-diffusion assumes tempered power law waiting times between particle jumps. Suppose that the nth particle jump $S(n) = X_1 + \cdots + X_n$ occurs at a random time $T_n = W_1 + \cdots + W_n$ where W_n are all independent with density $d(t) = B\beta t^{-\beta-1} e^{-\eta t} \mathbf{1}_{[B^{1/\beta},\infty)}(t)$ for some $0 < \beta < 1$. Theorem 2.1 yields random walk convergence to another stable Lévy motion D_t with Laplace transform $\tilde{g}(s,t) = e^{-t\psi_D(s)}$ where the Laplace symbol $\psi_D(s) = [(\eta + s)^\beta - \eta^\beta]$. The number of jumps by time $t > 0$ is given by the renewal process $N_t = \max\{n \ge 0 : T_n \le t\}$, the inverse of T_n (graph T_n versus n, then swap the axes, to get the graph of N_t versus t). Then the CTRW (continuous time random walk) defined by $S(N_t)$ gives the particle location at time $t > 0$.

A general result from CTRW limit theory [43, Theorem 2.1] shows that the CTRW converges at late time to $A(E_t)$, a tempered Lévy flight with the time variable replaced by an independent *inverse tempered stable subordinator* $E_t = \inf\{u > 0 : D_u > t\}$. Since $\mathbb{P}[E_t \le u] = \mathbb{P}[D_u \ge t]$, we can take derivatives and then Laplace transforms to see that the density $h(u,t)$ of E_t has Laplace transform

$$\tilde{h}(u,s) = \int_0^\infty e^{-st} h(u,t)\,dt = s^{-1}\psi_D(s) e^{-u\psi_D(s)},$$

see [43, Theorem 3.1] for details. Since $x = A(u)$ has density function $p(x,u)$ and $u = E_t$ has density function $h(u,t)$, the CTRW limit process $A(E_t)$ has density

$$q(x,t) = \int_0^\infty p(x,u) h(u,t)\,du.$$

Recall that the tempered Lévy density has Fourier transform $\hat{p}(k,u) = e^{-u\psi_A(k)}$ with Fourier symbol $\psi_A(k) = pD[(\lambda + ik)^\alpha - \lambda^\alpha - ik\alpha\lambda^{\alpha-1}] + qD[(\lambda - ik)^\alpha - \lambda^\alpha + ik\alpha\lambda^{\alpha-1}]$ for $1 < \alpha < 2$. Take Laplace and Fourier transforms to see that

$$\bar{q}(k,s) = \int_0^\infty \hat{p}(k,u)\tilde{h}(u,s)\,du = \frac{s^{-1}\psi_D(s)}{\psi_D(s) + \psi_A(k)},$$

then rearrange and invert to see that the limit density solves the space–time tempered fractional diffusion equation

$$\partial_t^{\beta,\eta} p(x,t) = pD\partial_x^{\alpha,\lambda} p(x,t) + qD\partial_{-x}^{\alpha,\lambda} p(x,t) + \delta(x) f(t) \tag{18}$$

where the forcing term $f(t) = \phi_D(t,\infty)$ has Laplace transform $\tilde{f}(s) = s^{-1}\psi_D(s)$, see [43, Theorem 4.1] for complete details. If $\eta = 0$, the boundary term reduces to $f(t) = t^{\beta-1}/\Gamma(1-\beta)$, as in the fractional kinetic equation of Zaslavsky [69]. Tempered jumps are the basis for the popular CGMY model in finance [10,11]. Tempered waiting times were applied to problems in ground water hydrology [44], and tempering in both variables was used for sediment transport in rivers [71]. Tempered fractional diffusion was first proposed by Cartea and del-Castillo-Negrete [12], and developed further in [3,44], see also [46, Sections 7.2 and 7.3]. A more general tempering scheme was developed in [13]. The theory of tempered stable processes was developed by Rosiński [57], and an exact simulation scheme for the sample paths of a tempered Lévy flight was presented in Cohen and Rosiński [14].

3. Tempered fractional calculus

Given any $\lambda > 0$, we define the positive tempered fractional integral of a suitable function $f(x)$ by

$$\mathbb{I}_+^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(u)(x-u)^{\alpha-1} e^{-\lambda(x-u)}\,du, \tag{19}$$

and the negative tempered fractional integral by

$$\mathbb{I}_-^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty f(u)(u-x)^{\alpha-1} e^{-\lambda(u-x)}\,du. \tag{20}$$

If $\lambda = 0$, these formulae reduce to the well-known Riemann–Liouville fractional integrals [46,58]. Hence we will call these operators the *Riemann–Liouville tempered fractional integrals*. Since these operators are convolutions, their Fourier transforms are products. Since $g(x) = x^{\alpha-1}e^{-\lambda x}\mathbf{1}_{[0,\infty)}(x)/\Gamma(\alpha)$ has Fourier transform $\hat{g}(k) = (\lambda + ik)^{-\alpha}$, it follows from the convolution property of the Fourier transform that $\mathbb{I}_+^{\alpha,\lambda} f(x)$ has Fourier transform $(\lambda + ik)^{-\alpha}\hat{f}(k)$. Likewise $\mathbb{I}_-^{\alpha,\lambda} f(x)$ has Fourier transform $(\lambda - ik)^{-\alpha}\hat{f}(k)$, and when $\lambda = 0$ we recover the well-known formulae for the Fourier transform of a Riemann–Liouville fractional integral [58].

For functions in the fractional Sobolev space

$$W^{\alpha,2}(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} \left(\lambda^2 + k^2\right)^{\alpha} \left|\hat{f}(k)\right|^2 dk < \infty \right\},$$

we define the *Riemann–Liouville tempered fractional derivatives* $\mathbb{D}_{\pm}^{\alpha,\lambda} f(x)$ to be the functions with Fourier transform $(\lambda \pm ik)^{\alpha}\hat{f}(k)$. Then integration and differentiation are inverse operators: $\mathbb{D}_{\pm}^{\alpha,\lambda}\mathbb{I}_{\pm}^{\alpha,\lambda} f(x) = f(x)$ for any $f \in L^2(\mathbb{R})$, and $\mathbb{I}_{\pm}^{\alpha,\lambda}\mathbb{D}_{\pm}^{\alpha,\lambda} f(x) = f(x)$ for any $f \in W^{\alpha,2}(\mathbb{R})$. We can relate back to the *normalized* tempered fractional derivative $\partial_{\pm x}^{\alpha,\lambda} f(x)$ from Section 2 with Fourier transform $[(\lambda \pm ik)^{\alpha} - \lambda^{\alpha}]\hat{f}(k)$ when $0 < \alpha < 1$, so that we can use (9) and (10) to write

$$\mathbb{D}_{\pm}^{\alpha,\lambda} f(x) = \partial_{\pm x}^{\alpha,\lambda} f(x) + \lambda^{\alpha} f(x) \quad \text{for } 0 < \alpha < 1. \tag{21}$$

