



Anomalous diffusion with ballistic scaling: A new fractional derivative

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ABSTRACT

Anomalous diffusion with ballistic scaling is characterized by a linear spreading rate with respect to time that scales like pure advection. Ballistic scaling may be modeled with a symmetric Riesz derivative if the spreading is symmetric. However, ballistic scaling coupled with a skewness is observed in many applications, including hydrology, nuclear physics, viscoelasticity, and acoustics. The goal of this paper is to find a governing equation for anomalous diffusion with ballistic scaling and arbitrary skewness. To address this problem, we propose a new operator called the *Zolotarev derivative*, which is valid for all orders $0 < \alpha \leq 2$. The Fourier symbol of this operator is related to the characteristic function of a stable random variable in the Zolotarev M parameterization. In the symmetric case, the Zolotarev derivative reduces to the well-known Riesz derivative. For $\alpha \neq 1$, the Zolotarev derivative is a linear combination of Riemann–Liouville fractional derivatives and a first derivative. For $\alpha = 1$, the Zolotarev derivative is a non-local operator that models ballistic anomalous diffusion. We prove that this operator is continuous with respect to α . We derive generator, Caputo, and Riemann–Liouville forms of this operator and provide two examples. The solutions of diffusion equations utilizing the Zolotarev derivative with an impulse initial condition are shifted and scaled stable densities in the M parameterization.

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1. Introduction

Anomalous super-diffusion is characterized by a spreading rate faster than the classical $t^{1/2}$ predicted by Fickian diffusion [1], where t is time. Super-diffusion with a spreading rate of $t^{1/\alpha}$ can be modeled by a space-fractional diffusion equation with order α . In particular, densities of stable Lévy motion with index $\alpha \neq 1$ solve space-fractional diffusion equations where the fractional derivative has order α [2,3]. The special case $\alpha = 1$ is *ballistic motion*, which scales like pure advection. The linear scaling may either indicate transport, or anomalous diffusion with a linear spreading rate [4]. For example, experimental evidence shows that the turbulent diffusion of contaminant plumes in rivers [5] and the spreading of mechanical pulses in viscoelastic materials [6] may exhibit ballistic scaling. Modeling this situation requires a derivative of order one which is neither the usual first derivative, nor the square root of the second derivative. In this paper, we introduce an appropriate derivative operator for this application.

The intimate connection between fractional partial differential equations (FPDEs), heavy-tailed probability distributions, and random walks is discussed in [7–9]. Solutions to diffusion-wave equations [10] and the general space–time fractional diffusion equation [11] also involve stable densities and the density of the inverse stable subordinator [12]. In particular, long

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particle jumps may be modeled with space-fractional PDEs, such as the space-fractional diffusion equation (FDE), whereas long waiting times may be modeled with time-fractional PDEs, such as the time-fractional diffusion equation.

All of these FPDEs assume that the index $\alpha \neq 1$. Nevertheless, stable densities are defined for $\alpha = 1$, which prompts the question: what is the governing equation of Lévy stable motion for $\alpha = 1$? In the symmetric case, the answer is known: simply replace the second derivative in the traditional diffusion equation with its square root, i.e., the symmetric Riesz derivative. However, many applications, such as hydrology [3], nuclear physics [13,14], pulse propagation in viscoelastic materials [15], seismology [16], ocean acoustics [17] and biomedical acoustics [18–20] require an asymmetric anomalous diffusion with stable index α either near one or identically one. For example, [20] derived a power-law wave equation (PLWE) that models frequency dependent attenuation of ultrasound in tissue, where the attenuation coefficient follows a power-law with respect to frequency. This PLWE does not allow the special case of linear attenuation even though empirical data [21,22] indicate that the power-law attenuation index for many types of tissue is approximately equal to $\alpha = 1$. Although several papers have proposed certain integro-differential equations for $\alpha = 1$ [13,17,18], the corresponding derivative operators have not been rigorously studied. Hence, the current paper will develop the governing equation of Lévy stable motion for $\alpha = 1$ along with their solutions.

