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A novel numerical method for the time variable fractional order mobile–immobile advection–dispersion model



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ABSTRACT

Evolution equations containing fractional derivatives can provide suitable mathematical models for describing anomalous diffusion and transport dynamics in complex systems that cannot be modeled accurately by normal integer order equations. Recently, researchers have found that many physical processes exhibit fractional order behavior that varies with time or space. The continuum of order in the fractional calculus allows the order of the fractional operator to be considered as a variable. In this paper, we consider the mobile–immobile advection–dispersion model with the Coimbra variable time fractional derivative which is preferable for modeling dynamical systems and is more efficient from the numerical standpoint. A novel implicit numerical method for the equation is proposed and the stability of the approximation is investigated. As for the convergence of the numerical method, we only consider a special case, i.e., the time fractional derivative is independent of the time variable t . The case where the time fractional derivative depends on both the time variable t and the space variable x will be considered in a future work. Finally, numerical examples are provided to show that the implicit difference approximation is computationally efficient.

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1. Introduction

Solute transport in rivers, streams and groundwater is controlled by the physical features or heterogeneity in different reaches. While the advection–dispersion equation and its extensions (e.g., the mobile–immobile or transient storage models based on a second order dispersion term) have been successfully used in the past, recent research highlights the need for transport models that can better describe the heterogeneity and connectivity of spatial features within a general network perspective of solute transport [1–5]. For example, breakthrough curves in karst aquifers and rivers can exhibit multiple peaks due to flow occlusion and diversion caused by in-channel features such as islands and pools [6,7]. Although earlier approaches to describe these complexities in breakthrough curves involved using different but constant model parameters in different reaches, models that explicitly acknowledge spatial heterogeneity may better describe solute transport processes. Fractional derivative models offer several approaches that are attractive to the solute transport modeling community. The aim of this paper is to describe one approach based on variable fractional order derivatives. Significant progress has already been made in the approximation of the time fractional order advection–dispersion equation, see for example [8–10]. Furthermore, numerical simulation of the fractional order mobile–immobile advection–dispersion model was considered by Liu et al. [11]. The time fractional order derivative in these papers is constant fractional order, not variable fractional

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order. As a step in this direction and to distinguish explicitly the mobile and immobile status using fractional dynamics, in this paper we consider a fractional-order, mobile–immobile advection–dispersion model with a Coimbra time variable fractional derivative for the total concentration.

Anomalous diffusion is ubiquitous in nature. Evolution equations containing fractional derivatives have been shown in the past to provide superior descriptions of anomalous diffusion and transport dynamics in complex systems. Recently, researchers have also found that many dynamic processes exhibit fractional order behavior that may vary with time or space, which indicates that variable-order calculus is a natural candidate to provide an effective mathematical framework for the description of complex dynamical problems. For example, variable-order has been applied to viscoelasticity by Coimbra [12], the processing of geographical data by Cooper and Cowan [13], signature verification by Tseng [14], and to diffusion by Sun et al. [15]. Samko and Ross [16] first discussed some properties and the inversion formula of the variable-order operator $(\frac{d}{dx})^{\alpha(x)}f(x)$ using the Riemann–Liouville definition and Fourier transforms. Some mapping properties in Hölder spaces and fractional operators of the Riemann–Liouville and Marchaud forms were generalized to the case of variable-order by Ross [17] and Samko [18]. Several classes of Markov processes with variable-order fractional transition probability densities on unbounded domains have been studied [19–22]. In particular, Kikuchi and Negoro [21] found the conditions for which general pseudodifferential operators on fractional Sobolev spaces of variable-order on \mathbb{R}^n form a Feller semigroup which has a transition density. Lorenzo and Hartley [23,24] suggested that the order of a Riemann–Liouville fractional derivative (integral) be allowed to vary as a function of time or space. Different definitions of variable fractional operators in various settings were discussed [12,25,26,15]. In a recent article, Ramirez and Coimbra [27] compared nine definitions of variable fractional order derivative (integral) operators based on a few criteria: (a) the variable-order operator must be able to return all intermediate values between 0 and 1 that correspond to the argument of the order of differentiation, (b) the variable-order operator must be effectively evaluated numerically, and (c) all derivatives of a true constant must be zero. They concluded that only the Coimbra variable fractional order derivative satisfied these criteria for modeling dynamic systems.

Research on the solution of variable fractional order partial differential equations is relatively new, and numerical approximation of these equations is still at an early stage of development. Lin et al. [28] studied the stability and convergence of an explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation. Zhuang et al. [29] discussed the stability and convergence of an Euler approximation for the variable-order space fractional advection–diffusion equation with a nonlinear source term. A variable-order anomalous subdiffusion equation was considered by Chen et al. [30].