Since $\partial_{\pm x}^{\alpha,\lambda} f(x)$ has Fourier transform $[(\lambda \pm ik)^{\alpha} - \lambda^{\alpha} \mp ik\alpha\lambda^{\alpha-1}]\hat{f}(k)$ for $1 < \alpha < 2$, we can also use (16) and (17) to write

$$\mathbb{D}_{\pm}^{\alpha,\lambda} f(x) = \partial_{\pm x}^{\alpha,\lambda} f(x) + \lambda^{\alpha} f(x) \pm \alpha\lambda^{\alpha-1} f'(x) \quad \text{for } 1 < \alpha < 2. \tag{22}$$

Using the shift property of the Fourier transform, we can also relate the Riemann–Liouville tempered fractional derivative to the (untempered) fractional derivative $\partial_x^{\alpha} f(x)$ from Section 2. Since $\int e^{-ikx}e^{\lambda x} f(x)\,dx = \hat{f}(k + i\lambda)$, and since $\partial_x^{\alpha} f(x)$ has Fourier transform $(ik)^{\alpha}\hat{f}(k)$, it follows that $\partial_x^{\alpha}[e^{\lambda x} f(x)]$ has Fourier transform $(ik)^{\alpha}\hat{f}(k + i\lambda)$. Another application of the shift property shows that $e^{-\lambda x}\partial_x^{\alpha}[e^{\lambda x} f(x)]$ has Fourier transform $(i(k - i\lambda))^{\alpha}\hat{f}((k - i\lambda) + i\lambda) = (ik + \lambda)^{\alpha}\hat{f}(k)$, and then the uniqueness of the Fourier transform implies that

$$\mathbb{D}_{+}^{\alpha,\lambda} f(x) = e^{-\lambda x}\partial_x^{\alpha}\left[e^{\lambda x} f(x)\right] \quad \text{for any } \alpha > 0. \tag{23}$$

Note that the formula (23) remains valid when α is a positive integer. One can also use (23) to connect $\partial_x^{\alpha} f(x)$ with the normalized tempered fractional derivative $\partial_x^{\alpha,\lambda} f(x)$, see [46, Section 7.2] for details.

Using the inner product $\langle f, g\rangle = \int f(x)\overline{g(x)}dx$ on $L^2(\mathbb{R})$, along with the Plancherel Theorem $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$, it follows easily that

$$\left\langle \mathbb{D}_{+}^{\alpha,\lambda} f, g \right\rangle = \int (\lambda + ik)^{\alpha}\hat{f}(k)\overline{\hat{g}(k)}dk = \int \hat{f}(k)\overline{(\lambda - ik)^{\alpha}\hat{g}(k)}dk = \left\langle f, \mathbb{D}_{-}^{\alpha,\lambda} g \right\rangle$$

so that the positive and negative Riemann–Liouville tempered fractional derivatives are adjoint operators in the Hilbert space $L^2(\mathbb{R})$. Essentially the same argument shows that the positive and negative Riemann–Liouville tempered fractional integrals are also $L^2(\mathbb{R})$ adjoints. The semigroup property

$$\mathbb{D}_{\pm}^{\alpha,\lambda}\mathbb{D}_{\pm}^{\beta,\lambda} f = \mathbb{D}_{\pm}^{\alpha+\beta,\lambda} f \quad \text{and} \quad \mathbb{I}_{\pm}^{\alpha,\lambda}\mathbb{I}_{\pm}^{\beta,\lambda} f = \mathbb{I}_{\pm}^{\alpha+\beta,\lambda} f$$

follows by another easy Fourier transform argument.

The space of rapidly decreasing functions $\mathcal{S}(\mathbb{R})$ consists of the infinitely differentiable functions $g : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} \left| x^n g^{(m)}(x) \right| < \infty,$$

where n, m are non-negative integers, and $g^{(m)}$ is the derivative of order m. The space of tempered distributions $\mathcal{S}'(\mathbb{R})$ consists of continuous linear functionals on $\mathcal{S}(\mathbb{R})$. The Fourier transform maps $\mathcal{S}'(\mathbb{R})$ into itself. If $f : \mathbb{R} \to \mathbb{R}$ is of polynomial growth, so that $\int |f(x)|(1 + |x|)^{-p}dx < \infty$ for some $p > 0$, then $T_f(\varphi) = \int f(x)\varphi(x)\,dx := \langle f, \varphi\rangle$ is a tempered distribution, also called a generalized function, with Fourier transform $\hat{T}_f(\varphi) = \langle \hat{f}, \varphi\rangle = \langle f, \hat{\varphi}\rangle = T_f(\hat{\varphi})$ for $\varphi \in \mathcal{S}(\mathbb{R})$. See Yosida [67, Ch. VI] for more details. For any tempered distribution f, the tempered fractional integral $\mathbb{I}_{+}^{\alpha,\lambda} f(x)$ exist as a convolution with the tempered distribution $g(x) = x^{\alpha-1}e^{-\lambda x}\mathbf{1}_{[0,\infty)}(x)/\Gamma(\alpha)$. The same holds for Riemann–Liouville fractional integrals (the case $\lambda = 0$), since $g(x) = x^{\alpha-1}\mathbf{1}_{[0,\infty)}(x)/\Gamma(\alpha)$ is of polynomial growth. Since $G(k) = (\lambda \pm ik)^{\alpha}$ is of polynomial growth, it is also a tempered distribution, and hence it follows from the Fourier inversion formula on $\mathcal{S}'(\mathbb{R})$ that $G = \hat{g}$ for some tempered distribution $g \in \mathcal{S}'(\mathbb{R})$. Then we may define the Riemann–Liouville tempered fractional derivative of a tempered distribution as the convolution with this function. If $n - 1 < \alpha < n$, we can also write $(\lambda + ik)^{\alpha}\hat{f}(k) = (\lambda + ik)^{n}(\lambda + ik)^{\alpha-n}\hat{f}(k)$, and inverting yields an alternative definition in terms of the Riemann–Liouville tempered fractional integral:

$$\mathbb{D}_{+}^{\alpha,\lambda} f(x) = \mathbb{D}_{+}^{n,\lambda}\mathbb{I}_{+}^{n-\alpha,\lambda} f(x)$$

where $\mathbb{D}_{+}^{n,\lambda} f(x) = e^{-\lambda x}[e^{\lambda x} f(x)]^{(n)}$ is a tempered derivative of integer order, defined by (23). There are many technical issues behind the fractional calculus of tempered distributions [58, Chapter 8], and it would be interesting to extend that theory for tempered fractional calculus.