The goal of this paper is to find a governing equation for anomalous diffusion with ballistic (linear) scaling and arbitrary skewness. From a probabilistic point of view, we will study the governing equation of Lévy stable motion for $\alpha = 1$. First, we recall the governing equation of Lévy stable motion for $\alpha \neq 1$, which is a space-fractional diffusion equation of order α . We then write the *point source* solution assuming an impulse initial condition in terms of stable distributions and show why these solutions are invalid for $\alpha = 1$. To alleviate this problem, we reparameterize the governing space-fractional diffusion equation and calculate these solutions in terms of stable densities in the Zolotarev M parameterization [23]. Since the Zolotarev M parameterization is continuous across the $\alpha = 1$ “barrier”, we derive a new operator, the *Zolotarev derivative of order one*. We derive generator, Riemann–Liouville, and Caputo forms of this operator, along with formulas for the Laplace transform. Concrete examples of the Zolotarev derivative are calculated, thereby illuminating practical applications of this operator.

2. The space-fractional diffusion equations and stable densities

This section reviews the connection between Riemann–Liouville derivatives, stable densities, and Lévy motion. We explicitly calculate the point source solution of the space-fractional diffusion equation for fractional orders excluding one. While this calculation has been performed in other works [11,24], our calculation demonstrates the singularity at $\alpha = 1$. Second, we explore numerically how this solution becomes singular as $\alpha \rightarrow 1$, motivating the remainder of the paper. Finally, we recall an alternative parameterization of stable densities, the *Zolotarev M parameterization*, and show how this parameterization eliminates the $\alpha = 1$ “barrier”.

2.1. Riemann–Liouville fractional derivatives

First, recall the definition of the Riemann–Liouville (RL) derivative on the real axis.

Definition 2.1. Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $\alpha \neq n$. The positively and negatively skewed Riemann–Liouville (RL) fractional derivatives are defined by [25, p. 87]

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x) = \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^x f(y)(x - y)^{n-1-\alpha} dy \tag{2.1a}$$

$$\frac{\partial^\alpha}{\partial (-x)^\alpha} f(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_x^\infty f(y)(y - x)^{n-1-\alpha} dy. \tag{2.1b}$$

Define the Fourier transform (FT) by

$$\hat{f}(k) = \mathcal{F}_x [f(x)] = \int_{-\infty}^\infty e^{-ikx} f(x) dx, \tag{2.2}$$

and assume $f^{(n)}(x)$ exists and $f(x), \dots, f^{(n)}(x) \in L^1$. Then we have [25, (2.3.27) and (2.3.28)]

$$\mathcal{F}_x \left[\frac{\partial^\alpha}{\partial x^\alpha} f(x) \right] = (ik)^\alpha \hat{f}(k) \tag{2.3a}$$

$$\mathcal{F}_x \left[\frac{\partial^\alpha}{\partial (-x)^\alpha} f(x) \right] = (-ik)^\alpha \hat{f}(k). \tag{2.3b}$$

Likewise, define the inverse FT via

$$\hat{f}(x) = \mathcal{F}_x^{-1} [\hat{f}(k)] = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ikx} \hat{f}(k) dk. \tag{2.4}$$

FPDEs involving space-fractional Riemann–Liouville derivatives are intimately connected with stable densities. Hence, we recall the generator form of the stable characteristic function and two particular parameterizations of stable laws in the following sections.

2.2. Stable densities: generator form and ST parameterization

Let $f(x)$ be an element of the Banach space $C_0(\mathbb{R})$ consisting of continuous real-valued functions on the real line that tend to zero at $\pm\infty$, with the supremum norm. Given a Lévy process $\{Z_t : t \geq 0\}$ [9, p. 57], define a family of linear operators

$$T_t f(x) = \mathbb{E} [f(x - Z_t)] \tag{2.5}$$

for $t \geq 0$. This family of operators T_t is a C_0 semigroup [9, p. 60]. The generator [9, Eq. (3.8)] of this semigroup is given by the following result from [9, Theorems 3.4 and 3.17]. Recall that if $f(x)$ is a probability density, its characteristic function is defined as $\hat{f}(-k)$.