In this paper we consider the following time variable fractional order mobile–immobile advection–dispersion model:

$$\beta_1 \frac{\partial C(x, t)}{\partial t} + \beta_2 D_t^{\gamma(x, t)} C(x, t) = -V \frac{\partial C(x, t)}{\partial x} + D \frac{\partial^2 C(x, t)}{\partial x^2} + f(x, t), \quad (x, t) \in \Omega = [0, L] \times [0, T], \quad (1)$$

with the following initial and boundary conditions:

$$\begin{aligned} C(x, 0) &= \phi(x), \quad 0 \leq x \leq L \\ C(0, t) &= \varphi_1(t), \quad C(L, t) = \varphi_2(t), \quad 0 \leq t \leq T \end{aligned} \quad (2)$$

where $\beta_1 \geq 0$, $\beta_2 \geq 0$, $V > 0$, $D > 0$, $0 < \underline{\gamma} \leq \gamma(x, t) \leq \bar{\gamma} \leq 1$.

The Coimbra variable-order derivative operators [12] in Eq. (1) are defined as follows:

$$D_t^{\gamma(x, t)} C(x, t) = \frac{1}{\Gamma(1 - \gamma(x, t))} \int_{0^+}^t (t - \sigma)^{-\gamma(x, t)} \frac{\partial C(x, \sigma)}{\partial \sigma} d\sigma + \frac{(C(x, t_{0+}) - C(x, t_{0-}))t^{-\gamma(x, t)}}{\Gamma(1 - \gamma(x, t))}, \quad (3)$$

where $0 < \gamma(x, t) < 1$. In the study of numerical approximation of the variable fractional order partial differential equation, it is worth mentioning that the Coimbra variable fractional derivative definition (3) is seldom used as the variable fractional order derivative.

The structure of the remainder of this paper is as follows: An implicit Euler numerical method for the time variable fractional order mobile–immobile advection–dispersion model is proposed in Section 2. The stability and convergence of the implicit Euler numerical method are discussed in Sections 3 and 4, respectively. Finally, numerical examples are given.

2. Implicit Euler approximation

Let $t_k = k\tau$, $k = 0, 1, 2, \dots, N$; $x_i = ih$, $i = 0, 1, 2, \dots, M$, where $\tau = T/N$ and $h = L/M$ are time and space steps respectively.

Suppose $C(\cdot, \cdot) \in C^{(2)}(\Omega)$, the operator $D^{\gamma(x, t)}C(x, t)$ can be discretized as

$$\begin{aligned} D_t^{\gamma(x_i, t_{k+1})} C(x_i, t_{k+1}) &= \frac{1}{\Gamma(1 - \gamma(x_i, t_{k+1}))} \int_0^{(k+1)\tau} \frac{\frac{\partial C(x_i, \eta)}{\partial \eta}}{(t_{k+1} - \eta)^{\gamma(x_i, t_{k+1})}} d\eta \\ &= \frac{1}{\Gamma(1 - \gamma(x_i, t_{k+1}))} \sum_{j=0}^k \frac{C(x_i, t_{j+1}) - C(x_i, t_j)}{\tau} \int_{j\tau}^{(j+1)\tau} \frac{1}{(t_{k+1} - \eta)^{\gamma(x_i, t_{k+1})}} d\eta + O(\tau) \\ &= \frac{1}{\Gamma(2 - \gamma_i^{k+1})} \sum_{j=0}^k \frac{C(x_i, t_{k+1-j}) - C(x_i, t_{k-j})}{\tau^{\gamma_i^{k+1}}} [(j+1)^{1-\gamma_i^{k+1}} - j^{1-\gamma_i^{k+1}}] + O(\tau) \end{aligned} \quad (4)$$

at the mesh point (x_i, t_{k+1}) . Denote

$$D_t^{\gamma_i^{k+1}} C(x_i, t_{k+1}) = \frac{\tau^{-\gamma_i^{k+1}}}{\Gamma(2 - \gamma_i^{k+1})} \sum_{j=0}^k (C(x_i, t_{k+1-j}) - C(x_i, t_{k-j})) b_{j,i}^{k+1}, \tag{5}$$

where $\gamma_i^k = \gamma(x_i, t_k)$ and $b_{j,i}^{k+1} = (j + 1)^{1-\gamma_i^{k+1}} - j^{1-\gamma_i^{k+1}}$.

Furthermore,

$$\begin{aligned} \frac{\partial C(x_i, t_{k+1})}{\partial t} &= \frac{C(x_i, t_{k+1}) - C(x_i, t_k)}{\tau} + O(\tau) \\ \frac{\partial C(x_i, t_{k+1})}{\partial x} &= \frac{C(x_i, t_{k+1}) - C(x_{i-1}, t_{k+1})}{h} + O(h) \\ \frac{\partial^2 C(x_i, t_{k+1})}{\partial x^2} &= \frac{C(x_{i+1}, t_{k+1}) - 2C(x_i, t_{k+1}) + C(x_{i-1}, t_{k+1})}{h^2} + O(h^2). \end{aligned}$$