4. Tempered fractional Brownian motion

As noted in Section 2, a random walk $S(n) = X_1 + \cdots + X_n$ with mean zero, finite variance particle jumps converge to a Brownian motion $B(t)$. For this Gaussian stochastic process, $B(0) = 0$, $B(t+s) - B(s)$ has the same density function as $B(t)$ (independent increment property) and $B(t+s) - B(s)$ is independent of $B(s)$ (independent increment property). The stochastic integral $\int f(x)B(dx)$ is defined for any $f \in L^2(\mathbb{R})$ as a Gaussian random variable with mean zero and variance $\langle f, f\rangle = \int f(x)^2 dx$, such the *Itô isometry* holds: the stochastic integrals $\int f(x)B(dx)$ and $\int g(x)B(dx)$ have covariance $\langle f, g\rangle = \int f(x)g(x)dx$. See [59] for more details. Then for any $H > 0$ and $\lambda > 0$ we can define the *tempered fractional Brownian motion*

$$B_{H,\lambda}(t) := \int \left[e^{-\lambda(t-x)_+}(t-x)_+^{H-1/2} - e^{-\lambda(0-x)_+}(0-x)_+^{H-1/2}\right] B(dx) \tag{24}$$

where $(x)_+ = \mathbf{1}_{[0,\infty)}(x)$, and $0^0 = 0$. The Gaussian stochastic process (24) has mean zero, and its increments are stationary, but not independent. Using the fact that $B(c\,dx) \simeq c^{1/2}B(dx)$ (same distribution), it is not hard to check that tempered fractional Brownian motion has a nice scaling property: $B_{H,\lambda}(ct) \simeq c^H B_{H,c\lambda}(t)$ for every $c > 0$. If $\lambda = 0$ and $0 < H < 1$ (required to keep the integrand in $L^2(\mathbb{R})$ when $\lambda = 0$), we get the *fractional Brownian motion* with *Hurst index* H. Kolmogorov [28] proposed this process with $H = 1/3$ as a model for turbulence. Mandelbrot and Van Ness [39] established the connection between fractional Brownian motion and fractional calculus. Next we explain that connection in the tempered case.

Stochastic integrals with respect to Brownian motion can also be constructed using white noise theory. Heuristically, one writes $\int f(x)B(dx) = \int f(x)W(x)dx$ where the white noise $W(x) = B'(x)$ is the (weak) derivative of the Brownian motion. This integral does not exist path-wise, since the paths of a Brownian motion are not differentiable. However, we can define $W(f) := \int f(x)W(x)dx = \langle f, W\rangle$ as a random distribution (generalized function) on $L^2(\mathbb{R})$ by specifying that this object has the same (Gaussian) distribution and covariance function as $\int f(x)B(dx)$ [30]. Taking $f(x) = \mathbf{1}_{[0,t]}(x)$, it follows that $B(t) = \int_0^t W(x)dx$, and in this sense, the white noise $W(x)$ is the (distributional) derivative of $B(x)$. Suppose that $H > 1/2$, and compute

$$\mathbb{I}_-^{H-1/2,\lambda}\mathbf{1}_{[0,t]}(x) - \lambda\mathbb{I}_-^{H+1/2,\lambda}\mathbf{1}_{[0,t]}(x) = e^{-\lambda(t-x)_+}(t-x)_+^{H-1/2} - e^{-\lambda(0-x)_+}(0-x)_+^{H-1/2}$$

for any $t > 0$. Using the fact that positive and negative Riemann–Liouville tempered fractional integrals are $L^2(\mathbb{R})$ adjoints, we can therefore use white noise theory to write

$$B_{H,\lambda}(t) = \left\langle \mathbf{1}_{[0,t]}(x), \mathbb{I}_+^{H-1/2,\lambda}W(x) - \lambda\mathbb{I}_+^{H+1/2,\lambda}W(x)\right\rangle$$

which shows that tempered fractional Brownian motion with $H > 1/2$ can be obtained by integrating a linear combination of tempered fractional integrals of a white noise. The same is true for $0 < H < 1/2$, if we interpret the tempered fractional integral of negative order as a tempered fractional derivative, see [49] for more details. Now if we take $\lambda = 0$ and $0 < H < 1$, we recover the fact that fractional Brownian motion is the fractional integral of a white noise [39,54].

Fractional Brownian motion with $1/2 < H < 1$ is a popular model for *long range dependence*. The increments $X_n = B_H(n) - B_H(n-1)$ form a stationary time series of Gaussian random variables with mean zero, and the covariance between X_n and X_{n+j} falls off like $|j|^{2H-2}$ as $j \to \infty$ [59, Proposition 7.2.10]. Tempered fractional Brownian motion with a small tempering parameter $\lambda \approx 0$ exhibits *semi-long range dependence*. The increments $X_n = B_{H,\lambda}(n) - B_{H,\lambda}(n-1)$ form a stationary Gaussian time series, the covariance between X_n and X_{n+j} falls off like $|j|^{2H-2}$ for moderate values of j, but then tempering makes the covariance fall off faster (short range dependence) as j goes to infinity. When $0 < H < 1/2$, both processes exhibit *negative dependence*, with a negative covariance between X_n and X_{n+j} for all j sufficiently large. See [47] for further details.

5. Tempered fractional difference operator

Tempered fractional derivatives can also be defined as the limit of a tempered fractional difference quotient.

Theorem 5.1. *If f and its derivatives up to order $n > 1 + \alpha$ exist and are absolutely integrable, then the Riemann–Liouville tempered fractional derivative*

$$\mathbb{D}_+^{\alpha,\lambda} f(x) = \lim_{h\to 0} h^{-\alpha}\Delta_\lambda^\alpha f(x) \tag{25}$$

where

$$\Delta_\lambda^\alpha f(x) = \sum_{j=0}^{\infty} w_j e^{-\lambda jh} f(x - jh) \quad \text{with } w_j := (-1)^j \binom{\alpha}{j} = \frac{(-1)^j\Gamma(1+\alpha)}{j!\Gamma(1+\alpha-j)}.$$

