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Stochastic integration for tempered fractional Brownian motion

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Abstract

Tempered fractional Brownian motion is obtained when the power law kernel in the moving average representation of a fractional Brownian motion is multiplied by an exponential tempering factor. This paper develops the theory of stochastic integrals for tempered fractional Brownian motion. Along the way, we develop some basic results on tempered fractional calculus.

Keywords: Tempered fractional Brownian motion; Fractional calculus; Stochastic integration; Fractional Sobolev space

1. Introduction

This paper develops the theory of stochastic integration for tempered fractional Brownian motion (TFBM). Our approach follows the seminal work of Pipiras and Taqqu [34] for fractional Brownian motion (FBM). An FBM is the fractional derivative (or integral) of a Brownian motion, in a sense made precise by [34]. A fractional derivative is a (distributional) convolution with a power law [29,32,37]. Recently, some authors have proposed a tempered fractional derivative [2,6] that multiplies the power law kernel by an exponential tempering factor. Tempering produces a more tractable mathematical object, and can be made arbitrarily light, so that the resulting operator approximates the fractional derivative to any desired degree of accuracy over

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a finite interval. Based on this work, the authors of this paper have recently proposed a tempered fractional Brownian motion (TFBM), see [28] for basic definitions and properties.

Kolmogorov [22] first defined FBM using the harmonizable representation, as a model for turbulence in the inertial range (moderate frequencies). Mandelbrot and Van Ness [26] later developed the moving average representation of FBM. Since then, FBM has found many diverse applications in almost every field of science and engineering [1,12,35]. Davenport [10] modified the power spectrum of FBM to obtain a model for wind speed, which is now widely used [24,31,33]. The authors showed in [28] that TFBM has the Davenport spectrum, and hence TFBM offers a useful extension of the Kolmogorov model for turbulence, to include low frequencies.

The structure of the paper is as follows. In Section 2 we prove some basic results on tempered fractional calculus, which will be needed in the sequel. In Section 3 we apply the methods of Section 2 to construct a suitable theory of stochastic integration for tempered fractional Brownian motion. Finally, in Section 4 we discuss model extensions, related results, and some open questions.

2. Tempered fractional calculus

In this section, we define tempered fractional integrals and derivatives, and establish their essential properties. These results will form the foundation of the stochastic integration theory developed in Section 3. We begin with the definition of a tempered fractional integral.

Definition 2.1. For any $f \in L^p(\mathbb{R})$ (where $1 \le p < \infty$), the positive and negative tempered fractional integrals are defined by

$$\mathbb{I}_{+}^{\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} f(u)(t-u)_{+}^{\alpha-1} e^{-\lambda(t-u)_{+}} du \tag{2.1}$$

and

$$\mathbb{I}_{-}^{\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} f(u)(u-t)_{+}^{\alpha-1} e^{-\lambda(u-t)_{+}} du \tag{2.2}$$

respectively, for any $\alpha > 0$ and $\lambda > 0$, where $\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx$ is the Euler gamma function, and $(x)_+ = xI(x > 0)$.

When $\lambda = 0$ these definitions reduce to the (positive and negative) Riemann–Liouville fractional integral [29,32,37], which extends the usual operation of iterated integration to a fractional order. When $\lambda = 1$, the operator (2.1) is called the Bessel fractional integral [37, Section 18.4].

Lemma 2.2. *For any $\alpha > 0$, $\lambda > 0$, and $p \ge 1$, $\mathbb{I}_{\pm}^{\alpha,\lambda}$ is a bounded linear operator on $L^p(\mathbb{R})$ such that*

$$\|\mathbb{I}_{\pm}^{\alpha,\lambda} f\|_p \le \lambda^{-\alpha} \|f\|_p \tag{2.3}$$

for all $f \in L^p(\mathbb{R})$.

Proof. Young's Theorem [37, p. 12] states that if $\phi \in L^1(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ then $\phi * f \in L^p(\mathbb{R})$ and the inequality

$$\|\phi * f\|_p \le \|\phi\|_1 \|f\|_p \tag{2.4}$$

holds for all $1 \le p < \infty$, where $*$ denotes the convolution

$$[f * \phi](t) = \int_{-\infty}^{+\infty} f(u)\phi(t-u)du = [\phi * f](t).$$

Obviously $\mathbb{I}_{\pm}^{\alpha,\lambda}$ is linear, and $\mathbb{I}_{\pm}^{\alpha,\lambda} f(t) = [f * \phi_\alpha^\pm](t)$ where

$$\begin{aligned} \phi_\alpha^+(t) &= \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t} \mathbf{1}_{(0,\infty)}(t) \\ \phi_\alpha^-(t) &= \frac{1}{\Gamma(\alpha)} (-t)^{\alpha-1} e^{-\lambda(-t)} \mathbf{1}_{(-\infty,0)}(t) \end{aligned} \tag{2.5}$$

for any $\alpha, \lambda > 0$. But

$$\|\phi_\alpha^\pm\|_1 = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{-\lambda t} dt = \frac{1}{\Gamma(\alpha)} \left[\lambda^{-\alpha}\Gamma(\alpha)\right] = \lambda^{-\alpha}$$

using the formula for the Laplace transform (moment generating function) of the gamma probability density, and then (2.3) follows from Young's Inequality (2.4). □

Next we prove a semigroup property for tempered fractional integrals, which follows easily from the following property of the convolution kernels in Definition 2.1.

Lemma 2.3. *For any $\lambda > 0$ the functions (2.5) satisfy*

$$\phi_\alpha^\pm * \phi_\beta^\pm = \phi_{\alpha+\beta}^\pm \tag{2.6}$$

for any $\alpha > 0$ and $\beta > 0$.

Proof. For $t > 0$ we have

$$\begin{aligned} \phi_\alpha^+ * \phi_\beta^+(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} s^{\beta-1} e^{-\lambda s}\, ds \\ &= \frac{1}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1} e^{-\lambda t} = \phi_{\alpha+\beta}^+(t) \end{aligned}$$

using the formula for the beta probability density. The proof for ϕ_α^- is similar. □

The following lemma establishes the semigroup property for tempered fractional integrals on $L^p(\mathbb{R})$. In the case $\lambda = 0$, the semigroup property for fractional integrals is well known (e.g., see Samko et al. [37, Theorem 2.5]).

Lemma 2.4. *For any $\lambda > 0$ we have*

$$\mathbb{I}_{\pm}^{\alpha,\lambda}\mathbb{I}_{\pm}^{\beta,\lambda} f = \mathbb{I}_{\pm}^{\alpha+\beta,\lambda} f \tag{2.7}$$

for all $\alpha, \beta > 0$ and all $f \in L^p(\mathbb{R})$.

Proof. Lemma 2.2 shows that both sides of (2.7) belong to $L^p(\mathbb{R})$ for any $f \in L^p(\mathbb{R})$, and then the result follows immediately from Lemma 2.3 along with the fact that $\mathbb{I}_{\pm}^{\alpha,\lambda} f(t) = [f * \phi_\alpha^\pm](t)$. □

The next result shows that positive and negative tempered fractional integrals are adjoint operators with respect to the inner product $\langle f, g\rangle_2 = \int f(x)g(x)\,dx$ on $L^2(\mathbb{R})$.

Lemma 2.5 (*Integration by Parts*)**.** *Suppose $f, g \in L^2(\mathbb{R})$. Then*

$$\langle f, \mathbb{I}_{+}^{\alpha,\lambda} g\rangle_2 = \langle \mathbb{I}_{-}^{\alpha,\lambda} f, g\rangle_2 \tag{2.8}$$

for any $\alpha > 0$ and any $\lambda > 0$.

Proof. Write

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)\mathbb{I}_{+}^{\alpha,\lambda} g(x)\,dx &= \int_{-\infty}^{+\infty} f(x)\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x} g(u)(x-u)^{\alpha-1}e^{-\lambda(x-u)}\,du\,dx\\ &= \int_{-\infty}^{+\infty} \frac{g(u)}{\Gamma(\alpha)}\int_{u}^{+\infty} f(x)(x-u)^{\alpha-1}e^{-\lambda(x-u)}\,dx\,du\\ &= \int_{-\infty}^{+\infty} \mathbb{I}_{-}^{\alpha,\lambda} f(x)g(x)\,dx\end{aligned}$$

and this completes the proof. □

Next we discuss the relationship between tempered fractional integrals and Fourier transforms. Recall that the Fourier transform

$$\mathcal{F}[f](k) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx} f(x)dx$$

for functions $f \in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ can be extended to an isometry (a linear onto map that preserves the inner product) on $L^2(\mathbb{R})$ such that

$$\widehat{f}(k) = \lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-n}^{n} e^{-ikx} f(x)\,dx \tag{2.9}$$

for any $f \in L^2(\mathbb{R})$, see for example [20, Theorem 6.6.4].

Lemma 2.6. *For any $\alpha > 0$ and $\lambda > 0$ we have*

$$\mathcal{F}[\mathbb{I}_{\pm}^{\alpha,\lambda} f](k) = \hat{f}(k)(\lambda \pm ik)^{-\alpha} \tag{2.10}$$

for all $f \in L^1(\mathbb{R})$ and all $f \in L^2(\mathbb{R})$.

Proof. The function ϕ_{α}^{+} in (2.5) has Fourier transform

$$\mathcal{F}[\phi_{\alpha}^{+}](k) = \frac{1}{\Gamma(\alpha)\sqrt{2\pi}}\int_{0}^{\infty} e^{-ikt}t^{\alpha-1}e^{-\lambda t}dt = \frac{1}{\sqrt{2\pi}}(\lambda + ik)^{-\alpha} \tag{2.11}$$

by the formula for the Fourier transform of a gamma density. For any two functions $f, g \in L^1(\mathbb{R})$, the convolution $f * g \in L^1(\mathbb{R})$ has Fourier transform $\sqrt{2\pi}\,\hat{f}(k)\hat{g}(k)$ (e.g., see [29, p. 65]), and then (2.10) follows. The argument for $\mathbb{I}_{-}^{\alpha,\lambda}$ is quite similar. If $f \in L^2(\mathbb{R})$, approximate by the L^1 function $f(x)\mathbf{1}_{[-n,n]}(x)$ and let $n \to \infty$. □

Remark 2.7. Recall that the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R})$ consists of the infinitely differentiable functions $g : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}}\left|x^n g^{(m)}(x)\right| < \infty,$$

where n, m are non-negative integers, and $g^{(m)}$ is the derivative of order m. The space $\mathcal{S}'(\mathbb{R})$ of continuous linear functionals on $\mathcal{S}(\mathbb{R})$ is called the space of tempered distributions. The Fourier transform, and inverse Fourier transform, can then be extended to linear continuous mappings of $\mathcal{S}'(\mathbb{R})$ into itself. If $f : \mathbb{R} \to \mathbb{R}$ is a measurable function with polynomial growth, so that $\int |f(x)|(1 + |x|)^{-p} dx < \infty$ for some $p > 0$, then $T_f(\varphi) = \int f(x)\varphi(x)\, dx := \langle f, \varphi\rangle_1$ is a tempered distribution, also called a generalized function. The Fourier transform of this generalized function is defined as $\hat{T}_f(\varphi) = \langle \hat{f}, \varphi\rangle_1 = \langle f, \hat{\varphi}\rangle_1 = T_f(\hat{\varphi})$ for $\varphi \in \mathcal{S}(\mathbb{R})$. See Yosida [43, Ch.VI] for more details. If f is a tempered distribution, then the tempered fractional integrals $\mathbb{I}^{\alpha,\lambda}_{\pm} f(x)$ exist as convolutions with the tempered distributions (2.5). The same holds for Riemann–Liouville fractional integrals (the case $\lambda = 0$), but that case is more delicate, because the power law kernel (2.5) with $\lambda = 0$ is not in $L^1(\mathbb{R})$.

Next we consider the inverse operator of the tempered fractional integral, which is called a tempered fractional derivative. For our purposes, we only require derivatives of order $0 < \alpha < 1$, and this simplifies the presentation.