Let C_i^k be the numerical approximation to $C(x_i, t_k)$, then we can obtain the following implicit difference approximation of Eq. (1):

$$\beta_1 \frac{C_i^{k+1} - C_i^k}{\tau} + \beta_2 \frac{\tau^{-\gamma_i^{k+1}}}{\Gamma(2 - \gamma_i^{k+1})} \sum_{j=0}^k (C_i^{k+1-j} - C_i^{k-j}) b_{j,i}^{k+1} = -V \frac{C_i^{k+1} - C_{i-1}^{k+1}}{h} + D \frac{C_{i+1}^{k+1} - 2C_i^{k+1} + C_{i-1}^{k+1}}{h^2} + f_i^{k+1},$$

where $f_i^k = f(x_i, t_k)$ ($k = 0, 1, \dots, N$; $i = 0, 1, \dots, M$). Denoting $r_i^{k+1} = \frac{\tau^{1-\gamma_i^{k+1}}}{\Gamma(2-\gamma_i^{k+1})}$, $\mu = \frac{V\tau}{h}$ and $\nu = \frac{D\tau}{h^2}$, the above formula can be rewritten as

$$\begin{aligned} &(\beta_1 + \beta_2 r_i^{k+1} + \mu + 2\nu) C_i^{k+1} - (\mu + \nu) C_{i-1}^{k+1} - \nu C_{i+1}^{k+1} \\ &= \beta_1 C_i^k - \beta_2 r_i^{k+1} \sum_{j=1}^k (b_{k+1-j,i}^{k+1} - b_{k-j,i}^{k+1}) C_i^j + \beta_2 r_i^{k+1} b_{k,i}^{k+1} C_i^0 + \tau f_i^{k+1} \end{aligned} \tag{6}$$

i.e.

$$\begin{aligned} &(\beta_1 + \beta_2 r_i^1 + \mu + 2\nu) C_i^1 - (\mu + \nu) C_{i-1}^1 - \nu C_{i+1}^1 = \beta_1 C_i^0 + \beta_2 r_i^1 C_i^0 + \tau f_i^1 \\ &(\beta_1 + \beta_2 r_i^{k+1} + \mu + 2\nu) C_i^{k+1} - (\mu + \nu) C_{i-1}^{k+1} - \nu C_{i+1}^{k+1} \\ &= \beta_1 C_i^k + \beta_2 r_i^{k+1} \sum_{j=0}^{k-1} (b_{j+1,i}^{k+1} - b_{j,i}^{k+1}) C_i^{k-j} + \beta_2 r_i^{k+1} b_{k,i}^{k+1} C_i^0 + \tau f_i^{k+1} \quad (k \geq 1), \end{aligned} \tag{7}$$

which are the implicit Euler schemes to the time variable fractional order mobile-immobile advection-dispersion model (1) and $i = 0, 1, \dots, M$; $k = 1, 2, \dots, N - 1$.

The boundary and initial conditions are discretized as

$$C_i^0 = \varphi(ih) = \varphi_i, \quad C_0^k = \varphi_1(k\tau), \quad C_M^k = \varphi_2(k\tau), \tag{8}$$

where $i = 0, 1, 2, \dots, M$; $k = 0, 1, 2, \dots, N$.

3. Stability of implicit Euler approximation

In this section, we will use the technique of mathematical induction to examine the stability of the approximate scheme (7)–(8).

Lemma 1. For $i = 1, 2, \dots, M$, $k = 1, 2, \dots, N$ and $0 < \underline{\gamma} \leq \gamma(x, t) \leq \bar{\gamma} < 1$, the coefficients $b_{j,i}^k$ satisfy

$$1 = b_{0,i}^k > b_{1,i}^k > b_{2,i}^k > \dots > 0.$$

Proof. Let $\varphi(x) = (x + 1)^{1-\gamma(x_i, t_k)} - x^{1-\gamma(x_i, t_k)}$ ($0 < \underline{\gamma} \leq \gamma(x_i, t_k) \leq \bar{\gamma} < 1, x > 0$), then

$$\varphi'(x) = (1 - \gamma(x_i, t_k)) [(x + 1)^{-\gamma(x_i, t_k)} - x^{-\gamma(x_i, t_k)}] < 0.$$

The result is therefore valid. \square

Definition 1 ([31,32]). For any arbitrary initial rounding error E^0 , if there exists positive number K , independent of h and τ , such that

$$\|E^k\| \leq K \|E^0\| \quad \text{or} \quad \|E^k\| \leq K$$

then the difference approximation is stable.

We suppose that \tilde{C}_i^k is the approximate solution of the Eqs. (7) and denote $\varepsilon_i^k = \tilde{C}_i^k - C_i^k$, where $i = 0, 1, \dots, M$ and $k = 0, 1, \dots, N$.