Proof. The proof is similar to [46, Proposition 2.1]. The fractional binomial formula

$$\sum_{k=0}^{\infty}\binom{\alpha}{k}z^k=(1+z)^\alpha$$

holds for any $\alpha>0$ and any complex $|z|\le 1$. Since $e^{-ikjh}\hat{f}(k)$ is the Fourier transform of $f(x-jh)$, the Fourier transform of $h^{-\alpha}\Delta_\lambda^\alpha f(x)$ is

$$\begin{aligned}
&h^{-\alpha}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}e^{-\lambda jh}e^{-ikjh}\hat{f}(k)\\
&=h^{-\alpha}\sum_{j=0}^{\infty}\binom{\alpha}{j}(-1)^j e^{-(\lambda+ik)jh}\hat{f}(k)\\
&=h^{-\alpha}\left(1-e^{-(\lambda+ik)h}\right)^\alpha\hat{f}(k)\to(\lambda+ik)^\alpha\hat{f}(k)
\end{aligned}$$

as $h\to 0$, using a Taylor expansion of the complex exponential function. Since f and its derivatives up to order $n>1+\alpha$ exist and are absolutely integrable, we have $\hat{f}(k)\le C/(1+|k|^n)$ for some $C>0$ [46, Lemma 2.4], and so $|(\lambda+ik)^\alpha\hat{f}(k)|\le C(\lambda^2+k^2)^{\alpha/2}/(1+|k|^n)$. Hence $(\lambda+ik)^\alpha\hat{f}(k)$ is absolutely integrable, and then the inversion formula for the Fourier transform [46, Theorem 1.4] implies that there exists a function $\mathbb{D}_+^{\alpha,\lambda}f(x)$ with this Fourier transform. Then the continuity theorem for the Fourier transform [46, Theorem 1.3] implies that (25) holds. □

The tempered fractional difference quotient can be applied to develop simple, effective numerical schemes to solve the tempered fractional diffusion equation. For physical reasons [7,8,17,62], the most important case in applications is $1<\alpha<2$. Consider the tempered fractional diffusion equation with drift

$$\partial_t p(x,t)=-v(x)\partial_x p(x,t)+D(x)\partial_x^{\alpha,\lambda}p(x,t)+f(x,t) \tag{26}$$

where $1<\alpha\le 2$, $v(x)\ge 0$, and $D(x)\ge 0$ on a finite domain $a\le x\le b$, with initial condition $p(x,0)=p_0(x)$, boundary conditions $p(a,t)=0$ and $p(b,t)=B(t)$, and forcing function $f(x,t)$ for all $t\ge 0$. Using Theorem 5.1 along with (22) and the fact that $h^{-\alpha}(1-e^{-\lambda h})^\alpha f(x)\to\lambda^\alpha f(x)$ as $h\to 0$, it follows that

$$\partial_x^{\alpha,\lambda}f(x)+\alpha\lambda^{\alpha-1}f'(x)=\lim_{h\to 0}h^{-\alpha}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}e^{-\lambda jh}f(x-jh)-h^{-\alpha}\left(1-e^{-\lambda h}\right)^\alpha f(x).$$

To write the details of our numerical code, set $t_i=i\Delta t$, $x_j=a+j\Delta x$, $h=\Delta x$, $p_{ij}=p(x_j,t_i)$, $D_j=D(x_j)$, $v_j=v(x_j)+D_j\alpha\lambda^{\alpha-1}$, and $f_{ij}=f(x_j,t_i)$. It is natural to consider the implicit Euler scheme

$$p_{i+1,j}-p_{ij}=\left[-v_j\frac{p_{i+1,j}-p_{i+1,j-1}}{h}+D_jh^{-\alpha}\sum_{m=0}^{j}g_m p_{i+1,j-m}+f_{ij}\right]\Delta t$$

using the weights

$$g_j=\begin{cases}w_je^{-jh\lambda} & \text{for } j\ge 1,\\ w_je^{-jh\lambda}-(1-e^{-h\lambda})^\alpha & \text{for } j=0.\end{cases}$$

However, this implicit scheme is unstable. The proof is very similar to [41, Proposition 2.3]. The explicit Euler scheme (replace $i+1$ by i on the right-hand side) is also unstable. The proof is essentially identical to [41, Proposition 2.1].

In order to obtain a stable convergent numerical method, it is necessary to use a shifted finite difference approximation

$$\partial_x^{\alpha,\lambda}f(x)+\alpha\lambda^{\alpha-1}f'(x)=\lim_{h\to 0}h^{-\alpha}\sum_{j=0}^{\infty}g_jf\left(x-(j-1)h\right), \tag{27}$$

where the exponentially tempered Grünwald weights are defined by

$$g_j=\begin{cases}w_je^{-(j-1)h\lambda} & \text{for } j\ne 1,\\ w_1-e^{h\lambda}(1-e^{-h\lambda})^\alpha & \text{for } j=1.\end{cases} \tag{28}$$

The proof that (27) holds is similar to Theorem 5.1, see [3, Proposition 3] for details. Now consider the implicit Euler scheme

$$p_{i+1,j}-p_{ij}=\left[-v_j\frac{p_{i+1,j}-p_{i+1,j-1}}{h}+D_jh^{-\alpha}\sum_{m=0}^{j+1}g_m p_{i+1,j-m+1}+f_{ij}\right]\Delta t. \tag{29}$$

Theorem 5.2. *The implicit Euler scheme (29) for the tempered fractional diffusion equation (26) using the weights (28) is consistent and unconditionally stable.*

Proof. The proof is similar to [41, Theorem 2.7]. The main difference is that the Grünwald weights w_j are replaced by the tempered weights g_j. It follows from the fractional binomial formula that

$$\sum_{j=0}^{\infty} g_j = e^{h\lambda}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}e^{-jh\lambda} - e^{h\lambda}\left(1-e^{-h\lambda}\right)^{\alpha} = 0,$$

and hence this numerical method mass-preserving. Since $1<\alpha<2$ we have $w_1=-\alpha$ and $w_j>0$ for all $j\neq 1$, so that $g_1<0$ and $g_j>0$ for all $j\neq 1$. Then

$$\sum_{j=0}^{K} g_j < 0$$

for any finite number K. The implicit Euler scheme can be written as a linear system of equations $AP_{i+1}=P_i+F_i\Delta t$ in matrix form where $P_i=[p_{i,0},\dots,p_{i,N}]'$, $F_i=[f_{i0},\dots,f_{iN}]'$, $N=(b-a)/h$, and the inequality $\sum_{j=0}^{K} g_j<0$ implies that every eigenvalue ζ of the implicit iteration matrix A satisfies $|\zeta|\geq 1$. Hence the spectral radius of the inverse matrix $\rho(A^{-1})\leq 1$, and the result follows. □

Note that the shifted finite difference approximation (27) is essentially a convolution with the exponentially tempered Grünwald weights. An application of Stirling's formula shows that $w_m \sim -\alpha m^{-\alpha-1}/\Gamma(1-\alpha)$ as $m\to\infty$ [46, Eq. (2.5)], and this connects the finite difference scheme (29) with the probability model from Section 2. In the probability model, the tempered fractional derivative codes power law jumps with density function $f(x)\approx x^{-\alpha-1}e^{-\lambda x}$ for x large. In the finite difference approximation of the tempered fractional derivative, the particle flux into location x from location $x-jh$ follows the same tempered power law model. Between time t_i and time $t_i+\Delta t$, the mass at location x_j is reduced by an amount proportional to the current particle mass density $p(x_j,t_i)$ at this location, and increased by an amount proportional to the mass at other locations $x_j-m\Delta x$ with a weight $g_m=w_m e^{-\lambda m\Delta x}$ that falls off like a tempered power law with distance $m\Delta x$.