Definition 2.8. The positive and negative tempered fractional derivatives of a function $f : \mathbb{R} \to \mathbb{R}$ are defined as

$$\mathbb{D}^{\alpha,\lambda}_{+} f(t) = \lambda^{\alpha} f(t) + \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f(t) - f(u)}{(t-u)^{\alpha+1}}\, e^{-\lambda(t-u)} du \tag{2.12}$$

and

$$\mathbb{D}^{\alpha,\lambda}_{-} f(t) = \lambda^{\alpha} f(t) + \frac{\alpha}{\Gamma(1-\alpha)} \int_{t}^{+\infty} \frac{f(t) - f(u)}{(u-t)^{\alpha+1}}\, e^{-\lambda(u-t)} du \tag{2.13}$$

respectively, for any $0 < \alpha < 1$ and any $\lambda > 0$.

If $\lambda = 0$, the definitions (2.12) and (2.13) reduce to the positive and negative Marchaud fractional derivatives [37, Section 5.4].

Note that tempered fractional derivatives cannot be defined pointwise for all functions $f \in L^p(\mathbb{R})$, since we need $|f(t) - f(u)| \to 0$ fast enough to counter the singularity of the denominator $(t-u)^{\alpha+1}$ as $u \to t$.

Next we establish the existence and compute the Fourier transform of tempered fractional derivatives on a natural domain.

Theorem 2.9. *Assume f and f' are in $L^1(\mathbb{R})$. Then the tempered fractional derivative $\mathbb{D}^{\alpha,\lambda}_{+} f(t)$ exists and*

$$\mathcal{F}[\mathbb{D}^{\alpha,\lambda}_{\pm} f](k) = \widehat{f}(k)(\lambda \pm ik)^{\alpha} \tag{2.14}$$

for any $0 < \alpha < 1$ *and any* $\lambda > 0$.

Proof. A standard argument from functional analysis (e.g., see [11, Proposition 2.2]) shows that if $f, f' \in L^1(\mathbb{R})$, then

$$I := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(t) - f(u)|}{|t-u|^{1+\alpha}}\, dt\, du < \infty \tag{2.15}$$

for any $0 < \alpha < 1$. To see this, write $I = I_1 + I_2$ where

$$I_1 := \int_{\mathbb{R}} \int_{\mathbb{R}\cap\{|t-u|<1\}} \frac{|f(t) - f(u)|}{|t-u|^{1+\alpha}}\, dt\, du$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \int_{\{|z|<1\}} \frac{|f(t)-f(z+t)|}{|z|^{1+\alpha}}\, dz\, dt \\
&\le \int_{\mathbb{R}} \int_{\{|z|<1\}} |z|^{-\alpha} \int_0^1 |f'(t+uz)|\, du\, dz\, dt = \frac{2}{1-\alpha} \|f'\|_{L^1(\mathbb{R})} < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 :&= \int_{\mathbb{R}} \int_{\mathbb{R}\cap\{|t-u|\ge 1\}} \frac{|f(t)-f(u)|}{|t-u|^{1+\alpha}}\, dt\, du \\
&\le \int_{\mathbb{R}} \int_{\{|z|\ge 1\}} \frac{|f(t)|+|f(z+t)|}{|z|^{1+\alpha}}\, dt\, dz = \frac{2}{\alpha} \|f\|_{L^1(\mathbb{R})} < \infty.
\end{aligned}
$$

Now it follows easily from (2.15) that $\mathbb{D}_{\pm}^{\alpha,\lambda} f$ exists for all $f, f' \in L^1(\mathbb{R})$. Define

$$
F(t) := \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f(t)-f(u)}{(t-u)^{\alpha+1}} e^{-\lambda(t-u)}\, du,
$$

and apply the Fubini Theorem, along with the shift property $\mathcal{F}[f(t-y)](k) = e^{-iky}\hat{f}(k)$ of the Fourier transform, to see that

$$
\begin{aligned}
\widehat{F}(k) &= \frac{\alpha}{\Gamma(1-\alpha)\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikt} \int_0^{\infty} \frac{f(t)-f(t-y)}{y^{\alpha+1}} e^{-\lambda y}\, dy\, dt \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} y^{-\alpha-1} e^{-\lambda y} \left(1-e^{-iky}\right) \widehat{f}(k)\, dy = \frac{I_\lambda(\alpha)}{\Gamma(1-\alpha)} \widehat{f}(k)
\end{aligned} \tag{2.16}
$$

where

$$
I_\lambda(\alpha) = \int_0^{+\infty} \left(e^{-\lambda y} - e^{-(\lambda+ik)y}\right) \alpha y^{-\alpha-1}\, dy.
$$

Integrate by parts with $u = e^{-\lambda y} - e^{-(\lambda+ik)y}$ to see that

$$
\begin{aligned}
I_\lambda(\alpha) &= \left[\left(e^{-\lambda y} - e^{-(\lambda+ik)y}\right)\left(-y^{-\alpha}\right)\right]\Big|_0^{\infty} \\
&\quad + \int_0^{\infty} y^{-\alpha} \left[-\lambda e^{-\lambda y} + (\lambda+ik) e^{-(\lambda+ik)y}\right] dy
\end{aligned}
$$

and note that the boundary terms vanish, since $e^{-\lambda y} - e^{-(\lambda+ik)y} = O(y)$ as $y \to 0$. Use the definition of the gamma function, and the formula for the Fourier transform of the gamma probability density, to compute that

$$
\begin{aligned}
I_\lambda(\alpha) &= -\lambda \int_0^{\infty} y^{-\alpha} e^{-\lambda y}\, dy + (\lambda+ik) \int_0^{\infty} y^{-\alpha} e^{-(\lambda+ik)y}\, dy \\
&= -\lambda^{\alpha} \Gamma(1-\alpha) + (\lambda+ik) \frac{\Gamma(1-\alpha)}{\lambda^{1-\alpha}} \left(1+\frac{ik}{\lambda}\right)^{\alpha-1} \\
&= \Gamma(1-\alpha)\left[(\lambda+ik)^{\alpha} - \lambda^{\alpha}\right].
\end{aligned}
$$

Then $\widehat{F}(k) = \widehat{f}(k)\,[(\lambda+ik)^{\alpha} - \lambda^{\alpha}]$, and hence $\mathcal{F}[\mathbb{D}_{+}^{\alpha,\lambda} f](k) = (\lambda+ik)^{\alpha} \hat{f}(k)$. The proof for $\mathcal{F}[\mathbb{D}_{-}^{\alpha,\lambda} f](k)$ is similar. □

Remark 2.10. Theorem 2.9 can also be proven, under somewhat stronger conditions, using the generator formula for infinitely divisible semigroups [29, Theorems 3.17 and 3.23(b)].

Next we extend the definition of tempered fractional derivatives to a suitable class of functions in $L^2(\mathbb{R})$. For any $\alpha > 0$ and $\lambda > 0$ we may define the fractional Sobolev space

$$W^{\alpha,2}(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (\lambda^2 + k^2)^{\alpha} |\hat{f}(k)|^2 \, dk < \infty \right\}, \tag{2.17}$$

which is a Banach space with norm $\|f\|_{\alpha,\lambda} = \|(\lambda^2 + k^2)^{\alpha/2} \hat{f}(k)\|_2$. The space $W^{\alpha,2}(\mathbb{R})$ is the same for any $\lambda > 0$ (typically we take $\lambda = 1$) and all the norms $\|f\|_{\alpha,\lambda}$ are equivalent, since $1 + k^2 \le \lambda^2 + k^2 \le \lambda^2(1 + k^2)$ for all $\lambda \ge 1$, and $\lambda^2 + k^2 \le 1 + k^2 \le \lambda^{-2}(1 + k^2)$ for all $0 < \lambda < 1$.

Definition 2.11. The positive (resp., negative) tempered fractional derivative $\mathbb{D}_{\pm}^{\alpha,\lambda} f(t)$ of a function $f \in W^{\alpha,2}(\mathbb{R})$ is defined as the unique element of $L^2(\mathbb{R})$ with Fourier transform $\widehat{f}(k)(\lambda \pm ik)^{\alpha}$ for any $\alpha > 0$ and any $\lambda > 0$.

Remark 2.12. The pointwise definition of the tempered fractional derivative in real space is more complicated when $\alpha > 1$. For example, when $1 < \alpha < 2$ we have

$$\mathbb{D}_{+}^{\alpha,\lambda} f(t) = \lambda^{\alpha} f(t) + \alpha \lambda^{\alpha-1} f'(x) + \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f(u) - f(t) + (t-u) f'(t)}{(t-u)^{\alpha+1}} e^{-\lambda(t-u)} du,$$

for all $f \in W^{1,2}(\mathbb{R})$, compare [29, Remark 7.11].

Lemma 2.13. *For any $\alpha > 0$, $\beta > 0$ and $\lambda > 0$ we have*

$$\mathbb{D}_{\pm}^{\alpha,\lambda} \mathbb{D}_{\pm}^{\beta,\lambda} f(t) = \mathbb{D}_{\pm}^{\alpha+\beta,\lambda} f(t)$$

for any $f \in W^{\alpha+\beta,2}(\mathbb{R})$.

Proof. It is obvious from (2.17) that $W^{\alpha,2}(\mathbb{R}) \subset W^{\beta,2}(\mathbb{R})$ for $\alpha > \beta$. It is clear from Definition 2.11 that $\mathbb{D}_{\pm}^{\beta,\lambda} f(t)$ exists and belongs to $W^{\alpha,2}(\mathbb{R})$ for any $f \in W^{\alpha+\beta,2}(\mathbb{R})$, and likewise, $\mathbb{D}_{\pm}^{\alpha,\lambda} f(t)$ exists and belongs to $L^2(\mathbb{R})$ for any $f \in W^{\alpha,2}(\mathbb{R})$. □

Lemma 2.14. *For any $\alpha > 0$ and $\lambda > 0$, we have*

$$\mathbb{D}_{\pm}^{\alpha,\lambda} \mathbb{I}_{\pm}^{\alpha,\lambda} f(t) = f(t) \tag{2.18}$$

for any function $f \in L^2(\mathbb{R})$, and

$$\mathbb{I}_{\pm}^{\alpha,\lambda} \mathbb{D}_{\pm}^{\alpha,\lambda} f(t) = f(t) \tag{2.19}$$

for any $f \in W^{\alpha,2}(\mathbb{R})$.

Proof. Given $f \in L^2(\mathbb{R})$, note that $g(t) = \mathbb{I}_{\pm}^{\alpha,\lambda} f(t)$ satisfies $\hat{g}(k) = \hat{f}(k)(\lambda \pm ik)^{-\alpha}$ by Lemma 2.6, and then it follows easily that $g \in W^{\alpha,2}(\mathbb{R})$. Definition 2.11 implies that

$$\mathcal{F}[\mathbb{D}_{\pm}^{\alpha,\lambda} \mathbb{I}_{\pm}^{\alpha,\lambda} f](k) = \mathcal{F}[\mathbb{D}_{\pm}^{\alpha,\lambda} g](k) = \widehat{g}(k)(\lambda \pm ik)^{\alpha} = \hat{f}(k), \tag{2.20}$$

and then (2.18) follows using the uniqueness of the Fourier transform. The proof of (2.19) is similar. □

Lemma 2.15. *Suppose* $f, g \in W^{\alpha,2}(\mathbb{R})$. *Then*

$$\langle f, \mathbb{D}_+^{\alpha,\lambda} g\rangle_2 = \langle \mathbb{D}_-^{\alpha,\lambda} f, g\rangle_2 \tag{2.21}$$

for any $\alpha > 0$ *and any* $\lambda > 0$.