Therefore, for $k = 0$, ε_i^k satisfies

$$(\beta_1 + \beta_2 r_i^1 + \mu + 2\nu)\varepsilon_i^1 - (\mu + \nu)\varepsilon_{i-1}^1 - \nu\varepsilon_{i+1}^1 = \beta_1 \varepsilon_i^0 + \beta_2 r_i^1 \varepsilon_i^0, \tag{9}$$

and for $k = 1, 2, \dots, N - 1$, ε_i^k satisfy

$$(\beta_1 + \beta_2 r_i^{k+1} + \mu + 2\nu)\varepsilon_i^{k+1} - (\mu + \nu)\varepsilon_{i-1}^{k+1} - \nu\varepsilon_{i+1}^{k+1} = \beta_1 \varepsilon_i^k + \beta_2 r_i^{k+1} \sum_{j=0}^{k-1} (b_{j,i}^{k+1} - b_{j+1,i}^{k+1})\varepsilon_i^{k-j} + \beta_2 r_i^{k+1} b_{k,i}^{k+1} \varepsilon_i^0, \tag{10}$$

where $i = 0, 1, \dots, M$.

Theorem 1. *The fractional implicit Euler scheme (7)–(8) is unconditionally stable.*

Proof. Let $\|E^k\|_\infty = |\varepsilon_i^k| = \max_{1 \leq i \leq M-1} |\varepsilon_i^k|$. Since $\beta_1, \beta_2, \mu, \nu > 0; r_i^k > 0 (i = 1, \dots, M - 1; k = 1, \dots, N)$, it can be obtained

$$\begin{aligned} (\beta_1 + \beta_2 r_i^1)\|E^1\|_\infty &= (\beta_1 + \beta_2 r_i^1)|\varepsilon_i^1| \\ &= (\beta_1 + \beta_2 r_i^1 + \mu + 2\nu)|\varepsilon_i^1| - (\mu + \nu)|\varepsilon_i^1| - \nu|\varepsilon_i^1| \\ &\leq (\beta_1 + \beta_2 r_i^1 + \mu + 2\nu)|\varepsilon_i^1| - (\mu + \nu)|\varepsilon_{i-1}^1| - \nu|\varepsilon_{i+1}^1| \\ &\leq |(\beta_1 + \beta_2 r_i^1 + \mu + 2\nu)\varepsilon_i^1 - (\mu + \nu)\varepsilon_{i-1}^1 - \nu\varepsilon_{i+1}^1| \\ &= |(\beta_1 + \beta_2 r_i^1)\varepsilon_i^0| \leq (\beta_1 + \beta_2 r_i^1)\|E^0\|_\infty, \end{aligned}$$

i.e.

$$\|E^1\|_\infty \leq \|E^0\|_\infty.$$

Assume that $\|E^j\|_\infty \leq \|E^0\|_\infty, j = 2, 3, \dots, k$, then using the Lemma 1 we have

$$\begin{aligned} (\beta_1 + \beta_2 r_i^{k+1})\|E^{k+1}\|_\infty &= (\beta_1 + \beta_2 r_i^{k+1})|\varepsilon_i^{k+1}| \\ &= (\beta_1 + \beta_2 r_i^{k+1} + \mu + 2\nu)|\varepsilon_i^{k+1}| - (\mu + \nu)|\varepsilon_i^{k+1}| - \nu|\varepsilon_i^{k+1}| \\ &\leq (\beta_1 + \beta_2 r_i^{k+1} + \mu + 2\nu)|\varepsilon_i^{k+1}| - (\mu + \nu)|\varepsilon_{i-1}^{k+1}| - \nu|\varepsilon_{i+1}^{k+1}| \\ &\leq |(\beta_1 + \beta_2 r_i^{k+1} + \mu + 2\nu)\varepsilon_i^{k+1} - (\mu + \nu)\varepsilon_{i-1}^{k+1} - \nu\varepsilon_{i+1}^{k+1}| \\ &= \left| \beta_1 \varepsilon_i^k + \beta_2 r_i^{k+1} \sum_{j=0}^{k-1} (b_{j,i}^{k+1} - b_{j+1,i}^{k+1})\varepsilon_i^{k-j} + \beta_2 r_i^{k+1} b_{k,i}^{k+1} \varepsilon_i^0 \right| \\ &\leq \beta_1 \|E^k\|_\infty + \beta_2 r_i^{k+1} \sum_{j=0}^{k-1} (b_{j,i}^{k+1} - b_{j+1,i}^{k+1})\|E^{k-j}\|_\infty + \beta_2 r_i^{k+1} b_{k,i}^{k+1} \|E^0\|_\infty \\ &\leq \left[\beta_1 + \beta_2 r_i^{k+1} \sum_{j=0}^{k-1} (b_{j,i}^{k+1} - b_{j+1,i}^{k+1}) + \beta_2 r_i^{k+1} b_{k,i}^{k+1} \right] \|E^0\|_\infty \\ &= (\beta_1 + \beta_2 r_i^{k+1})\|E^0\|_\infty, \end{aligned}$$

i.e.

$$\|E^{k+1}\|_\infty \leq \|E^0\|_\infty.$$

Hence the conclusion is valid according to Definition 1. \square

4. Convergence of implicit Euler approximation

In this section we will consider the convergence of the approximate scheme (7)–(8) for a special case, i.e., when the time fractional derivative is independent of time variable t .