Example 5.3. Consider the tempered fractional diffusion equation (26) with $v(x)=0$, $x\in[0,1]$, $\alpha=1.5$, $\lambda=0.5$, initial condition $u(x,0)=x^{\beta}e^{-\lambda x}/\Gamma(\beta+1)$ with $\beta=2.5$, diffusion coefficient $c(x)=x^{\alpha}\Gamma(1+\beta-\alpha)/\Gamma(\beta+1)$, forcing function

$$q(x,t)=e^{-\lambda x-t}\frac{\Gamma(1+\beta-\alpha)}{\Gamma(\beta+1)}\left(\frac{(1-\alpha)\lambda^{\alpha}x^{\alpha+\beta}}{\Gamma(\beta+1)}+\frac{\alpha\lambda^{\alpha-1}x^{\alpha+\beta-1}}{\Gamma(\beta)}-\frac{2x^{\beta}}{\Gamma(1+\beta-\alpha)}\right)$$

and boundary conditions $u(0,t)=0$ and $u(1,t)=B(t):=e^{-\lambda-t}/\Gamma(\beta+1)$. The exact solution is given by $u(x,t)=x^{\beta}e^{-\lambda x-t}/\Gamma(1+\beta)$. A similar example was considered in [3, Example 7]. Set $\bar F_i=[f_{i1},\dots,f_{i,N-2},f_{i,N-1}+D_{N-1}h^{-\alpha}g_0B(t_{i+1})]'$, $p_{i0}=0$, $p_{i,N}=B(t_i)$, and let $\bar A$ denote the $(N-1)\times(N-1)$ matrix with entries

$$\bar A_{ij}=\begin{cases}1+v_ih^{-1}\Delta t-D_ih^{-\alpha}\Delta t\,g_1 & \text{when } j=i,\\ -v_ih^{-1}\Delta t-D_ih^{-\alpha}\Delta t\,g_2 & \text{when } j=i-1,\\ -D_ih^{-\alpha}\Delta t g_{i-j+1} & \text{when } j<i-1 \text{ or } j=i+1,\\ 0 & \text{otherwise.}\end{cases}$$

Then the vector of interior points $\bar P_i=[p_{i,1},\dots,p_{i,N-1}]'$ solves the iteration equation $\bar A\bar P_{i+1}=\bar P_i+\bar F_i\Delta t$. The additional term in the last entry of $\bar F_i$ accounts for cutting off the super-diagonal entry of the full iteration matrix A. The algorithm was implemented in MATLAB (code is available from the authors upon request), see Table 1 for a summary of results.

The implicit Euler method in Theorem 5.2 has error $O(\Delta x+\Delta t)$, consistent with Table 1. A Crank–Nicolson method for solving the tempered fractional diffusion equation (26) with $1<\alpha\leq 2$ is developed in [3]. The method is stable and consistent, and second order convergent in time. However, because the finite difference approximation (27) is only first order accurate, the method is $O(\Delta t^2+\Delta x)$. An application of Richardson extrapolation yields $O(\Delta t^2+\Delta x^2)$, see [3] for more details. It is also possible to construct particle tracking solutions based on the Monte Carlo simulation scheme in [3, Section 4], see [44,71]. A wide variety of effective numerical methods have been developed to solve fractional diffusion equations, see for example [18,21,32–34,56,64,68]. It would certainly be interesting to extend these methods to tempered fractional diffusion equations. Even in the untempered case, many interesting open problems remain, including a rigorous proof that fractional boundary value problems are well-posed, see [19] for a nice discussion.

Table 1
Comparison of maximum errors and error rate for Example 5.3 at time $t = 1.0$.

Δt	Δx	Max error	Error rate
.05000	.05000	.003560	–
.02500	.02500	.001900	1.87
.01250	.01250	.000982	1.94
.00625	.00625	.000499	1.97

6. ARTFIMA model

Another useful application of the fractional difference operator arises in time series analysis. Given a sequence of independent Gaussian random variables Z_n with mean zero and variance σ^2 (white noise sequence), a fractional noise sequence can be constructed by setting

$$X_n = \Delta^{-\alpha} Z_n = \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} Z_{n-j}$$

where typically $-1/2 < \alpha < 1/2$. Since $\Delta^\alpha \Delta^\beta = \Delta^{\alpha+\beta}$ in general (this follows from the fractional binomial formula), one can also write $Z_n = \Delta^\alpha X_n$. Since the fractional difference is the discrete analogue of the fractional derivative, a fractional noise is closely related to fractional Brownian motion. Indeed, the sum $X_1 + \cdots + X_t \approx B_H(t)$ when $H + \alpha = 1/2$. More precisely, we have

$$n^{-H}(X_1 + \cdots + X_{[nt]}) \quad \Rightarrow \quad B_H(t) \quad \text{as}\, n \to \infty$$

for all $t > 0$, see [66, Theorem 4.6.1]. The fractional noise can also exhibit long range dependence. The covariance between X_j and X_{j+h} falls off like $|j|^{2H-2}$ as $j \to \infty$. In applications, a fractional difference is applied to a time series data set, and the order α of the fractional difference is chosen to remove long range correlations from the data. See [48] for a recent application to ground water hydrology.

Fractional differencing can be combined with the standard ARMA(p, q) model for time series, to yield the ARFIMA(p, α, q) time series model. The ARMA(p, q) model is

$$X_t - \sum_{j=1}^{p} \phi_j X_{t-j} = Z_t + \sum_{i=1}^{q} \theta_i Z_{t-i}$$

and we say that X_t follows an ARFIMA(p, α, q) if $\Delta^\alpha X_t$ follows an ARMA(p, q) model. The spectral density $f_X(k)$ of a stationary time series X_t is the discrete Fourier transform of its covariance function. If $\gamma(h)$ is the covariance between X_t and X_{t+h}, the spectral density

$$f_X(k) = \frac{1}{2\pi} \sum_{j=-\infty}^{+\infty} e^{ikh} \gamma(h).$$

The spectral density of the noise sequence Z_n is a constant $f_Z(k) = \sigma^2/(2\pi)$, which is the reason for the term "white noise," as all frequencies have equal weight. Using the backward shift operator notation $BX_t = X_{t-1}$, one can write the ARMA(p, q) model in the compact form $\Phi(B)X_t = \Theta(B)Z_t$ where $\Phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ and $\Theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$. Then it follows from the theory of linear filters that the spectral density of an ARMA(p, q) time series is given by

$$f_X(k) = \frac{|\Theta(e^{-ik})|^2}{|\Phi(e^{-ik})|^2} f_Z(k)$$

using the complex absolute value. Indeed, for any time invariant linear filter $X_t = \sum_j \psi_j Z_{t-j}$ a standard argument shows that $f_X(k) = |\Psi(e^{-ik})|^2 f_Z(k)$ where $\Psi(z) = \sum_j \psi_j z^j$, see for example [9]. For an ARMA(p, q) time series, the filter function is $\Psi(z) = \Theta(z)/\Phi(z)$. Since

$$\Delta^\alpha = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} B^j = (1 - B)^\alpha$$

in operator notation, a similar argument shows that the spectral density of an ARFIMA(p, α, q) time series is given by

$$f_X(k) = \frac{\sigma^2}{2\pi} \frac{|\Theta(e^{-ik})|^2}{|\Phi(e^{-ik})|^2} \left|1 - e^{-ik}\right|^{-2\alpha}.$$