Proof. Apply the Plancherel Theorem along with Definition 2.11 to see that

$$\begin{aligned}\langle f, \mathbb{D}_+^{\alpha,\lambda} g\rangle_2 &= \int f(x)\overline{\mathbb{D}_+^{\alpha,\lambda} g(x)}\,dx = \langle \hat{f}, (\lambda + ik)^\alpha \hat{g}\rangle_2 = \langle (\lambda - ik)^\alpha \hat{f}, \hat{g}\rangle_2\\ &= \langle \mathbb{D}_-^{\alpha,\lambda} f, g\rangle_2\end{aligned}$$

and this completes the proof. □

Remark 2.16. One can also prove (2.21) for $f, f', g, g' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ using integration by parts, compare [44, Appendix A.1].

A slightly different tempered fractional derivative

$$\begin{aligned}\mathbf{D}_+^{\alpha,\lambda} f(t) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f(t) - f(u)}{(t-u)^{\alpha+1}} e^{-\lambda(t-u)} du\\ \mathbf{D}_-^{\alpha,\lambda} f(t) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{t}^{+\infty} \frac{f(t) - f(u)}{(u-t)^{\alpha+1}} e^{-\lambda(u-t)} du\end{aligned} \tag{2.22}$$

was proposed by Cartea and del-Castillo-Negrete [6] for a problem in physics, and studied further by Baeumer and Meerschaert [2,29] using tools from probability theory and semigroups. When $0 < \alpha < 1$, $\mathcal{F}[\mathbf{D}_\pm^{\alpha,\lambda} f](k) = \widehat{f}(k)[(\lambda \pm ik)^\alpha - \lambda^\alpha]\hat{f}(k)$ for suitable functions f. The additional λ^α term makes the evolution equation

$$\frac{\partial}{\partial t} u(x,t) = [p\mathbf{D}_+^{\alpha,\lambda} + q\mathbf{D}_-^{\alpha,\lambda}]u(x,t) \tag{2.23}$$

for $p, q \geq 0$ mass preserving, which can easily be seen by considering the Fourier transform $\hat{u}(k,t) = \exp(t[(\lambda \pm ik)^\alpha - \lambda^\alpha])$ of point source solutions to the tempered fractional diffusion equation (2.23). Now $x \mapsto u(x,t)$ are the probability density functions of a tempered stable Lévy process, as in Rosiński [36]. That process arises as the long-time scaling limit of a random walk with exponentially tempered power law jumps, see Chakrabarty and Meerschaert [7]. The tempered fractional diffusion equation (2.23) has been applied to contaminant plumes in underground aquifers, and sediment transport in rivers [30,45,46].

Remark 2.17. Tempered fractional derivatives are a natural analogues of integer (and fractional) order derivatives. For suitable functions $f(x)$, the Fourier transform of the derivative $f'(x)$ is $(ik)\hat{f}(k)$ (e.g., see [29, p. 8]), and one can define the fractional derivative $\mathbb{D}_\pm^\alpha f(t)$ as the function with Fourier transform $(ik)^\alpha \hat{f}(k)$. Definition 2.11 extends to tempered fractional derivatives.

3. Stochastic integrals

In this section, we apply tempered fractional calculus to define stochastic integrals with respect to tempered fractional Brownian motion (TFBM). First we recall the moving average representation of TFBM as a stochastic integral with respect to Brownian motion, from [28]. Let $\{B(t)\}_{t\in\mathbb{R}}$ be a real-valued Brownian motion on the real line, a process with stationary independent increments such that $B(t)$ has a Gaussian distribution with mean zero and variance $|t|$ for all $t \in \mathbb{R}$. Define an independently scattered Gaussian random measure $B(dx)$ with control measure $m(dx) = dx$ by setting $B[a, b] = B(b) - B(a)$ for any real numbers $a < b$, and then extending to all Borel sets. Since Brownian motion sample paths are almost surely of unbounded variation, the measure $B(dx)$ is not almost surely σ-additive, but it is a σ-additive measure in the sense of mean square convergence. Then the stochastic integrals $I(f) := \int f(x)B(dx)$ are defined for all functions $f : \mathbb{R} \to \mathbb{R}$ such that $\int f(x)^2 dx < \infty$, as Gaussian random variables with mean zero and covariance $\mathbb{E}[I(f)I(g)] = \int f(x)g(x)dx$. See for example [38, Chapter 3] or [29, Section 7.6].

Definition 3.1. Given an independently scattered Gaussian random measure $B(dx)$ on $\mathbb{R}$ with control measure $m(dx) = dx$, for any $\alpha < 1/2$ and $\lambda > 0$, the stochastic integral

$$B_{\alpha,\lambda}(t) = \int_{-\infty}^{+\infty} \left[e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha} - e^{-\lambda(-x)_+}(-x)_+^{-\alpha} \right] B(dx) \tag{3.1}$$

where $(x)_+ = xI(x > 0)$, and $0^0 = 0$, will be called a *tempered fractional Brownian motion* (TFBM).

Tempered fractional Brownian motion has a pleasant scaling property

$$\left\{ B_{\alpha,\lambda}(ct) \right\}_{t\in\mathbb{R}} \overset{f.d.}{=} \left\{ c^H B_{\alpha,c\lambda}(t) \right\}_{t\in\mathbb{R}} \quad \text{for any } c > 0, \tag{3.2}$$

where $H = 1/2 - \alpha$ and $\overset{f.d.}{=}$ indicates equality of all finite dimensional distributions [28, Proposition 2.2]. When $\lambda = 0$ and $-1/2 < \alpha < 1/2$, the right-hand side of (3.1) is a fractional Brownian motion (FBM), a self-similar Gaussian stochastic process with Hurst scaling index H (e.g., see Embrechts and Maejima [13]). When $\lambda = 0$ and $\alpha < -1/2$, the right-hand side of (3.1) does not exist, since the integrand is not in $L^2(\mathbb{R})$. However, TFBM with $\lambda > 0$ and $\alpha < -1/2$ is well-defined, because the exponential tempering keeps the integrand in $L^2(\mathbb{R})$. When $1/2 < H < 1$, the increments of FBM exhibit long range dependence, see [38, Proposition 7.2.10]. Increments of TFBM with $1/2 < H < 1$ exhibit *semi-long range dependence*, their autocorrelation function falling off like $|j|^{2H-2}$ over moderate lags, but then eventually falling off faster as $|j| \to \infty$. When $0 < H < 1/2$ the increments of both FBM and TFBM exhibit anti-persistence, also called negative dependence, since their autocorrelation function is negative for all large lags. See [28, Remark 4.1] for more details.

Stochastic integration theory for FBM is complicated by the fact that FBM is not a semimartingale [34]. If $\alpha < -1/2$ and $\lambda > 0$, or if $\alpha = 0$ and $\lambda > 0$, we will now show that TFBM is a semimartingale, and hence one can define stochastic integrals $I(f) := \int f(x)B_{\alpha,\lambda}(dx)$ in the standard manner, via the Itô stochastic calculus (e.g., see Kallenberg [19, Chapter 15]).

Theorem 3.2. *A tempered fractional Brownian motion $\{B_{\alpha,\lambda}(t)\}_{t\geq 0}$ with $\alpha < -1/2$ and $\lambda > 0$ is a continuous semimartingale with the canonical decomposition*

$$B_{\alpha,\lambda}(t) = -\lambda \int_0^t M_{\alpha,\lambda}(s)\, ds - \alpha \int_0^t M_{\alpha+1,\lambda}(s)\, ds \tag{3.3}$$

where

$$M_{\alpha,\lambda}(t) := \int_{-\infty}^{+\infty} e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha}\, B(dx). \tag{3.4}$$

Moreover, $\{B_{\alpha,\lambda}(t)\}_{t\ge 0}$ *is a finite variation process. The same is true if* $\alpha = 0$ *and* $\lambda > 0$.

Proof. Let $\{\mathcal{F}_t^B\}_{t\ge 0}$ be the σ-algebra generated by $\{B_s : 0 \le s \le t\}$. Given a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(t) = 0$ for all $t < 0$, and

$$g(t) = C + \int_0^t h(s)\, ds \quad \text{for all } t > 0 \tag{3.5}$$

for some $C \in \mathbb{R}$ and some $h \in L^2(\mathbb{R})$, a result of Cheridito [8, Theorem 3.9] shows that the Gaussian stationary increment process

$$Y_t^g := \int_{\mathbb{R}} [g(t-u) - g(-u)]\, B(du), \quad t \ge 0 \tag{3.6}$$

is a continuous $\{\mathcal{F}_t^B\}_{t\ge 0}$ semimartingale with canonical decomposition

$$Y_t^g = g(0)B_t + \int_0^t \int_{-\infty}^s h(s-u)B(du)ds, \tag{3.7}$$

and conversely, that if (3.6) defines a semimartingale on $[0, T]$ for some $T > 0$, then g satisfies these properties. Define $g(t) = 0$ for $t \le 0$ and

$$g(t) := e^{-\lambda t} t^{-\alpha} \quad \text{for } t > 0. \tag{3.8}$$

It is easy to check that the function $g(t-u) - g(-u)$, which is the integrand in (3.1), is square integrable over the entire real line for any $\alpha < 1/2$ and $\lambda > 0$. Next observe that (3.5) holds with $C = 0$, $h(s) = 0$ for $s < 0$ and

$$h(s) := \frac{d}{ds}[e^{-\lambda s} s^{-\alpha}] = -\lambda e^{-\lambda s} s^{-\alpha} - \alpha e^{-\lambda s} s^{-\alpha-1} \in L^2(\mathbb{R}) \tag{3.9}$$

for any $\alpha < -1/2$ and $\lambda > 0$. Then it follows from [8, Theorem 3.9] that TFBM is a continuous semimartingale with canonical decomposition

$$B_{\alpha,\lambda} = \int_0^t \int_{-\infty}^s -\lambda e^{-\lambda(s-u)}(s-u)^{-\alpha} - \alpha e^{-\lambda(s-u)}(s-u)^{-\alpha-1} B(du)\, ds \tag{3.10}$$

which reduces to (3.3). Since $C = 0$, Theorem 3.9 in [8] implies that $\{B_{\alpha,\lambda}(t)\}$ is a finite variation process. The proof for $\alpha = 0$ is similar, using $g(t) = e^{-\lambda t}$ for $t > 0$. □

Remark 3.3. When $\alpha = 0$ and $\lambda > 0$, the Gaussian stochastic process (3.4) is an Ornstein–Uhlenbeck process. When $\alpha < -1/2$ and $\lambda > 0$, it is a one dimensional Matérn stochastic process [3,14,16], also called a "fractional Ornstein–Uhlenbeck process" in the physics literature [25]. It follows from Knight [21, Theorem 6.5] that $M_{\alpha,\lambda}(t)$ is a semimartingale in both cases.

Cheridito [8, Theorem 3.9] provides a necessary and sufficient condition for the process (3.6) to be a semimartingale, and then it is not hard to check that TFBM is *not a semimartingale* in the remaining cases when $-1/2 < \alpha < 0$ or $0 < \alpha < 1/2$. Next we will investigate the problem of stochastic integration with deterministic integrands in these two cases. Our approach follows that of Pipiras and Taqqu [34].

Next we establish a link between TFBM and tempered fractional calculus.

Lemma 3.4. *For a tempered fractional Brownian motion* (3.1) *with* $\lambda > 0$*, we have:*

(i) *When* $-1/2 < \alpha < 0$, *we can write*

$$B_{\alpha,\lambda}(t) = \Gamma(\kappa+1) \int_{-\infty}^{+\infty} \left[\mathbb{I}_{-}^{\kappa,\lambda} \mathbf{1}_{[0,t]}(x) - \lambda \mathbb{I}_{-}^{\kappa+1,\lambda} \mathbf{1}_{[0,t]}(x) \right] B(dx) \tag{3.11}$$

where $\kappa = -\alpha$.