Let $C(x_i, t_k)$ be the exact solution of the space–time variable fractional order advection–dispersion Eqs. (1)–(2) and C_i^k be the exact solution of the discrete Eqs. (7)–(8) at mesh points (x_i, t_k) respectively, where $i = 1, 2, \dots, M$ and $k = 1, 2, \dots, N$. Define $e_i^k = C(x_i, t_k) - C_i^k (i = 1, 2, \dots, M; k = 1, 2, \dots, N)$ and $e^k = (e_1^k, e_2^k, \dots, e_M^k)^T$. Obviously, $e^0 = \mathbf{0}$. Substituting

$C(x_i, t_k)$ into the Eq. (1) and C_i^k into (7), then subtracting (7) from (1), we get

$$\begin{aligned}
 &(\beta_1 + \beta_2 r_i + \mu + 2\nu)e_i^1 - (\mu + \nu)e_{i-1}^1 - \nu e_{i+1}^1 = R_i^1 \\
 &(\beta_1 + \beta_2 r_i + \mu + 2\nu)e_i^{k+1} - (\mu + \nu)e_{i-1}^{k+1} - \nu e_{i+1}^{k+1} = \beta_1 e_i^k + \beta_2 r_i \sum_{j=0}^{k-1} (b_{j,i} - b_{j+1,i})e_i^{k-j} + R_i^{k+1} \quad (k \geq 1)
 \end{aligned}
 \tag{11}$$

where $\gamma_i = \gamma(x_i)$, $r_i = \frac{\tau^{1-\gamma_i}}{\Gamma(2-\gamma_i)}$, $b_{j,i} = (j + 1)^{1-\gamma_i} - j^{1-\gamma_i}$, and $R_i^k = O(\tau^2 + \tau h) \leq C\beta_2 r_i(\tau^{1+\gamma_i} + \tau^\gamma h)$, $k = 1, 2, \dots$

Theorem 2. *The fractional implicit difference scheme (7)–(8) is convergent, and the solution C_i^k of the discrete scheme (7)–(8) and the solution $C(x_i, t_k)$ of the Eqs. (1)–(2) satisfy*

$$\|C(\cdot, t_k) - C^k\|_\infty \leq C(\tau + h), \quad i = 1, 2, \dots, M - 1; \quad k = 1, 2, \dots, N,
 \tag{12}$$

when the time fractional derivative is independent of the time variable t , where C is a constant independent of τ and h .

Proof. Let $\|e^k\|_\infty = |e_i^k| = \max_{1 \leq i \leq M-1} |e_i^k|$, we obtain

$$\begin{aligned}
 (\beta_1 + \beta_2 r_i)|e_i^1| &= (\beta_1 + \beta_2 r_i + \mu + 2\nu)|e_i^1| - (\mu + \nu)|e_i^1| - \nu|e_i^1| \\
 &\leq (\beta_1 + \beta_2 r_i + \mu + 2\nu)|e_i^1| - (\mu + \nu)|e_{i-1}^1| - \nu|e_{i+1}^1| \\
 &\leq |(\beta_1 + \beta_2 r_i + \mu + 2\nu)e_i^1 - (\mu + \nu)e_{i-1}^1 - \nu e_{i+1}^1| \\
 &= |R_i^1|,
 \end{aligned}$$

i.e.

$$|e_i^1| \leq \frac{C\beta_2 r_i}{\beta_1 + \beta_2 r_i}(\tau^{1+\gamma_i} + \tau^\gamma h) \leq C(\tau^{1+\gamma_i} + \tau^\gamma h).$$

Suppose that $|e_i^j| \leq C(b_{j-1,i})^{-1}(\tau^{1+\gamma_i} + \tau^\gamma h)$, $j \leq k$, by Lemma 1 we have

$$\begin{aligned}
 (\beta_1 + \beta_2 r_i)|e_i^{k+1}| &= (\beta_1 + \beta_2 r_i + \mu + 2\nu)|e_i^{k+1}| - (\mu + \nu)|e_i^{k+1}| - \nu|e_i^{k+1}| \\
 &\leq (\beta_1 + \beta_2 r_i + \mu + 2\nu)|e_i^{k+1}| - (\mu + \nu)|e_{i-1}^{k+1}| - \nu|e_{i+1}^{k+1}| \\
 &\leq |(\beta_1 + \beta_2 r_i + \mu + 2\nu)e_i^{k+1} - (\mu + \nu)e_{i-1}^{k+1} - \nu e_{i+1}^{k+1}| \\
 &= \left| \beta_1 e_i^k + \beta_2 r_i \sum_{j=0}^{k-1} (b_{j,i} - b_{j+1,i})e_i^{k-j} + R_i^{k+1} \right| \\
 &\leq C\beta_1(b_{k-1,i})^{-1}(\tau^{1+\gamma_i} + \tau^\gamma h) + \beta_2 r_i \sum_{j=0}^{k-1} (b_{j,i} - b_{j+1,i}) \\
 &\quad \cdot C(b_{k-j-1,i})^{-1}(\tau^{1+\gamma_i} + \tau^\gamma h) + C\beta_2 r_i(\tau^{1+\gamma_i} + \tau^{1+\gamma_i} h) \\
 &\leq C \left[\beta_1 + \beta_2 r_i \sum_{j=0}^{k-1} (b_{j,i} - b_{j+1,i}) + \beta_2 r_i b_{k,i} \right] \cdot (b_{k,i})^{-1}(\tau^{1+\gamma_i} + \tau^\gamma h) \\
 &= C(\beta_1 + \beta_2 r_i)(b_{k,i})^{-1}(\tau^{1+\gamma_i} + \tau^\gamma h)
 \end{aligned}$$