As $k \to 0$, the fractional term $|1 - e^{-ik}|^{-2\alpha} \sim |k|^{-2\alpha}$ blows up at a power law rate. This diverging spectral density is the hallmark of long range dependence.

The tempered fractional difference

$$X_n = \Delta_\lambda^\alpha Z_n = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} e^{-\lambda j} Z_{n-j}$$

can provide a more flexible model for (semi-)long range dependence. We say that X_t follows an ARTFIMA(p, α, λ, q) if $\Delta_\lambda^\alpha X_t$ follows an ARMA(p, q) model. Since

$$\Delta_\lambda^\alpha = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} e^{-\lambda j} B^j = \left(1 - e^{-\lambda} B\right)^\alpha$$

in operator notation, it follows that the spectral density of an ARTFIMA(p, α, λ, q) time series is given by

$$f_X(k) = \frac{\sigma^2}{2\pi} \frac{|\Theta(e^{-ik})|^2}{|\Phi(e^{-ik})|^2} \left|1 - e^{-(\lambda + ik)}\right|^{-2\alpha}. \tag{30}$$

For small values of the tempering parameter $\lambda \approx 0$, this spectral density grows like a power law as k decreases, but then remains bounded as $k \to 0$ because of tempering. More details on ARTFIMA(p, α, λ, q) time series will be included in a forthcoming paper by the authors.

7. Turbulence

Kolmogorov [28] invented fractional Brownian motion as a model for turbulence in the inertial range. The main idea is that the spectral density grows like a power law as the frequency decreases, and a scaling argument leads to $H = 1/3$. This anti-persistent model has been verified in many real world experiments, for a range of frequencies. Davenport [16] modified this model in his study of wind speed at low altitudes, recognizing that the observed spectral density cannot actually diverge at low frequencies, outside the inertial range. The Davenport model for wind turbulence is $s_t = \mu + X_t$ where $\mu = \mathbb{E}[s_t]$ is the average wind speed, and the stochastic process X_t has the normalized spectral density

$$4800 D V_{10} \frac{x^2}{(1 + x^2)^{4/3}} \tag{31}$$

where V_{10} is the mean velocity (m/s) at an altitude of 10 m, D is the corresponding drag coefficient, and $x = 1200k/V_{10}$ where k is the wave number, see also Li and Kareem [31].

The spectral density of a stochastic process is the Fourier transform of its covariance function. The increment process $X_t = B_{H,\lambda}(t + 1) - B_{H,\lambda}(t)$ of a tempered fractional Brownian motion has spectral density

$$h(k) = \frac{C|e^{-ik} - 1|^2}{(\lambda^2 + k^2)^{H+1/2}} \tag{32}$$

for all real k, where $C > 0$ is some constant [47, Remark 4.3]. Since $e^{-ik} - 1 \sim -ik$ as $k \to 0$, this leads to the low frequency approximation

$$h(k) \approx C \frac{k^2}{(\lambda^2 + k^2)^{H+1/2}},$$

which shows that the stochastic process X_t has the Davenport spectrum with $H = 5/6$ and $\lambda = V_{10}/1200$. The process X_t is called a *tempered fractional Gaussian noise*.

Fig. 3 plots the spectral density of this tempered fractional Gaussian noise, along with the corresponding fractional Gaussian noise (the case $\lambda = 0$). Here we take $\lambda = 0.01$ to represent an average wind speed of 12 m/s. The low frequency range corresponds to large eddy production. In the high frequency range, energy is dissipated via shear stress. In the intermediate (inertial) range, energy is transferred from smaller to larger eddies, and this is where the Kolmogorov model pertains. In that model, the spectral density is proportional to $|k|^{1-2H}$ for moderate frequencies. For more details, see for example Pérez Beaupuits et al. [6].

In the inertial range, it can be seen from Fig. 3 that the spectrum of tempered fractional Gaussian noise is almost indistinguishable from that of (untempered) fractional Gaussian noise. However, there is a significant difference at low frequencies. The spectral density of fractional Gaussian noise diverges to infinity, which is not physical. However, the spectral density of tempered fractional Gaussian noise remains bounded, providing a more reasonable model for turbulence, consistent with the Davenport spectrum.

The Davenport model for wind turbulence is widely used to design windmill farms, offshore oil platforms, and large antennae [6,16,25,53], but to date the model provides only a spectrum for wind speed data. Tempered fractional Gaussian

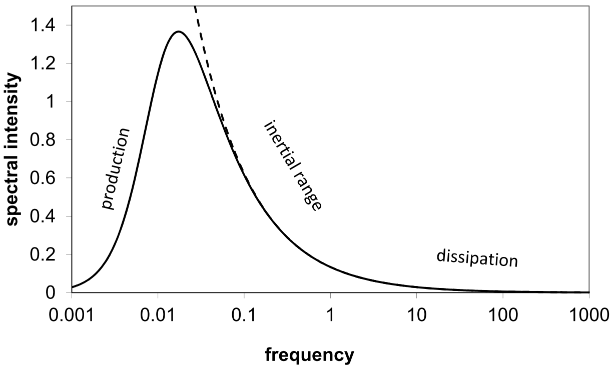

Fig. 3. Spectral density of tempered fractional Gaussian noise with $H = 5/6$ and $\lambda = 0.01$ (solid line) and of the corresponding fractional Gaussian noise with $H = 5/6$ (dashed line).

noise, the increments of a tempered fractional Brownian motion, provides a stochastic process model in the time domain, which can also be useful in applications. For example, it can be used to simulate realistic wind speed data. It may also be useful to apply tempered fractional Brownian motion with $H \approx 1/3$ as a model for turbulence in a more general setting. In the inertial range, it agrees well with the original Kolmogorov model (fractional Brownian motion), but it would seem to provide a more reasonable physical model at low frequencies.