(ii) *When* $0 < \alpha < 1/2$, *we can write*

$$B_{\alpha,\lambda}(t) = \Gamma(1-\alpha) \int_{-\infty}^{+\infty} \left[\mathbb{D}_{-}^{\alpha,\lambda} \mathbf{1}_{[0,t]}(x) - \lambda \mathbb{I}_{-}^{1-\alpha,\lambda} \mathbf{1}_{[0,t]}(x) \right] B(dx). \tag{3.12}$$

Proof. To prove part (i), write the kernel function from (3.1) in the form

$$\begin{aligned} g_{t,\lambda}(x) &:= e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha} - e^{-\lambda(-x)_+}(-x)_+^{-\alpha} \\ &= \int_0^t \frac{d\left[e^{-\lambda(u-x)_+}(u-x)_+^{\kappa}\right]}{du} du \\ &= -\lambda \int_{-\infty}^{+\infty} \mathbf{1}_{[0,t]}(u) e^{-\lambda(u-x)_+}(u-x)_+^{(\kappa+1)-1} du \\ &\quad + \kappa \int_{-\infty}^{+\infty} \mathbf{1}_{[0,t]}(u) e^{-\lambda(u-x)_+}(u-x)_+^{\kappa-1} du \end{aligned}$$

and apply the definition (2.2) of the tempered fractional integral.

To prove part (ii), it suffices to show that the integrand

$$g_{t,\lambda}(x) = e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha} - e^{-\lambda(0-x)_+}(0-x)_+^{-\alpha} =: \phi_t(x) - \phi_0(x)$$

in (3.1) equals the integrand in (3.12). We will prove this using Fourier transforms. A substitution $u = t - x$ shows that

$$\widehat{\phi_t}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-ikx} e^{-\lambda(t-x)}(t-x)^{-\alpha}\, dx = \frac{e^{-ikt}\Gamma(1-\alpha)}{\sqrt{2\pi}(\lambda-ik)^{1-\alpha}}$$

using the formula for the Fourier transform of the gamma density, and hence

$$\widehat{g_{t,\lambda}}(k) = \widehat{\phi_t}(k) - \widehat{\phi_0}(k) = \Gamma(1-\alpha) \frac{e^{-ikt}-1}{\sqrt{2\pi}(\lambda-ik)^{1-\alpha}}. \tag{3.13}$$

On the other hand, from Lemma 2.6 and Theorem 2.9 we obtain

$$\begin{aligned} \mathcal{F}[\mathbb{D}_{-}^{\alpha,\lambda}\mathbf{1}_{[0,t]} - \lambda \mathbb{I}_{-}^{1-\alpha,\lambda}\mathbf{1}_{[0,t]}](k) &= [(\lambda-ik)^{\alpha} - \lambda(\lambda-ik)^{\alpha-1}] \cdot \frac{e^{-ikt}-1}{(-ik)\sqrt{2\pi}} \\ &= (\lambda-ik)^{\alpha-1} \cdot \frac{e^{-ikt}-1}{\sqrt{2\pi}} \end{aligned} \tag{3.14}$$

where we have used the formula (which is easy to verify)

$$\hat{h}(k) = \mathcal{F}[\mathbf{1}_{[a,b)}](k) = \frac{e^{-ikb}-e^{-ika}}{(-ik)\sqrt{2\pi}}, \tag{3.15}$$

and then the desired result follows by uniqueness of the Fourier transform. □

Next we explain the connection between the fractional calculus representations (3.11) and (3.12). Substitute $\kappa = -\alpha$ into (3.11) and note that the resulting formula differs from (3.12) only in that the tempered fractional integral $\mathbb{I}_-^{-\alpha,\lambda}$ is replaced by the tempered fractional derivative $\mathbb{D}_-^{\alpha,\lambda}$. Lemma 2.14 shows that $\mathbb{I}_-^{\alpha,\lambda}$ and $\mathbb{D}_-^{\alpha,\lambda}$ are inverse operators, and hence it makes sense to define $\mathbb{I}_\pm^{-\alpha,\lambda} := \mathbb{D}_\pm^{\alpha,\lambda}$ when $0 < \alpha < 1$. Now Eqs. (3.11) and (3.12) are equivalent.

Next we discuss a general construction for stochastic integrals with respect to TFBM. For a standard Brownian motion $\{B(t)\}_{t\in\mathbb{R}}$ on $(\Omega, \mathcal{F}, P)$, the stochastic integral $\mathcal{I}(f) := \int f(x)B(dx)$ is defined for any $f \in L^2(\mathbb{R})$, and the mapping $f \mapsto \mathcal{I}(f)$ defines an isometry from $L^2(\mathbb{R})$ into $L^2(\Omega)$, called the *Itô isometry*:

$$\langle \mathcal{I}(f), \mathcal{I}(g)\rangle_{L^2(\Omega)} = \mathrm{Cov}[\mathcal{I}(f), \mathcal{I}(g)] = \int f(x)g(x)\,dx = \langle f, g\rangle_{L^2(\mathbb{R})}. \tag{3.16}$$

Since this isometry maps $L^2(\mathbb{R})$ onto the space $\overline{\mathrm{Sp}}(B) = \{\mathcal{I}(f) : f \in L^2(\mathbb{R})\}$, we say that these two spaces are isometric. For any elementary function (step function)

$$f(u) = \sum_{i=1}^{n} a_i \mathbf{1}_{[t_i,t_{i+1})}(u), \tag{3.17}$$

where a_i, t_i are real numbers such that $t_i < t_j$ for $i < j$, it is natural to define the stochastic integral

$$\mathcal{I}^{\alpha,\lambda}(f) = \int_{\mathbb{R}} f(x)B_{\alpha,\lambda}(dx) = \sum_{i=1}^{n} a_i \left[B_{\alpha,\lambda}(t_{i+1}) - B_{\alpha,\lambda}(t_i)\right], \tag{3.18}$$

and then it follows immediately from (3.11) that for $f \in \mathcal{E}$, the space of elementary functions, the stochastic integral

$$\mathcal{I}^{\alpha,\lambda}(f) = \int_{\mathbb{R}} f(x)B_{\alpha,\lambda}(dx) = \Gamma(\kappa+1)\int_{\mathbb{R}} \left[\mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda\mathbb{I}_-^{\kappa+1,\lambda} f(x)\right] B(dx)$$

is a Gaussian random variable with mean zero, such that for any $f, g \in \mathcal{E}$ we have

$$\begin{aligned}\langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g)\rangle_{L^2(\Omega)} &= \mathbb{E}\left(\int_{\mathbb{R}} f(x)B_{\alpha,\lambda}(dx)\int_{\mathbb{R}} g(x)B_{\alpha,\lambda}(dx)\right)\\ &= \Gamma(\kappa+1)^2\int_{\mathbb{R}}\left[\mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda\mathbb{I}_-^{\kappa+1,\lambda} f(x)\right]\left[\mathbb{I}_-^{\kappa,\lambda} g(x) - \lambda\mathbb{I}_-^{\kappa+1,\lambda} g(x)\right] dx,\end{aligned} \tag{3.19}$$

in view of (3.11) and the Itô isometry (3.16). The linear space of Gaussian random variables $\left\{\mathcal{I}^{\alpha,\lambda}(f),\, f \in \mathcal{E}\right\}$ is contained in the larger linear space

$$\overline{\mathrm{Sp}}(B_{\alpha,\lambda}) = \left\{X : \mathcal{I}^{\alpha,\lambda}(f_n) \to X \text{ in } L^2(\Omega) \text{ for some sequence } (f_n) \text{ in } \mathcal{E}\right\}. \tag{3.20}$$

An element $X \in \overline{\mathrm{Sp}}(B_{\alpha,\lambda})$ is mean zero Gaussian with variance

$$\mathrm{Var}(X) = \lim_{n\to\infty} \mathrm{Var}[\mathcal{I}^{\alpha,\lambda}(f_n)],$$

and X can be associated with an equivalence class of sequences of elementary functions (f_n) such that $\mathcal{I}^{\alpha,\lambda}(f_n) \to X$ in $L^2(\mathbb{R})$. If $[f_X]$ denotes this class, then X can be written in an integral form as

$$X = \int_{\mathbb{R}} [f_X] dB_{\alpha,\lambda} \tag{3.21}$$

and the right hand side of (3.21) is called the stochastic integral with respect to TFBM on the real line (see, for example, Huang and Cambanis [17], page 587). In the special case of a Brownian motion $\lambda = \alpha = 0$, $\mathcal{I}^{\alpha,\lambda}(f_n) \to X$ along with the Itô isometry (3.16) implies that (f_n) is a Cauchy sequence, and then since $L^2(\mathbb{R})$ is a (complete) Hilbert space, there exists a unique $f \in L^2(\mathbb{R})$ such that $f_n \to f$ in $L^2(\mathbb{R})$, and we can write $X = \int_{\mathbb{R}} f(x)B(dx)$. However, if the space of integrands is not complete, then the situation is more complicated. We begin with the case $-1/2 < \alpha < 0$, where the corresponding FBM is long range dependent.

3.1. *Case 1: semi-long range dependence*

Here we investigate stochastic integrals with respect to TFBM in the case $-1/2 < \alpha < 0$, so that $1/2 < H < 1$ in (3.2). Eq. (3.19) suggests the appropriate space of integrands for TFBM, in order to obtain a nice isometry that maps into the space $\overline{\mathrm{Sp}}(B_{\alpha,\lambda})$ of stochastic integrals.

Theorem 3.5. *Given $-1/2 < \alpha < 0$ and $\lambda > 0$, let $\kappa = -\alpha$. Then the class of functions*

$$\mathcal{A}_1 := \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} \left| \mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f(x) \right|^2 dx < \infty \right\}, \tag{3.22}$$

is a linear space with inner product

$$\langle f, g \rangle_{\mathcal{A}_1} := \langle F, G \rangle_{L^2(\mathbb{R})} \tag{3.23}$$

where

$$\begin{aligned} F(x) &= \Gamma(\kappa+1)[\mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f(x)] \\ G(x) &= \Gamma(\kappa+1)[\mathbb{I}_-^{\kappa,\lambda} g(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} g(x)]. \end{aligned} \tag{3.24}$$

The set of elementary functions $\mathcal{E}$ is dense in the space $\mathcal{A}_1$. The space $\mathcal{A}_1$ is not complete.

The proof of Theorem 3.5 requires one simple lemma, which shows that $\mathbb{I}_-^{\kappa,\lambda} - \lambda \mathbb{I}_-^{\kappa+1,\lambda}$ is a bounded linear operator on $L^p(\mathbb{R})$ for any $1 \le p < \infty$.

Lemma 3.6. *Under the assumptions of Theorem 3.5, suppose $1 \le p < \infty$. Then for any $f \in L^p(\mathbb{R})$ we have*

$$\| \mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f(x) \|_p \le C \| f \|_p \tag{3.25}$$

where C is a constant depending only on α and λ.

Proof. It follows from Lemma 2.2 that $\mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f(x) \in L^p(\mathbb{R})$ and that

$$\| \mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f(x) \|_p \le \| \mathbb{I}_-^{\kappa,\lambda} f(x) \|_p + \lambda \| \mathbb{I}_-^{\kappa+1,\lambda} f(x) \|_p \le 2\lambda^{-\kappa} \| f \|_p$$

for any $f \in L^p(\mathbb{R})$. □

Remark 3.7. It follows from Lemma 3.6 that $\mathcal{A}_1$ contains every function in $L^2(\mathbb{R})$, and hence they are the same set, but endowed with a different inner product. The inner product on the space $\mathcal{A}_1$ is required to obtain a nice isometry.