i.e.

$$|e_i^{k+1}| \leq C(b_{k,i})^{-1}(\tau^{1+\gamma_i} + \tau^\gamma h).$$

Because

$$\lim_{k \rightarrow \infty} \frac{(b_{k,i})^{-1}}{k^\gamma} = \lim_{k \rightarrow \infty} \frac{k^{-\gamma}}{(k+1)^{1-\gamma} - k^{1-\gamma}} = \lim_{k \rightarrow \infty} \frac{k^{-1}}{(1 + \frac{1}{k})^{1-\gamma} - 1} = \lim_{k \rightarrow \infty} \frac{k^{-1}}{(1-\gamma)k^{-1}} = \frac{1}{1-\gamma}$$

then $\|e^{k+1}\|_\infty \leq C(\tau + h)$ when $k\tau \leq T$. \square

5. Numerical examples

In order to demonstrate the effectiveness of our theoretical analysis, three examples are now presented.

Table 1
Numerical solution, exact solution and absolute error at $T = 1.0$ of (13).

x	Numerical solution	Exact solution	Error
0.1	0.16184371	0.16200000	0.00015629
0.2	0.51059931	0.51200000	0.00140069
0.3	0.87902481	0.88200000	0.00297519
0.4	1.14770234	1.15200000	0.00429766
0.5	1.24502781	1.25000000	0.00497219
0.6	1.14719659	1.15200000	0.00480341
0.7	0.87818473	0.88200000	0.00381527
0.8	0.50972531	0.51200000	0.00227469
0.9	0.16127925	0.16200000	0.00072075

Table 2
Maximum errors of the numerical solution and the effect of grid size reduction at time $T = 1.0$ of (13).

$h = \tau$	Max error	Error rate
1/50	9.4391×10^{-3}	
1/100	5.0134×10^{-3}	1.88
1/200	2.5613×10^{-3}	1.96
1/400	1.2781×10^{-3}	2.00

Example 1. Consider the following problem:

$$\begin{cases} \beta_1 \frac{\partial C(x, t)}{\partial t} + \beta_2 D_t^{\gamma(x, t)} C(x, t) = -V \frac{\partial C(x, t)}{\partial x} + D \frac{\partial^2 C(x, t)}{\partial x^2} + f(x, t), & (x, t) \in \Omega = [0, 1] \times [0, T], \\ C(x, 0) = 10x^2(1-x)^2, & 0 \leq x \leq 1, \\ C(0, t) = 0, \quad C(1, t) = 0, & 0 \leq t \leq T, \end{cases} \quad (13)$$

where $f(x, t) = f_1(x, t) + f_2(x, t) + f_3(x, t) + f_4(x, t)$ and

$$\begin{aligned} f_1(x, t) &= 10\beta_1 x^2(1-x)^2, & f_2(x, t) &= \frac{10\beta_2 x^2(1-x)^2 t^{1-\gamma(x, t)}}{\Gamma(2-\gamma(x, t))} \\ f_3(x, t) &= 10V(t+1)(2x-6x^2+4x^3), \\ f_4(x, t) &= -10D(t+1)(2-12x+12x^2). \end{aligned}$$

The exact solution in Example 1 is $C(x, t) = 10(t+1)x^2(1-x)^2$. Let $\beta_1 = \beta_2 = V = D = 1$ and $\gamma(x, t) = 1 - 0.5e^{-xt}$, when $h = \tau = 1/100$, Table 1 gives the numerical solution, exact solution and the absolute error. Fig. 1 shows the comparison of the numerical solution and exact solution (13) at $T = 1.0$. It can be seen that the numerical solution is in good agreement with the exact solution. Table 2 presents the absolute value of the maximum errors of the numerical solution and error rate at $T = 1.0$ of example 1. From this table it is easy to conclude that the order of convergence of the implicit numerical scheme (7)–(8) is first-order. Fig. 2 shows the solution behavior of (13) at $T = 1.0, T = 0.75, T = 0.5$ and $T = 0.25$, respectively.