The ARTFIMA process from Section 6 is the discrete time analogue of a tempered fractional Gaussian noise. Hence it could also provide a useful model for turbulence, when applied for example to a time series of turbulent velocities. From (30) we see that the ARTFIMA$(0, \alpha, \lambda, 0)$ spectral density is proportional to $|1 - e^{-(\lambda+ik)}|^{-2\alpha}$ and so, for small values of the tempering parameter $\lambda \approx 0$, the spectral density is approximately $|k|^{1-2H}$ for small frequencies k, recalling that $\alpha = 1/2 - H$. As $k \to 0$, the spectral density remains bounded, because of the tempering.

8. Summary

Fractional calculus is a very useful analytical toolbox, with many diverse applications to all areas of science and engineering. This paper is promoting a new variation called tempered fractional calculus, as a more flexible alternative with considerable promise for practical applications. A fractional derivative (or integral) is a (distributional) convolution with a power law. A tempered fractional derivative multiplies that power law kernel by an exponential factor. Fractional diffusion equations govern random walks with a power law probability density for the particle jumps. Multiplying that power law by an exponential tempering factor leads to a tempered fractional diffusion equation. Point source solutions to those equations are tempered stable probability densities, with semi-heavy tails that transition from power law to Gaussian. The emerging theory of tempered fractional calculus provides a sound mathematical basis for applications. Tempered fractional Brownian motion, the tempered fractional derivative or integral of a Brownian motion, extends the popular fractional Brownian motion, and provides a flexible model for semi-long range dependence that transitions from long to short range correlations. The tempered fractional difference operator can provide the basis for effective numerical schemes to solve tempered fractional differential equations. It is also useful as a model for time series analysis, where ARTFIMA(p, α, λ, q) models incorporate semi-long range dependence in a natural way. Tempered fractional Brownian motion provides a new stochastic process model for turbulence, consistent with the Davenport spectrum that extends the Kolmogorov theory of turbulence beyond the inertial range, and the ARTFIMA(p, α, λ, q) model provides a useful discrete time analogue.

Acknowledgement

We would like to thank Prof. Wojciech Charemza, Department of Economics, University of Leicester, for supplying the data used in Fig. 2.

References

[1] B. Baeumer, M.M. Meerschaert, D.A. Benson, S.W. Wheatcraft, Subordinated advection–dispersion equation for contaminant transport, Water Resour. Res. 37 (2001) 1543–1550.
[2] B. Baeumer, M. Kovács, M.M. Meerschaert, Fractional reproduction–dispersal equations and heavy tail dispersal kernels, Bull. Math. Biol. 69 (2007) 2281–2297.
[3] B. Baeumer, M.M. Meerschaert, Tempered stable Lévy motion and transient super-diffusion, J. Comput. Appl. Math. 233 (2010) 243–248.
[4] E. Barkai, Y. Garini, R. Metzler, Strange kinetics of single molecules in living cells, Phys. Today 65 (2012) 29.
[5] O.E. Barndorff-Nielsen, Processes of normal inverse Gaussian type, Finance Stoch. 2 (1998) 41–68.
[6] J.P. Pérez Beaupuits, A. Otárola, F.T. Rantakyrö, R.C. Rivera, S.J.E. Radford, L.-Å Nyman, Analysis of wind data gathered at Chajnantor, ALMA Memo 497 (2004).
[7] D. Benson, S. Wheatcraft, M. Meerschaert, Application of a fractional advection–dispersion equation, Water Resour. Res. 36 (2000) 1403–1412.