Proof of Theorem 3.5. The proof is similar to [34, Theorem 3.2]. To show that $\mathcal{A}_1$ is an inner product space, we will check that $\langle f, f\rangle_{\mathcal{A}_1} = 0$ implies $f = 0$ almost everywhere. If $\langle f, f\rangle_{\mathcal{A}_1} = 0$, then in view of (3.23) and (3.24) we have $\langle F, F\rangle_2 = 0$, so $F(x) = \Gamma(1+\kappa)[\mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda\mathbb{I}_-^{\kappa+1,\lambda} f(x)] = 0$ for almost every $x \in \mathbb{R}$. Then

$$\mathbb{I}_-^{\kappa,\lambda} f(x) = \lambda\mathbb{I}_-^{\kappa+1,\lambda} f(x) \quad \text{for almost every } x \in \mathbb{R}. \tag{3.26}$$

Apply $\mathbb{D}_-^{\kappa,\lambda}$ to both sides of Eq. (3.26) and use Lemma 2.4 along with Lemma 2.14 to get

$$f(x) = \mathbb{D}_-^{\kappa,\lambda}\mathbb{I}_-^{\kappa,\lambda} f(x) = \mathbb{D}_-^{\kappa,\lambda}\lambda\mathbb{I}_-^{\kappa+1,\lambda} f(x) = \lambda\left[\mathbb{D}_-^{\kappa,\lambda}\mathbb{I}_-^{\kappa,\lambda}\right]\mathbb{I}_-^{1,\lambda} f(x) = \lambda\mathbb{I}_-^{1,\lambda} f(x)$$

for almost every $x \in \mathbb{R}$, and in view of the definition (2.1) this is equivalent to

$$f(x) = \lambda\int_x^{+\infty} f(u)e^{-\lambda(u-x)}du = \lambda e^{\lambda x}\int_x^{+\infty} f(u)e^{-\lambda u}du \tag{3.27}$$

for almost every $x \in \mathbb{R}$. Observe that the functions $f(u)$ and $e^{-\lambda u}$ are in $L^2[x,\infty)$ for any $x \in \mathbb{R}$ and then, by the Cauchy–Schwarz inequality, the function $f(u)e^{-\lambda u}$ is in $L^1[x,\infty)$. It follows that $\int_x^{+\infty} f(u)e^{-\lambda u}du$ is absolutely continuous, and so the function $f(x)$ in (3.27) is also absolutely continuous. Taking the derivative on both sides of (3.27) using the Lebesgue Differentiation Theorem (e.g., see [42, Theorem 7.16]) we get

$$f'(x) = \lambda f(x) - \lambda e^{\lambda x} f(x)e^{-\lambda x} = 0 \quad \text{for almost every } x \in \mathbb{R}.$$

Then for any $a, b \in R$ we have

$$f(b) = f(a) + \int_a^b f'(x)\,dx = f(a)$$

and so $f(x)$ is a constant function. Since $f \in L^2(\mathbb{R})$, it follows that $f(x) = 0$ for all $x \in \mathbb{R}$, and hence $\mathcal{A}_1$ is an inner product space.

Next, we want to show that the set of elementary functions $\mathcal{E}$ is dense in $\mathcal{A}_1$. For any $f \in \mathcal{A}_1$, we also have $f \in L^2(\mathbb{R})$, and hence there exists a sequence of elementary functions (f_n) in $L^2(\mathbb{R})$ such that $\|f - f_n\|_2 \to 0$. But

$$\|f - f_n\|_{\mathcal{A}_1} = \langle f - f_n, f - f_n\rangle_{\mathcal{A}_1} = \langle F - F_n, F - F_n\rangle_2 = \|F - F_n\|_2,$$

where $F_n(x) = \mathbb{I}_-^{\kappa,\lambda} f_n(x) - \lambda\mathbb{I}_-^{\kappa+1,\lambda} f_n(x)$ and $F(x)$ is given by (3.24). Lemma 3.6 implies that

$$\|f - f_n\|_{\mathcal{A}_1} = \|F - F_n\|_2 = \|\mathbb{I}_-^{\kappa,\lambda}(f - f_n) - \lambda\mathbb{I}_-^{\kappa+1,\lambda}(f - f_n)\|_2 \le C\|f - f_n\|_2$$

for some $C > 0$, and since $\|f - f_n\|_2 \to 0$, it follows that the set of elementary functions is dense in $\mathcal{A}_1$.

Finally, we provide an example to show that $\mathcal{A}_1$ is not complete. The functions

$$\widehat{f_n}(k) = |k|^{-p}\mathbf{1}_{\{1<|k|<n\}}(k), \qquad p > 0,$$

are in $L^2(\mathbb{R})$, $\overline{\widehat{f_n}(k)} = \widehat{f_n}(-k)$, and hence they are the Fourier transforms of functions $f_n \in L^2(\mathbb{R})$. Apply Lemma 2.6 to see that the corresponding functions $F_n(x) = \Gamma(\kappa+1)[\mathbb{I}_-^{\kappa,\lambda} f_n(x) - \lambda\mathbb{I}_-^{\kappa+1,\lambda} f_n(x)]$ from (3.24) have Fourier transform

$$\mathcal{F}[F_n](k) = \Gamma(1-\alpha)[(\lambda - ik)^{\alpha} - \lambda(\lambda - ik)^{\alpha-1}]\hat{f}_n(k) = \frac{-ik\Gamma(1-\alpha)}{(\lambda - ik)^{1-\alpha}}\hat{f}_n(k). \tag{3.28}$$

Since $\alpha < 0$, it follows that

$$\|F_n\|_2^2 = \|\hat{F}_n\|_2^2 = \Gamma(1-\alpha)^2 \int_{-\infty}^{\infty} \left|\widehat{f_n}(k)\right|^2 \frac{k^2}{(\lambda^2+k^2)^{1-\alpha}} < \infty$$

for each n, which shows that $f_n \in \mathcal{A}_1$. Now it is easy to check that $f_n - f_m \to 0$ in $\mathcal{A}_1$, as $n, m \to \infty$, whenever $p > 1/2+\alpha$, so that (f_n) is a Cauchy sequence. Choose $p \in (1/2+\alpha, 1/2)$ and suppose that there exists some $f \in \mathcal{A}_1$ such that $\|f_n - f\|_{\mathcal{A}_1} \to 0$ as $n \to \infty$. Then

$$\int_{-\infty}^{\infty} \left|\widehat{f_n}(k) - \widehat{f}(k)\right|^2 \frac{k^2}{(\lambda^2+k^2)^{1-\alpha}} \to 0 \tag{3.29}$$

as $n \to \infty$, and since, for any given $m \geq 1$, the value of $\widehat{f_n}(k)$ does not vary with $n > m$ whenever $k \in [-m, m]$, it follows that $\hat{f}(k) = |k|^{-p} 1_{\{|k|>1\}}$ on any such interval. Since m is arbitrary, it follows that $\hat{f}(k) = |k|^{-p} 1_{\{|k|>1\}}$, but this function is not in $L^2(\mathbb{R})$, so $\hat{f}(k) \notin \mathcal{A}_1$, which is a contradiction. Hence $\mathcal{A}_1$ is not complete, and this completes the proof. □

We now define the stochastic integral with respect to TFBM for any function in $\mathcal{A}_1$ in the case where $1/2 < H < 1$ in (3.2).

Definition 3.8. For any $-1/2 < \alpha < 0$ and $\lambda > 0$, we define

$$\int_{\mathbb{R}} f(x) B_{\alpha,\lambda}(dx) := \Gamma(\kappa+1) \int_{\mathbb{R}} \left[\mathbb{I}_{-}^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_{-}^{\kappa+1,\lambda} f(x)\right] B(dx) \tag{3.30}$$

for any $f \in \mathcal{A}_1$, where $\kappa = -\alpha$.

Theorem 3.9. *For any $-1/2 < \alpha < 0$ and $\lambda > 0$, the stochastic integral $\mathcal{I}^{\alpha,\lambda}$ in (3.30) is an isometry from $\mathcal{A}_1$ into $\overline{\mathrm{Sp}}(B_{\alpha,\lambda})$. Since $\mathcal{A}_1$ is not complete, these two spaces are not isometric.*

Proof. It follows from Lemma 3.6 that the stochastic integral (3.30) is well-defined for any $f \in \mathcal{A}_1$. Proposition 2.1 in Pipiras and Taqqu [34] implies that, if $\mathcal{D}$ is an inner product space such that $(f, g)_{\mathcal{D}} = \langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g)\rangle_{L^2(\Omega)}$ for all $f, g \in \mathcal{E}$, and if $\mathcal{E}$ is dense $\mathcal{D}$, then there is an isometry between $\mathcal{D}$ and a linear subspace of $\overline{\mathrm{Sp}}(B_{\alpha,\lambda})$ that extends the map $f \to \mathcal{I}^{\alpha,\lambda}(f)$ for $f \in \mathcal{E}$, and furthermore, $\mathcal{D}$ is isometric to $\overline{\mathrm{Sp}}(B_{\alpha,\lambda})$ itself if and only if $\mathcal{D}$ is complete. Using the Itô isometry and the definition (3.30), it follows from (3.23) that for any $f, g \in \mathcal{A}_1$ we have

$$\langle f, g\rangle_{\mathcal{A}_1} = \langle F, G\rangle_{L^2(\mathbb{R})} = \langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g)\rangle_{L^2(\Omega)},$$

and then the result follows from Theorem 3.5. □

3.2. Case 2: anti-persistence

Next we investigate stochastic integrals with respect to TFBM in the case $0 < \alpha < 1/2$, so that $0 < H < 1/2$ in (3.2). It follows from (3.12) that the stochastic integral (3.18) can be written in the form

$$\mathcal{I}^{\alpha,\lambda}(f) = \int_{\mathbb{R}} f(x) B_{\alpha,\lambda}(dx) = \Gamma(1-\alpha) \int_{-\infty}^{\infty} \left[\mathbb{D}_{-}^{\alpha,\lambda} f(x) - \lambda \mathbb{I}_{-}^{1-\alpha,\lambda} f(x)\right] B(dx)$$

for any $f \in \mathcal{E}$, the space of elementary functions. Then $\mathcal{I}^{\alpha,\lambda}(f)$ is a Gaussian random variable with mean zero, such that

$$\begin{aligned}\langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g)\rangle_{L^2(\Omega)} &= \mathbb{E}\left(\int_{\mathbb{R}} f(x)B_{\alpha,\lambda}(dx)\int_{\mathbb{R}} g(x)B_{\alpha,\lambda}(dx)\right)\\ &= \Gamma(1-\alpha)^2\int_{\mathbb{R}}\left[\mathbb{D}_-^{\alpha,\lambda}f(x)-\lambda\mathbb{I}_-^{1-\alpha,\lambda}f(x)\right]\left[\mathbb{D}_-^{\alpha,\lambda}g(x)-\lambda\mathbb{I}_-^{1-\alpha,\lambda}g(x)\right]dx \end{aligned} \tag{3.31}$$

for any $f, g \in \mathcal{E}$, using (3.12) and the Itô isometry (3.16). Eq. (3.31) suggests the following space of integrands for TFBM in the case $0 < H < 1/2$. Recall that $W^{\alpha,2}(\mathbb{R})$ is the fractional Sobolev space (2.17).

Theorem 3.10. *For any $0 < \alpha < 1/2$ and $\lambda > 0$, the class of functions*

$$\mathcal{A}_2 := \left\{ f \in W^{\alpha,2}(\mathbb{R}) : \varphi_f = \mathbb{D}_-^{\alpha,\lambda} f - \lambda \mathbb{I}_-^{1-\alpha,\lambda} f \text{ for some } \varphi_f \in L^2(\mathbb{R}). \right\} \tag{3.32}$$

is a linear space with inner product

$$\langle f, g\rangle_{\mathcal{A}_2} := \langle F, G\rangle_{L^2(\mathbb{R})} \tag{3.33}$$

where

$$\begin{aligned} F(x) &= \Gamma(1-\alpha)\left[\mathbb{D}_-^{\alpha,\lambda}f(x)-\lambda\mathbb{I}_-^{1-\alpha,\lambda}f(x)\right]\\ G(x) &= \Gamma(1-\alpha)\left[\mathbb{D}_-^{\alpha,\lambda}g(x)-\lambda\mathbb{I}_-^{1-\alpha,\lambda}g(x)\right]. \end{aligned} \tag{3.34}$$

The set of elementary functions $\mathcal{E}$ is dense in the space $\mathcal{A}_2$. The space $\mathcal{A}_2$ is not complete.

We begin with two lemmas. The first lemma shows that the set $\mathcal{A}_2$ contains every function in $W^{\alpha,2}(\mathbb{R})$, and hence they are the same set, but different spaces, since they have different inner products.