Example 2. Consider the following problem:

$$\begin{cases} \beta_1 \frac{\partial C(x, t)}{\partial t} + \beta_2 D_t^{\gamma(x, t)} C(x, t) = -V \frac{\partial C(x, t)}{\partial x} + D \frac{\partial^2 C(x, t)}{\partial x^2} + f(x, t), & (x, t) \in \Omega = [0, 1] \times [0, 1], \\ C(x, 0) = 5x(1-x), & 0 \leq x \leq 1, \\ C(0, t) = 0, \quad C(1, t) = 0, & 0 \leq t \leq 1, \end{cases} \quad (14)$$

where $f(x, t) = f_1(x, t) + f_2(x, t) + f_3(x, t)$ and

$$\begin{aligned} f_1(x, t) &= 5\beta_1 x(1-x), & f_2(x, t) &= \frac{5\beta_2 x(1-x) t^{1-\gamma(x, t)}}{\Gamma(2-\gamma(x, t))} \\ f_3(x, t) &= 5V(t+1)(1-2x) + 10D(t+1). \end{aligned}$$

The exact solution in Example 2 is $C(x, t) = 5(t+1)x(1-x)$. Let $\beta_1 = \beta_2 = V = D = 1$ and $\gamma(x, t) = 0.8 + 0.005 \cos(xt) \sin(x)$, Table 3 presents the absolute value of the maximum errors of the numerical solution and error rate of (14). Fig. 3 plots the computational errors according to different time steps τ . It can be seen that the order of convergence is first-order. The three-dimensional numerical solution of Example 2 is shown in Fig. 4.

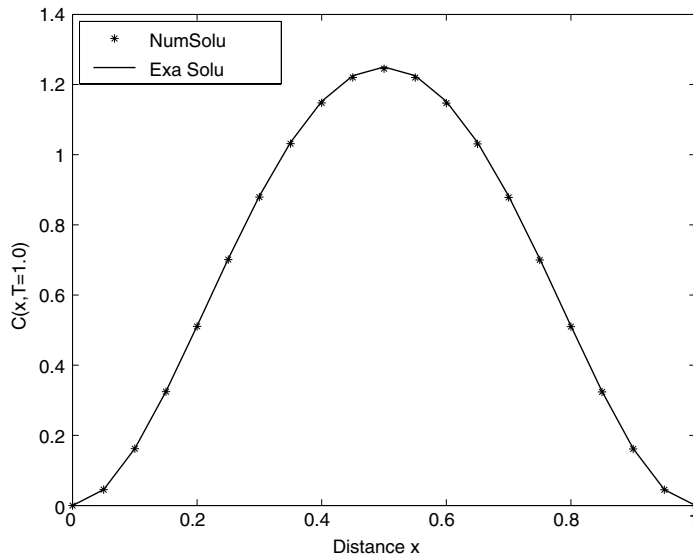


Fig. 1. Comparison of the numerical solution and the exact solution at $T = 1.0$ of (13).

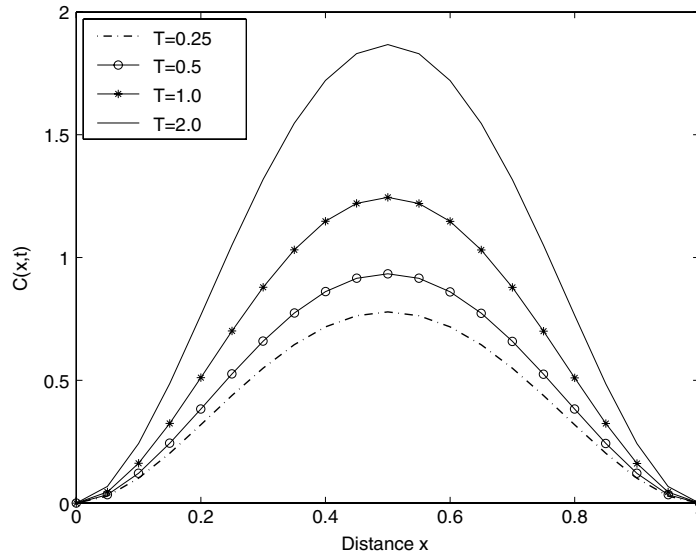


Fig. 2. Solution behavior of (13) at $T = 0.25, T = 0.5, T = 1.0$ and $T = 2.0$.

Example 3. Consider the following problem:

$$\begin{cases} \frac{\partial C(x, t)}{\partial t} + D_t^{\gamma(x,t)} C(x, t) = -\frac{\partial C(x, t)}{\partial x} + \frac{\partial^2 C(x, t)}{\partial x^2}, & (x, t) \in \Omega = [0, 1] \times [0, 1], \\ C(x, 0) = \sin(\pi x), & 0 \leq x \leq 1, \\ C(0, t) = 0, \quad C(1, t) = 0, & 0 \leq t \leq 1, \end{cases} \quad (15)$$

where $\gamma(x, t) = 0.8 + 0.05e^{-x} \sin t$.