[8] D. Benson, R. Schumer, M. Meerschaert, S. Wheatcraft, Fractional dispersion, Lévy motions, and the MADE tracer tests, Transp. Porous Media 42 (2001) 211–240.
[9] P.J. Brockwell, R.A. Davis, Time Series: Theory and Methods, 2nd ed., Springer-Verlag, New York, 1991.
[10] P. Carr, H. Geman, D.B. Madan, M. Yor, The fine structure of asset returns: an empirical investigation, J. Bus. 75 (2002) 303–325.
[11] P. Carr, H. Geman, D.B. Madan, M. Yor, Stochastic volatility for Lévy processes, Math. Finance 13 (2003) 345–382.
[12] Á. Cartea, D. del-Castillo-Negrete, Fluid limit of the continuous-time random walk with general Lévy jump distribution functions, Phys. Rev. E 76 (2007) 041105.
[13] A. Chakrabarty, M.M. Meerschaert, Tempered stable laws as random walk limits, Stat. Probab. Lett. 81 (2011) 989–997.
[14] S. Cohen, J. Rosiński, Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered stable processes, Bernoulli 13 (2007) 195–210.
[15] J.H. Cushman, T.R. Ginn, Fractional advection–dispersion equation: a classical mass balance with convolution-Fickian flux, Water Resour. Res. 36 (2000) 3763–3766.
[16] A.G. Davenport, The spectrum of horizontal gustiness near the ground in high winds, Q. J. R. Meteorol. Soc. 87 (1961) 194–211.
[17] Z.-Q. Deng, L. Bengtsson, V.P. Singh, Parameter estimation for fractional dispersion model for rivers, Environ. Fluid Mech. 6 (2006) 451–475.
[18] K. Diethelm, N.J. Ford, A.D. Freed, Yu. Luchko, Algorithms for the fractional calculus: a selection of numerical methods, Comput. Methods Appl. Mech. Eng. 194 (2005) 743–773.
[19] Q. Du, M. Gunzburger, R.B. Lehoucq, K. Zhou, Analysis and approximation of nonlocal diffusion problems with volume constraints, SIAM Rev. 54 (2012) 667–696.
[20] A. Einstein, On the movement of small particles suspended in a stationary liquid demanded by the molecular kinetic theory of heat, Ann. Phys. 17 (1905) 549–560.
[21] V.J. Ervin, J.P. Roop, Variational solution of fractional advection dispersion equations on bounded domains in R^d, Numer. Methods Partial Differ. Equ. 23 (2007) 256–281.
[22] S. Fedotov, A. Iomin, Migration and proliferation dichotomy in tumor-cell invasion, Phys. Rev. Lett. 98 (2007) 8101.
[23] R. Gorenflo, F. Mainardi, E. Scalas, M. Raberto, Fractional calculus and continuous-time finance. III. The diffusion limit, in: Mathematical Finance, Konstanz, 2000, in: Trends Math., Birkhäuser, Basel, 2001, pp. 171–180.
[24] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, 2000.
[25] J.-J. Jang, G. Jyh-Shinn, Analysis of maximum wind force for offshore structure design, J. Mar. Sci. Technol. 7 (1999) 43–51.
[26] J.-H. Jeon, H. Martinez-Seara Monne, M. Javanainen, R. Metzler, Anomalous diffusion of phospholipids and cholesterols in a lipid bilayer and its origins, Phys. Rev. Lett. 109 (2012) 188103.
[27] A. Jurlewicz, A. Wyłomańska, P. Żebrowski, Coupled continuous-time random walk approach to the Rachev–Rüschendorf model for financial data, Physica A 388 (2009) 407–418.
[28] A.N. Kolmogorov, Wiener spiral and some other interesting curves in Hilbert space, Dokl. Akad. Nauk SSSR 26 (1940) 115–118.
[29] I. Koponen, Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process, Phys. Rev. E 52 (1995) 1197–1199.
[30] H.H. Kuo, White Noise Distribution Theory, CRC Press, Boca Raton, Florida, 1996.
[31] Y. Li, A. Kareem, ARMA systems in wind engineering, Probab. Eng. Mech. 5 (1990) 49–59.
[32] F. Liu, V. Ahn, I. Turner, P. Zhuang, Numerical simulation for solute transport in fractal porous media, ANZIAM J. 45 (2004) C461–C473.
[33] F. Liu, V. Ahn, I. Turner, Numerical solution of the space fractional Fokker–Planck equation, J. Comput. Appl. Math. 166 (2004) 209–219.
[34] V.E. Lynch, B.A. Carreras, D. del-Castillo-Negrete, K.M. Ferreira-Mejias, H.R. Hicks, Numerical methods for the solution of partial differential equations of fractional order, J. Comput. Phys. 192 (2003) 406–421.
[35] M. Magdziarz, A. Weron, K. Weron, Fractional Fokker–Planck dynamics: stochastic representation and computer simulation, Phys. Rev. E 75 (2007) 016708.
[36] R.L. Magin, Fractional Calculus in Bioengineering, Begell House, 2006.
[37] F. Mainardi, M. Raberto, R. Gorenflo, E. Scalas, Fractional calculus and continuous-time finance II: the waiting-time distribution, Physica A 287 (2000) 468–481.
[38] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models, World Scientific, 2010.
[39] B. Mandelbrot, J. Van Ness, Fractional Brownian motion, fractional noises and applications, SIAM Rev. 10 (1968) 422–437.
[40] R.N. Mantegna, H.E. Stanley, Stochastic process with ultraslow convergence to a Gaussian: the truncated Lévy flight, Phys. Rev. Lett. 73 (1994) 2946–2949.
[41] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection–dispersion flow equations, J. Comput. Appl. Math. 172 (2004) 65–77.
[42] M.M. Meerschaert, E. Scalas, Coupled continuous time random walks in finance, Physica A 370 (2006) 114–118.
[43] M.M. Meerschaert, H.-P. Scheffler, Triangular array limits for continuous time random walks, Stoch. Process. Appl. 118 (2008) 1606–1633.
[44] M.M. Meerschaert, Y. Zhang, B. Baeumer, Tempered anomalous diffusions in heterogeneous systems, Geophys. Res. Lett. 35 (2008) L17403–L17407.
[45] M.M. Meerschaert, Fractional calculus, anomalous diffusion, and probability, in: R. Metzler, J. Klafter (Eds.), Fractional Dynamics, World Scientific, Singapore, 2012, pp. 265–284.
[46] M.M. Meerschaert, A. Sikorskii, Stochastic Models for Fractional Calculus, De Gruyter, Berlin, 2012.
[47] M.M. Meerschaert, F. Sabzikar, Tempered fractional Brownian motion, Stat. Probab. Lett. 83 (2013) 2269–2275.
[48] M.M. Meerschaert, M. Dogan, R.L. Van Dam, D.W. Hyndman, D.A. Benson, Hydraulic conductivity fields: Gaussian or not?, Water Resour. Res. 49 (2013) 4730–4737.
[49] M.M. Meerschaert, F. Sabzikar, Stochastic integration for tempered fractional Brownian motion, Stoch. Process. Appl. 124 (2014) 2363–2387.
[50] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339 (2000) 1–77.
[51] R. Metzler, J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, J. Phys. A 37 (2004) R161–R208.
[52] K. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley and Sons, New York, 1993.
[53] D.J. Norton, Mobile offshore platform wind loads, in: Proc. 13th Offshore Techn. Conf., vol. 4, 1981, pp. 77–88, OTC 4123.
[54] V. Pipiras, M. Taqqu, Integration questions related to fractional Brownian motion, Probab. Theory Relat. Fields 118 (2000) 251–291.
[55] A. Piryatinska, A.I. Saichev, W.A. Woyczynski, Models of anomalous diffusion: subdiffusive and superdiffusive cases, Physica A 349 (2005) 375–424.
[56] J.P. Roop, Computational aspects of FEM approximation of fractional advection dispersion equations on bounded domains in $\mathbb{R}^2$, J. Comput. Appl. Math. 193 (2006) 243–268.
[57] J. Rosiński, Tempering stable processes, Stoch. Process. Appl. 117 (2007) 677–707.
[58] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Gordon and Breach, 1993.
[59] G. Samorodnitsky, M. Taqqu, Stable Non-Gaussian Random Processes, Chapman and Hall, New York, 1994.
[60] E. Scalas, R. Gorenflo, F. Mainardi, Fractional calculus and continuous-time finance, Physica A 284 (2000) 376–384.

[61] E. Scalas, Five years of continuous-time random walks in econophysics, in: The Complex Networks of Economic Interactions, Springer, Berlin, 2006, pp. 3–16.

[62] R. Schumer, D.A. Benson, M.M. Meerschaert, S.W. Wheatcraft, Eulerian derivation of the fractional advection–dispersion equation, J. Contam. Hydrol. 48 (2001) 69–88.

[63] I.M. Sokolov, J. Klafter, Anomalous diffusion spreads its wings, Phys. World 18 (2005) 29–32.

[64] C. Tadjeran, M.M. Meerschaert, H.-P. Scheffler, A second order accurate numerical approximation for the fractional diffusion equation, J. Comput. Phys. 213 (2006) 205–213.

[65] V.E. Tarasov, Fractional vector calculus and fractional Maxwell's equations, Ann. Phys. 323 (2008) 2756–2778.

[66] W. Whitt, Stochastic-Process Limits, Springer, New York, 2002.

[67] K. Yosida, Functional Analysis, sixth edition, Springer, 1980.

[68] S.B. Yuste, L. Acedo, An explicit finite difference method and a new von Neumann type stability analysis for fractional diffusion equations, SIAM J. Numer. Anal. 42 (2005) 1862–1874.

[69] G. Zaslavsky, Fractional kinetic equation for Hamiltonian chaos, Physica D 76 (1994) 110–122.

[70] Y. Zhang, M.M. Meerschaert, Gaussian setting time for solute transport in fluvial systems, Water Resour. Res. 47 (2011) W08601.

[71] Y. Zhang, M.M. Meerschaert, A.I. Packman, Linking fluvial bed sediment transport across scales, Geophys. Res. Lett. 39 (2012) L20404.