Lemma 3.11. *Under the assumptions of Theorem 3.10, every $f \in W^{\alpha,2}(\mathbb{R})$ is an element of $\mathcal{A}_2$.*

Proof. Given $f \in W^{\alpha,2}(\mathbb{R})$, we need to show that

$$\varphi_f = \mathbb{D}_-^{\alpha,\lambda} f - \lambda \mathbb{I}_-^{1-\alpha,\lambda} f \tag{3.35}$$

for some $\varphi_f \in L^2(\mathbb{R})$. From the definition (2.17) we see that $\int(\lambda^2+k^2)^\alpha|\hat{f}(k)|^2\,dk < \infty$. Define $h_1(k) = (\lambda - ik)^\alpha \hat{f}(k)$ and note that h_1 is the Fourier transform of some function $\varphi_1 \in L^2(\mathbb{R})$. Define $h_2(k) := (\lambda - ik)^{\alpha-1}\widehat{f}(k)$, and observe that

$$\begin{aligned}\int |h_2(k)|^2\,dk &= \int |\widehat{f}(k)|^2(\lambda^2+k^2)^{\alpha-1}\,dk\\ &= \int \frac{|h_1(k)|^2}{\lambda^2+k^2}\,dk < \infty,\end{aligned}$$

since $h_1 \in L^2(\mathbb{R})$ and $1/(\lambda^2+k^2)$ is bounded. Hence there is another function $\varphi_2 \in L^2(\mathbb{R})$ such that $h_2 = \hat{\varphi}_2$. Define $\varphi_f := \varphi_1 - \lambda\varphi_2$ so that

$$\widehat{\varphi_f}(k) = \widehat{\varphi_1}(k) - \lambda\widehat{\varphi_2}(k) = \widehat{f}(k)(\lambda - ik)^\alpha - \widehat{f}(k)\lambda(\lambda - ik)^{\alpha-1}. \tag{3.36}$$

Since $f \in W^{\alpha,2}(\mathbb{R}) \subset L^2(\mathbb{R})$, we can apply Definition 2.11 and Lemma 2.6 to see that (3.35) holds. □

Lemma 3.12. *Under the assumptions of Theorem 3.10, if $f \in W^{\alpha,2}(\mathbb{R})$, then there exists a sequence of elementary functions (f_n) such that $f_n \to f$ in $L^2(\mathbb{R})$, and also*

$$\int_{-\infty}^{+\infty} |\widehat{f_n}(k) - \widehat{f}(k)|^2 |k|^{2\alpha} dk \to 0 \quad \text{as } n \to \infty. \tag{3.37}$$

Proof. Eq. (3.37) is proven in [34, Lemma 5.1]. For any $L > 0$, that proof constructs a sequence of elementary functions f_n such that $\hat{f}_n(k) \to \mathbf{1}_{[-1,1]}(k)$ almost everywhere on $-L \le x \le L$, and shows that $|\hat{f}_n(k)| \le C \min\{1, |k|^{-1}\}$ for all $k \in \mathbb{R}$ and all $n \ge 1$. In the notation of that paper, we have $\hat{f}_n(k) = k^{-1} U_n(k)$. Apply the dominated convergence theorem to see that

$$\int_{-L}^{+L} |\hat{f}_n(k) - \mathbf{1}_{[-1,1]}(k)|^2 dk \to 0$$

and note that

$$\int_{|k|>L} |\hat{f}_n(k) - \mathbf{1}_{[-1,1]}(k)|^2 dk \le 2C^2 \int_L^{\infty} \frac{dk}{k^2} \le \frac{2C^2}{L}.$$

Since L is arbitrary, it follows that $\hat{f}_n(k) \to \mathbf{1}_{[-1,1]}(k)$ in $L^2(\mathbb{R})$, and then the result follows as in [34, Lemma 5.1]. □

Proof of Theorem 3.10. For $f \in \mathcal{A}_2$ we define

$$\|f\|_{\mathcal{A}_2} = \sqrt{\langle f, f\rangle_{\mathcal{A}_2}} = \sqrt{\langle \varphi_f, \varphi_f\rangle_2} = \|\varphi_f\|_2 \tag{3.38}$$

where φ_f is given by (3.35). Next, use (3.36) to see that

$$\widehat{\varphi_f}(k) = (-ik)(\lambda - ik)^{\alpha-1} \widehat{f}(k). \tag{3.39}$$

To verify that (3.33) is an inner product, note that if $\langle f, f\rangle_{\mathcal{A}_2} = 0$ then

$$\|f\|_{\mathcal{A}_2}^2 = \|\varphi_f\|_2^2 = \|\widehat{\varphi_f}\|_2^2 = \int_{-\infty}^{\infty} |\widehat{f}(k)|^2 \frac{k^2}{(\lambda^2 + k^2)^{1-\alpha}} \, dk \tag{3.40}$$

equals zero, which implies that $|\widehat{f}(k)| = 0$ almost everywhere, and then $f = 0$ almost everywhere. This proves that (3.35) is an inner product.

Next we show that $\mathcal{E}$ is dense in $\mathcal{A}_2$. Apply Lemma 3.12 to obtain a sequence (f_n) in $\mathcal{E}$ such that $\|f_n - f\|_2 \to 0$ and (3.37) holds. It is easy to check using (3.15) that any elementary function is an element of $W^{\alpha,2}(\mathbb{R})$, and then Lemma 3.11 implies that it is also an element of $\mathcal{A}_2$. Now use (3.40) to write

$$\begin{aligned} \|f_n - f\|_{\mathcal{A}_2}^2 = & \int_{-\infty}^{+\infty} \left|\widehat{f_n}(k) - \widehat{f}(k)\right|^2 (k^2 + \lambda^2)^{\alpha} \, dk \\ & - \lambda^2 \int_{-\infty}^{+\infty} \left|\widehat{f_n}(k) - \widehat{f}(k)\right|^2 \frac{1}{(\lambda^2 + k^2)^{1-\alpha}} \, dk. \end{aligned}$$

Since $1/(\lambda^2 + k^2)^{1-\alpha}$ is bounded, it follows easily using (3.37) and $\|f_n - f\|_2 \to 0$ that $\|f_n - f\|_{\mathcal{A}_2} \to 0$, and hence $\mathcal{E}$ is dense in $\mathcal{A}_2$.

Finally, we want to show that $\mathcal{A}_2$ is not complete. The proof is similar to that of Theorem 3.5. The functions

$$\widehat{f_n}(k) = |k|^{-p}\mathbf{1}_{\{1/n<|k|<1\}}(k)$$

are the Fourier transforms of some functions $f_n \in L^2(\mathbb{R})$. Clearly $f_n \in W^{\alpha,2}(\mathbb{R})$, and then it follows from Lemma 2.6 and Theorem 2.9 that the corresponding functions $F_n(x) = \Gamma(1-\alpha)[\mathbb{D}_-^{\alpha,\lambda} f_n(x) - \lambda\mathbb{I}_-^{1-\alpha,\lambda} f_n(x)]$ from (3.34) have Fourier transform (3.28), that is,

$$\mathcal{F}[F_n](k) = \Gamma(1-\alpha)\frac{-ik}{(\lambda - ik)^{1-\alpha}}\hat{f}_n(k).$$

Then

$$\|f_n\|_{\mathcal{A}_2}^2 = \|F_n\|_2^2 = \|\widehat{F_n}\|_2^2 = \Gamma(1-\alpha)^2\int_{-\infty}^{\infty}\left|\widehat{f_n}(k)\right|^2\frac{k^2}{(\lambda^2+k^2)^{1-\alpha}}\,dk < \infty$$

for any $p < 3/2$, so that $f_n \in \mathcal{A}_2$. Now it is easy to check that $f_n - f_m \to 0$ in $\mathcal{A}_2$, as $n, m \to \infty$, so that (f_n) is a Cauchy sequence. Suppose $1/2 < p < 3/2$ and that $\|f_n - f\|_{\mathcal{A}_2} \to 0$ for some $f \in \mathcal{A}_2$. Then $\hat{f}(k) = |k|^{-p}1_{\{0<|k|<1\}}$, but this $\hat{f}$ is not in $L^2(\mathbb{R})$, so $\hat{f} \notin \mathcal{A}_2$, and hence $\mathcal{A}_2$ is not complete. □

We now define the stochastic integral with respect to TFBM for any function in $\mathcal{A}_2$ in the case where $0 < H < 1/2$ in (3.2).

Definition 3.13. For any $0 < \alpha < 1/2$ and $\lambda > 0$, we define

$$\mathcal{I}^{\alpha,\lambda}(f) = \int_{\mathbb{R}} f(x)B_{\alpha,\lambda}(dx) := \Gamma(1-\alpha)\int_{\mathbb{R}}\left[\mathbb{D}_-^{\alpha,\lambda} f(x) - \lambda\mathbb{I}_-^{1-\alpha,\lambda} f(x)\right] B(dx) \quad (3.41)$$

for any $f \in \mathcal{A}_2$.

Theorem 3.14. *For any $0 < \alpha < 1/2$ and $\lambda > 0$, the stochastic integral $\mathcal{I}^{\alpha,\lambda}$ is an isometry from $\mathcal{A}_2$ into $\overline{\mathrm{Sp}}(B_{\alpha,\lambda})$. Since $\mathcal{A}_2$ is not complete, these two spaces are not isometric.*

Proof. The proof is similar to that of Theorem 3.9. It follows from Lemma 3.11 that the stochastic integral (3.41) is well-defined for any $f \in \mathcal{A}_2$. Use Proposition 2.1 in Pipiras and Taqqu [34], and note that the Itô isometry, the definition (3.41), and Eq. (3.33) imply that for any $f, g \in \mathcal{A}_2$ we have

$$\langle f, g\rangle_{\mathcal{A}_2} = \langle F, G\rangle_{L^2(\mathbb{R})} = \langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g)\rangle_{L^2(\Omega)}.$$

Then the result follows from Theorem 3.10. □

3.3. *Harmonizable representation*

By now it should be clear that the Fourier transform plays an important role in the theory of stochastic integration for TFBM. Here we apply the harmonizable representation of TFBM to unify the two cases $-1/2 < \alpha < 0$ and $0 < \alpha < 1/2$.

For any $-1/2 < \alpha < 1/2$ and any $\lambda > 0$, Proposition 3.1 in [28] shows that TFBM has the harmonizable representation

$$B_{\alpha,\lambda}(t) = \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-itk}-1}{(\lambda - ik)^{1-\alpha}}\hat{B}(dk)$$

where $\hat{B} = \hat{B}_1 + i\hat{B}_2$ is a complex-valued Gaussian random measure constructed as follows. Let $\hat{B}_1$ and $\hat{B}_2$ be two independent Brownian motions on the positive real line with $\mathbb{E}[(\hat{B}_i(t))^2] = t/2$ for $i = 1, 2$, and define two independently scattered Gaussian random measures by setting $\hat{B}_i[a, b] = \hat{B}_i(b) - \hat{B}_i(a)$, extend to Borel subsets of the positive real line, and then extend to the entire real line by setting $\hat{B}_1(A) = \hat{B}_1(-A)$, $\hat{B}_2(A) = -\hat{B}_2(-A)$.