The three-dimensional numerical solution of Example 3 is shown in Fig. 5 which shows that the system exhibits a typical mobile-immobile behavior.

6. Conclusion

In this paper, an implicit Euler approximation for the time variable fractional order mobile-immobile advection-dispersion model has been described and demonstrated. We prove that the implicit Euler approximation is unconditionally

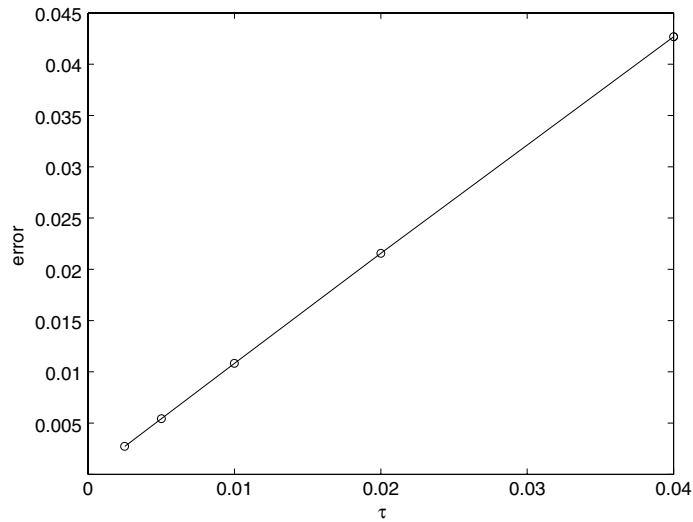


Fig. 3. Numerical error as a function of the time step τ of (14).

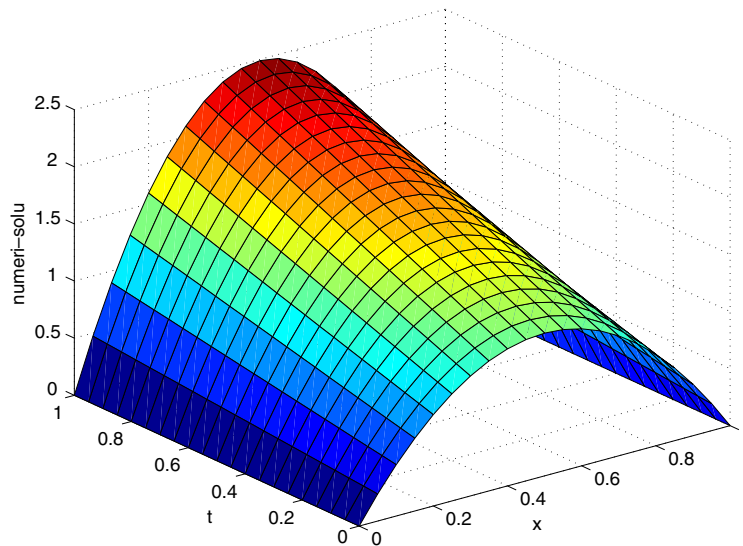


Fig. 4. Three-dimensional numerical solution of (14).

Table 3
Maximum errors of the numerical solution and the effect of grid size reduction of (14).

$h = \tau$	Max error	Error rate
1/50	2.1562×10^{-2}	
1/100	1.0825×10^{-2}	1.99
1/200	5.4267×10^{-3}	1.99
1/400	2.7164×10^{-3}	2.00

stable. As convergence of the implicit Euler approximation in a special case is also discussed. Numerical examples are provided to show that the implicit Euler approximation is computationally efficient.

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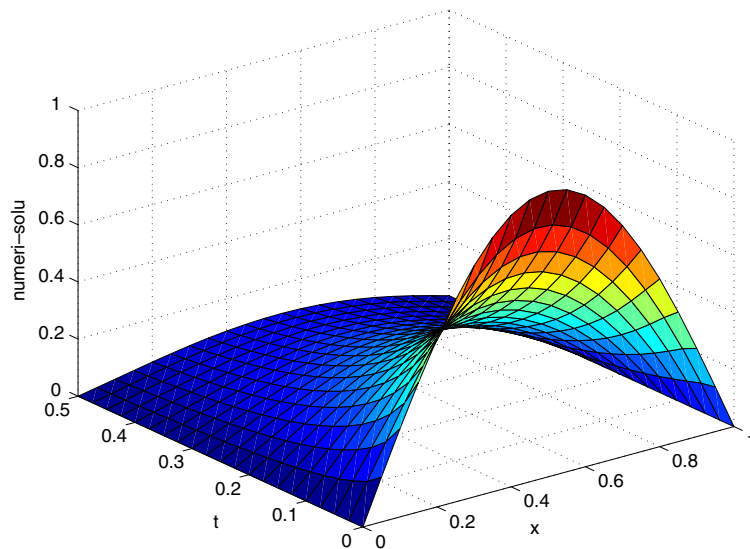


Fig. 5. Three-dimensional numerical solution of (15).

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