Apply the formula (3.15) for the Fourier transform of an indicator function to write this harmonizable representation in the form

$$B_{\alpha,\lambda}(t) = \Gamma(1-\alpha)\int_{-\infty}^{+\infty} \widehat{\mathbf{1}}_{[0,t]}(k)\frac{(-ik)}{(\lambda - ik)^{1-\alpha}}\hat{B}(dk).$$

It follows easily that for any elementary function (3.17) we may write

$$\mathcal{I}^{\alpha,\lambda}(f) = \Gamma(1-\alpha)\int_{-\infty}^{\infty} \widehat{f}(k)\frac{(-ik)}{(\lambda - ik)^{1-\alpha}}\hat{B}(dk), \tag{3.42}$$

and then for any elementary functions f and g we have

$$\langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g)\rangle_{L^2(\Omega)} = \Gamma(1-\alpha)^2\int_{-\infty}^{\infty} \widehat{f}(k)\overline{\widehat{g}(k)}\frac{k^2}{(\lambda^2 + k^2)^{1-\alpha}}dk. \tag{3.43}$$

Theorem 3.15. *For any $\alpha \in (-1/2, 0) \cup (0, 1/2)$ and $\lambda > 0$, the class of functions*

$$\mathcal{A}_3 := \left\{ f \in L^2(\mathbb{R}) : \int \left|\widehat{f}(k)\right|^2 \frac{k^2}{(\lambda^2 + k^2)^{1-\alpha}}\, dk < \infty \right\} \tag{3.44}$$

is a linear space with the inner product

$$\langle f, g\rangle_{\mathcal{A}_3} = \Gamma(1-\alpha)^2\int_{-\infty}^{+\infty} \widehat{f}(k)\overline{\widehat{g}(k)}\frac{k^2}{(\lambda^2 + k^2)^{1-\alpha}}dk. \tag{3.45}$$

The set of elementary functions $\mathcal{E}$ is dense in the space $\mathcal{A}_3$. The space $\mathcal{A}_3$ is not complete.

Proof. The proof combines Theorems 3.5 and 3.10 using the Plancherel Theorem. First suppose that $0 < \alpha < 1/2$ and recall that $\varphi_f = \mathbb{D}_{-}^{\alpha,\lambda} f - \lambda \mathbb{I}_{-}^{1-\alpha,\lambda} f$ is a function with Fourier transform

$$\begin{aligned}\hat{\varphi}_f &= [(\lambda - ik)^{\alpha} - \lambda(\lambda - ik)^{\alpha-1}]\hat{f} = [\lambda - ik - \lambda](\lambda - ik)^{\alpha-1}\hat{f}\\ &= (-ik)(\lambda - ik)^{\alpha-1}\hat{f}.\end{aligned}$$

Then it follows from the Plancherel Theorem that

$$\begin{aligned}\langle f, g\rangle_{\mathcal{A}_2} = \Gamma(1-\alpha)^2\langle \varphi_f, \varphi_g\rangle_2 &= \Gamma(1-\alpha)^2\langle \hat{\varphi}_f, \hat{\varphi}_g\rangle_2\\ &= \Gamma(1-\alpha)^2\int_{-\infty}^{+\infty} \widehat{f}(k)\overline{\widehat{g}(k)}\frac{k^2}{(\lambda^2 + k^2)^{1-\alpha}}dk = \langle f, g\rangle_{\mathcal{A}_3}\end{aligned}$$

and hence the two inner products are identical. If $f \in \mathcal{A}_3$, then

$$\begin{aligned}\int_{-\infty}^{+\infty} |\hat{f}(k)|^2(\lambda^2 + k^2)^{\alpha}\, dk &= \int_{-\infty}^{+\infty} |\hat{f}(k)|^2\frac{k^2}{(\lambda^2 + k^2)^{1-\alpha}}\, dk\\ &\quad + \lambda^2\int_{-\infty}^{+\infty} |\hat{f}(k)|^2\frac{1}{(\lambda^2 + k^2)^{1-\alpha}}\, dk.\end{aligned} \tag{3.46}$$

The first integral on the right-hand side is finite by (3.44), and the second is finite since $1/(\lambda^2+k^2)^{1-\alpha}$ is bounded. Then it follows from the definition (2.17) that $f \in W^{\alpha,2}(\mathbb{R})$. Conversely, if $f \in W^{\alpha,2}(\mathbb{R})$ then since

$$\frac{k^2}{(\lambda^2+k^2)^{1-\alpha}} = \frac{k^2}{\lambda^2+k^2}(\lambda^2+k^2)^{\alpha} \le (\lambda^2+k^2)^{\alpha}$$

it follows immediately that $f \in \mathcal{A}_3$, and hence $W^{\alpha,2}(\mathbb{R})$ and $\mathcal{A}_3$ are the same set of functions. Then it follows from Lemma 3.11 that $\mathcal{A}_2$ and $\mathcal{A}_3$ are identical when $0<\alpha<1/2$, and the conclusions of Theorem 3.15 follow from Theorem 3.10 in this case.

If $-1/2<\alpha<0$, then the function $k^2/(\lambda^2+k^2)^{1-\alpha}$ is bounded by a constant $C(\alpha,\lambda)$ that depends only on α and λ, so for any $f \in L^2(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \left|\widehat{f}(k)\right|^2 \frac{k^2}{(\lambda^2+k^2)^{1-\alpha}}\,dk \le C(\alpha,\lambda)\int_{\mathbb{R}} \left|\widehat{f}(k)\right|^2\,dk < \infty \tag{3.47}$$

and hence $f \in \mathcal{A}_3$. Since $\mathcal{A}_3 \subset L^2(\mathbb{R})$ by definition, this proves that $L^2(\mathbb{R})$ and $\mathcal{A}_3$ are the same set of functions, and then it follows from Lemma 3.6 that $\mathcal{A}_1$ and $\mathcal{A}_3$ are the same set of functions in this case. Let $\kappa=-\alpha$ and note that $\varphi_f = \mathbb{I}_-^{\kappa,\lambda} f - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f$ is again a function with Fourier transform

$$\hat{\varphi}_f = [(\lambda-ik)^{\alpha} - \lambda(\lambda-ik)^{\alpha-1}]\hat{f} = (-ik)(\lambda-ik)^{\alpha-1}\hat{f}.$$

Then it follows from the Plancherel Theorem that

$$\begin{aligned}\langle f,g\rangle_{\mathcal{A}_1} = \Gamma(\kappa+1)^2\langle \varphi_f,\varphi_g\rangle_2 &= \Gamma(1-\alpha)^2\langle \hat{\varphi}_f,\hat{\varphi}_g\rangle_2 \\ &= \Gamma(1-\alpha)^2\int_{-\infty}^{+\infty} \widehat{f}(k)\overline{\widehat{g}(k)}\frac{k^2}{(\lambda^2+k^2)^{1-\alpha}}dk = \langle f,g\rangle_{\mathcal{A}_3}\end{aligned}$$

and hence the two inner products are identical. Then the conclusions of Theorem 3.15 follow from Theorem 3.5 in this case as well. □

Definition 3.16. For any $\alpha \in (-1/2,0)\cup(0,1/2)$ and $\lambda>0$, we define

$$\mathcal{I}^{\alpha,\lambda}(f) = \Gamma(1-\alpha)\int_{-\infty}^{\infty} \widehat{f}(k)\frac{(-ik)}{(\lambda-ik)^{1-\alpha}}\hat{B}(dk) \tag{3.48}$$

for any $f \in \mathcal{A}_3$.

Theorem 3.17. *For any $\alpha \in (-1/2,0)\cup(0,1/2)$ and $\lambda>0$, the stochastic integral $\mathcal{I}^{\alpha,\lambda}$ in (3.48) is an isometry from $\mathcal{A}_3$ into $\overline{\mathrm{Sp}}(B_{\alpha,\lambda})$. Since $\mathcal{A}_3$ is not complete, these two spaces are not isometric.*

Proof. The proof of Theorem 3.15 shows that $\mathcal{A}_1$ and $\mathcal{A}_3$ are identical when $-1/2<\alpha<0$, and $\mathcal{A}_2$ and $\mathcal{A}_3$ are identical when $0<\alpha<1/2$. Then the result follows immediately from Theorems 3.9 and 3.14. □

4. Discussion

In this section, we collect some remarks and extensions.

4.1. General TFBM

For any $p, q \geq 0$ with $p + q > 0$, we can extend Definition 3.1 and write

$$B^{p,q}_{\alpha,\lambda}(t) = p \int_{-\infty}^{+\infty} \left[e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha} - e^{-\lambda(-x)_+}(-x)_+^{-\alpha} \right] B(dx)$$
$$+ q \int_{-\infty}^{+\infty} \left[e^{-\lambda(x-t)_+}(x-t)_+^{-\alpha} - e^{-\lambda(x)_+}(x)_+^{-\alpha} \right] B(dx). \tag{4.1}$$

When $q = 0$, the process is causal, and hence appropriate for typical applications in time series analysis. The case $q > 0$ is useful in spatial statistics. For FBM (the case $\lambda = 0$), the right-hand side of (3.1) with $q > 0$ is the same process (with the same finite dimensional distributions) as another FBM with $q = 0$ [38, p. 322 and Exercise 7.2]. However, this is not true for TFBM. In fact, the stochastic process $B^{p,q}_{\alpha,\lambda}$ given by (4.1) has covariance function

$$E\left[B^{p,q}_{\alpha,\lambda}(t) B^{p,q}_{\alpha,\lambda}(s) \right] = \frac{1}{2} \left[C_t^2 |t|^{1-2\alpha} + C_s^2 |s|^{1-2\alpha} - C_{t-s}^2 |t-s|^{1-2\alpha} \right] \tag{4.2}$$

where

$$C_t^2 = (p^2 + q^2) \left[\frac{2\Gamma(1-2\alpha)}{(2\lambda t)^{1-2\alpha}} - \frac{2\Gamma(1-\alpha)}{\sqrt{\pi}\, 2\lambda t^{(\frac{1}{2}-\alpha)}} K_{\frac{1}{2}-\alpha}(\lambda t) \right] - 2pq e^{-\lambda t} \frac{\Gamma(1-\alpha)^2}{\Gamma(2-2\alpha)},$$

and $K_\nu(x)$ is modified Bessel function of the second kind. In this paper, to ease notation, we have only considered the causal TFBM (3.1). However, all of the results developed here extend easily to the more general case (4.1).

4.2. White noise approach

Heuristically, the TFBM (3.11) with $1/2 < H < 1$ in (3.2) can be written in terms of tempered fractional integrals of the white noise $W(x)dx = B(dx)$, since in view of (2.8) we can write

$$B_{\alpha,\lambda}(t) = \Gamma(\kappa+1) \int_{-\infty}^{+\infty} \left[\mathbb{I}_+^{\kappa,\lambda} W(x) - \lambda \mathbb{I}_+^{\kappa+1,\lambda} W(x) \right] \mathbf{1}_{[0,t]}(x)\, dx.$$

In the same way, when $0 < H < 1/2$ we can write

$$B_{\alpha,\lambda}(t) = \Gamma(1-\alpha) \int_{-\infty}^{+\infty} \left[\mathbb{D}_+^{\alpha,\lambda} W(x) - \lambda \mathbb{I}_+^{1-\alpha,\lambda} W(x) \right] \mathbf{1}_{[0,t]}(x)\, dx,$$

using Lemma 2.15. These ideas could be made rigorous using white noise theory [23]. Setting $\lambda = 0$, we recover the fact that FBM is the fractional integral or derivative of a Brownian motion [34, p. 261]. The white noise approach is preferred in engineering applications (e.g., see [4]).

4.3. Reproducing kernel Hilbert space

The reproducing kernel Hilbert space (RKHS) of TFBM provides another approach to stochastic integration that produces an isometric space of deterministic integrands. The RKHS for FBM was computed in [4,34]. For any mean zero Gaussian process $\{X_t\}_{t\in\mathbb{R}}$ with covariance function $R(s,t) = \mathbb{E}[X_s X_t]$, the RKHS of X is the unique Hilbert space $\mathbb{H}(X)$ of measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $R(\cdot, t) \in \mathbb{H}(X)$ for all $t \in \mathbb{R}$, and $\langle f, R(\cdot, t)\rangle_{\mathbb{H}(X)} = f(t)$ for all $t \in \mathbb{R}$ and $f \in \mathbb{H}(X)$ [15,41]. As noted in [15], if there exists a measure space $(\Lambda, \mathfrak{B}, \nu)$ and a set of functions $\{f_t\} \subset L^2(\mathbb{R}, \nu)$ such that

$$R(s,t) = \int_\Lambda f_s(x) f_t(x) \nu(dx) \quad \text{for all } s, t \in \mathbb{R}. \tag{4.3}$$

Then $\mathbb{H}(X)$ consists of the functions $g(t) = \int f_t(x) g^*(x) \nu(dx)$ for $g^* \in \overline{\mathrm{Sp}}\{f_t\}$, the closure in $L^2(\mathbb{R}, \nu)$ of the set of linear combinations of functions f_t. Then $\mathbb{H}(X)$ is a Hilbert space with the inner product

$$\langle g, h\rangle_{\mathbb{H}(X)} = \int_\Lambda g^*(x) h^*(x) \nu(dx).$$

Let $\overline{\mathrm{Sp}}(X)$ denote the closure of the set of linear combinations of random variables $\{X_t\}$ in the space $L^2(\Omega)$. The mapping $\mathcal{J}$ that sends

$$\sum_{j=1}^J a_j R(\cdot, t_j) \mapsto \sum_{j=1}^J a_j X_{t_j}$$

is an isometry that maps $\mathbb{H}(X)$ onto $\overline{\mathrm{Sp}}(X)$, and hence these two Hilbert spaces are isometric. Then $\mathcal{J}(f)$ is the stochastic integral of any $f \in \mathbb{H}(X)$.

For TFBM with $-1/2 < \alpha < 0$, let $\kappa = -\alpha$. Since $B_{\alpha,\lambda}(t) = \int_\mathbb{R} \mathbf{1}_{[0,t]}(x) B_{\alpha,\lambda}(dx)$, it follows immediately from the definition (3.30) that TFBM has covariance function

$$R(s,t) = \Gamma(\kappa+1)^2 \int_\mathbb{R} \left[\mathbb{I}_-^{\kappa,\lambda}\mathbf{1}_{[0,s]}(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda}\mathbf{1}_{[0,s]}\right]\left[\mathbb{I}_-^{\kappa,\lambda}\mathbf{1}_{[0,t]}(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda}\mathbf{1}_{[0,t]}\right] dx,$$

and hence the RKHS $\mathbb{H}(B_{\alpha,\lambda})$ consists of functions

$$g(t) = \Gamma(k+1) \int_\mathbb{R} \left[\mathbb{I}_-^{k,\lambda} - \lambda \mathbb{I}_-^{k+1,\lambda}\right] 1_{[0,t]}(x) g^*(x) dx$$

for $g^* \in L_2(\mathbb{R})$, with the inner product

$$\langle g, h\rangle_{\mathbb{H}(X)} = \int_\mathbb{R} g^*(x) h^*(x) dx = \langle g^*, h^*\rangle_{L^2(\mathbb{R})}. \tag{4.4}$$

For TFBM with $0 < \alpha < 1/2$ and $\lambda > 0$, the RKHS $\mathbb{H}(B_{\alpha,\lambda})$ consists of functions

$$g(t) = \Gamma(1-\alpha)^2 \int_\mathbb{R} \left[\mathbb{D}_-^{\alpha,\lambda} - \lambda \mathbb{I}_-^{1-\alpha,\lambda}\right] 1_{[0,t]}(x) g^*(x) dx$$

for $g^* \in L_2(\mathbb{R})$, with the same inner product (4.4). The proof is similar to [34, Section 6]. Complete details will be provided in the forthcoming paper [27]. Here we take $\Lambda = L^2(\mathbb{R})$, with ν the Lebesgue measure on $\mathbb{R}$. The main technical difficulty is to show that $L_2(\mathbb{R}) = \overline{\mathrm{Sp}}\{f_t\}$, where $f_t(x) = \Gamma(k+1)[\mathbb{I}_-^{k,\lambda} - \lambda \mathbb{I}_-^{k+1,\lambda}]1_{[0,t]}(x)$ in the case $-1/2 < \alpha < 0$, and $f_t(x) = \Gamma(1-\alpha)[\mathbb{D}_-^{\alpha,\lambda} - \lambda \mathbb{I}_-^{1-\alpha,\lambda}]1_{[0,t]}(x)$ for $0 < \alpha < 1/2$.

4.4. Tempered distributions as integrands

Jolis [18] proved that the exact domain of the Wiener integral for a fractional Brownian motion $B_H(t)$ is given by

$$\Lambda^H = \left\{ f \in \mathcal{S}'(\mathbb{R}) = \int_{\mathbb{R}} |\widehat{f}(k)|^2 \, |k|^{1-2H} \, dk < \infty \right\}$$

where $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions. This gives an isometry using the inner product (for a standard FBM)

$$\langle f, g \rangle = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \int \hat{f}(k)\overline{\hat{g}(k)}|k|^{1-2H} dk,$$

that makes Λ^H isometric to $\overline{\mathrm{Sp}}(B_H)$. She also proved that this space contains distributions that cannot be represented by locally integrable functions in the case of long range dependence ($1/2 < H < 1$). Tudor [39] extended this result to subfractional Brownian motion. The distributional approach is useful in the study of partial differential equations with a Gaussian forcing term [5,9,40].

Following along these lines, we conjecture that the exact domain of the Wiener integral with respect to TFBM is given by the distributional fractional Sobolev space

$$\Lambda^{\alpha,\lambda} = \left\{ f \in \mathcal{S}'(\mathbb{R}) : \int_{\mathbb{R}} |\widehat{f}(k)|^2 \, (\lambda^2 + k^2)^\alpha \, dk < \infty \right\}$$

with the inner product

$$\langle f, g \rangle = C_{\alpha,\lambda} \int \hat{f}(k)\overline{\hat{g}(k)}(\lambda^2 + k^2)^\alpha dk.$$

Proving this using [18, Theorem 3.5] would require computing the second derivative of the variance function (4.2) and taking the (inverse) Fourier transform of the result. This computation seems difficult, due to the Bessel function term.

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References

[1] R. Adler, R. Feldman, M. Taqqu, A Practical Guide to Heavy Tails: Statistical Techniques and Applications, Springer, 1998.
[2] B. Baeumer, M.M. Meerschaert, Tempered stable Lévy motion and transient super-diffusion, J. Comput. Appl. Math. 233 (2010) 2438–2448.
[3] S. Banerjee, A.E. Gelfand, On smoothness properties of spatial processes, J. Multivariate Anal. 84 (2003) 85–100.
[4] R.J. Barton, V.H. Poor, Signal detection in fractional Gaussian noise, IEEE Trans. Inform. Theory 34 (5) (1998) 943–955.
[5] F. Biagini, Y. Hu, B. Øksendal, T. Zhang, Stochastic Calculus for Fractional Brownian Motion and Applications, Springer, 2010.
[6] Á. Cartea, D. del-Castillo-Negrete, Fluid limit of the continuous-time random walk with general Lévy jump distribution functions, Phys. Rev. E 76 (2007) 041105.

[7] A. Chakrabarty, M.M. Meerschaert, Tempered stable laws as random walk limits, Statist. Probab. Lett. 81 (8) (2011) 989–997.
[8] P. Cheridito, Gaussian moving averages, semimartingales and option pricing, Stochastic Process. Appl. 109 (1) (2004) 47–68.
[9] R.C. Dalang, D. Khoshnevisan, F. Rassoul-Agha, A Minicourse on Stochastic Partial Differential Equations, in: Lecture Notes in Mathematics, vol. 1962, Springer, 2009.
[10] A.G. Davenport, The spectrum of horizontal gustiness near the ground in high winds, Q. J. Royal Meteorol. Soc. 87 (1961) 194–211.
[11] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012) 521–573.
[12] P. Doukhan, G. Oppenheim, M.S. Taqqu, Theory and Applications of Long-Range Dependence, Springer, 2003.
[13] P. Embrechts, M. Maejima, Selfsimilar Processes, in: Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2002.
[14] T. Gneiting, W. Kleiber, M. Schlather, Matérn cross-covariance functions for multivariate random fields, J. Amer. Statist. Assoc. 105 (2010) 1167–1177.
[15] G. Grenander, Abstract Inference, Wiley, 1981.
[16] M.S. Handcock, M.L. Stein, A Bayesian analysis of kriging, Technometrics 35 (1993) 403–410.
[17] S.T. Huang, S. Cambanis, Stochastic and multiple Wiener integrals for Gaussian processes, Ann. Probab. 6 (1978) 585–614.
[18] M. Jolis, The Wiener integral with respect to second order processes with stationary increments, J. Math. Anal. Appl. 336 (2010) 607–620.
[19] O. Kallenberg, Foundations of Modern Probability, second ed., Springer, New York, 2002.
[20] W. Kierat, U. Sztaba, Distributions, Integral Transforms and Applications, Taylor and Francis, CRC Press, 2003.
[21] F. Knight, Foundations of the Prediction Process, Clarendon Press, Oxford, 1992.
[22] A.N. Kolmogorov, Wiener spiral and some other interesting curves in Hilbert space, Dokl. Akad. Nauk SSSR 26 (1940) 115–118.
[23] H.H. Kuo, White Noise Distribution Theory, CRC Press, Boca Raton, Florida, 1996.
[24] Y. Li, A. Kareem, ARMA systems in wind engineering, Probab. Eng. Mech. 5 (1990) 49–59.
[25] S.C. Lim, L.P. Teo, Weyl and Riemann–Liouville multifractional Ornstein–Uhlenbeck processes, J. Phys. A 40 (2007) 6035–6060.
[26] B. Mandelbrot, J. Van Ness, Fractional Brownian motion, fractional noises and applications, SIAM Rev. 10 (1968) 422–437.
[27] V. Mandrekar, M.M. Meerschaert, F. Sabzikar, RKHS representation for tempered fractional Brownian motion, 2013, Preprint.
[28] M.M. Meerschaert, F. Sabzikar, Tempered fractional Brownian motion, Statist. Probab. Lett. 83 (10) (2013) 2269–2275.
[29] M.M. Meerschaert, A. Sikorskii, Stochastic Models for Fractional Calculus, De Gruyter, Berlin/Boston, 2012.
[30] M.M. Meerschaert, Y. Zhang, B. Baeumer, Tempered anomalous diffusion in heterogeneous systems, Geophys. Res. Lett. 35 (2008) L17403.
[31] D.J. Norton, Mobile offshore platform wind loads, in: Proc. 13th Offshore Techn. Conf., OTC 4123, 4, 1981, pp. 77–88.
[32] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, 1974.
[33] J.P. Pérez Beaupuits, A. Otárola, F.T. Rantakyrö, R.C. Rivera, S.J.E. Radford, L.-Å. Nyman, Analysis of wind data gathered at Chajnantor, ALMA Memo 497 (2004).
[34] V. Pipiras, M. Taqqu, Integration questions related to fractional Brownian motion, Probab. Theory Related Fields 118 (2000) 251–291.
[35] G. Rangarajan, M. Ding, Processes with Long-Range Correlations: Theory and Applications, Springer, 2003.
[36] J. Rosiński, Tempering stable processes, Stochastic Process. Appl. 117 (2007) 677–707.
[37] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Gordon and Breach, 1993.
[38] G. Samorodnitsky, M. Taqqu, Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance, Chapman & Hall, 1994.
[39] C. Tudor, Inner product spaces of integrands associated to subfractional Brownian motion, Statist. Probab. Lett. 78 (2008) 2201–2209.
[40] C. Tudor, Analysis of Variations for Self-similar Processes: A Stochastic Calculus Approach, Springer, 2013.
[41] H., L. Weinert, Reproducing kernel Hilbert spaces: applications in statistical signal processing, Hutchinson Ross (1982).
[42] R.L. Wheeden, A. Zygmund, Measure and Integral, Marcel Dekker, New York, 1977.
[43] K. Yosida, Functional Analysis, sixth ed., Springer, 1980.

[44] Y. Zhang, D.A. Benson, M.M. Meerschaert, H.-P. Scheffler, On using random walks to solve the space-fractional advection–dispersion equations, J. Stat. Phys. 123 (2006) 89–110.

[45] Y. Zhang, M.M. Meerschaert, Gaussian setting time for solute transport in fluvial systems, Water Resour. Res. 47 (2011) W08601.

[46] Y. Zhang, M.M. Meerschaert, A.I. Packman, Linking fluvial bed sediment transport across scales, Geophys. Res. Lett. 39 (2012) L20404.