Theorem 2.2. Suppose that Z_t is a Lévy processes with characteristic function $e^{t\psi(k)}$, where the log characteristic function (LCF)

$$\psi(k) = ika - \frac{k^2b}{2} + \int \left(e^{iky} - 1 - \frac{iky}{1+y^2} \right) \phi(dy), \tag{2.6}$$

where $\phi(dy)$ is the Lévy measure. Then the generator of the semigroup given by (2.5) is

$$Lf(x) = -af'(x) + \frac{b}{2}f''(x) + \int \left(f(x-y) - f(x) + \frac{yf'(x)}{1+y^2} \right) \phi(dy) \tag{2.7}$$

for any f such that $f, f', f'' \in C_0(\mathbb{R})$. If we also have $f, f', f'' \in L^1(\mathbb{R})$, then $\psi(-k)\hat{f}(k)$ is the Fourier transform of $Lf(x)$.

From the generator form (2.6), stable characteristic functions may be derived in several parameterizations. First, we recall the popular Samorodnitsky and Taqqu (ST) parameterization of stable densities [26].

Definition 2.3. A stable random variable X with index $0 < \alpha \leq 2$, skewness $-1 \leq \beta \leq 1$, scale $\sigma > 0$ and location $\mu \in \mathbb{R}$ in the ST parameterization [26, Equation (1.1.6)] has characteristic function $\mathbb{E} [e^{ikX}] = e^{\phi(k; \alpha, \beta, \sigma, \mu)}$ where:

$$\phi(k; \alpha, \beta, \sigma, \mu) = \begin{cases} -\sigma^\alpha |k|^\alpha \left[1 - i\beta \operatorname{sgn}(k) \tan\left(\frac{\pi\alpha}{2}\right) \right] + ik\mu & \alpha \neq 1; \\ -\sigma |k| \left[1 + \frac{2i\beta}{\pi} \operatorname{sgn}(k) \ln|k| \right] + ik\mu & \alpha = 1. \end{cases} \tag{2.8}$$

In the special case of a standard stable law with $\sigma = 1$ and $\mu = 0$ we will also write $\phi_{\alpha, \beta}(k) = \phi(k; \alpha, \beta, 1, 0)$, so that:

$$\phi_{\alpha, \beta}(k) = \begin{cases} -|k|^\alpha \left(1 - i\beta \operatorname{sgn}(k) \tan\left(\frac{\pi\alpha}{2}\right) \right) & \alpha \neq 1; \\ -|k| \left(1 + \frac{2i\beta}{\pi} \operatorname{sgn}(k) \ln|k| \right) & \alpha = 1. \end{cases} \tag{2.9}$$

For X as in Definition 2.3, the random variable $(X - \mu)/\sigma$ is standard stable with LCF $\phi(k; \alpha, \beta, 1, 0)$ for $\alpha \neq 1$, and $\phi(k; \alpha, \beta, 1, 2\frac{\beta}{\pi} \ln \sigma)$ for $\alpha = 1$ [26, Propositions 1.2.2 and 1.2.3]. A standard stable density in the ST parameterization is given by

$$g_{\alpha, \beta}(x) = \mathcal{F}_x^{-1} [\exp(\phi_{\alpha, \beta}(-k))] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} [\exp(\phi_{\alpha, \beta}(-k))] dk. \tag{2.10}$$

Remark 2.4. If the skewness parameter $\beta \neq 0$, the ST parameterization is discontinuous at $\alpha = 1$ due to the $\tan(\pi\alpha/2)$ term in (2.9), which was pointed out in [23, p. 11]. This discontinuity poses a problem in the numerical evaluation of stable densities in this parameterization for stability parameters near $\alpha = 1$ [27].

2.3. Space-fractional diffusion equation

Now consider the space-fractional diffusion equation

$$\frac{\partial}{\partial t} u(x, t) = p \frac{\partial^\alpha}{\partial x^\alpha} u(x, t) + q \frac{\partial^\alpha}{\partial (-x)^\alpha} u(x, t) \tag{2.11}$$

where $1 < \alpha \leq 2$ and p, q are nonnegative real numbers such that $p + q = 1$, subject to an impulse initial condition $u(x, 0) = \delta(x)$. If $0 < \alpha < 1$, then the right hand side of (2.11) is multiplied by -1 [Remark 5.10, 9]. Space-fractional

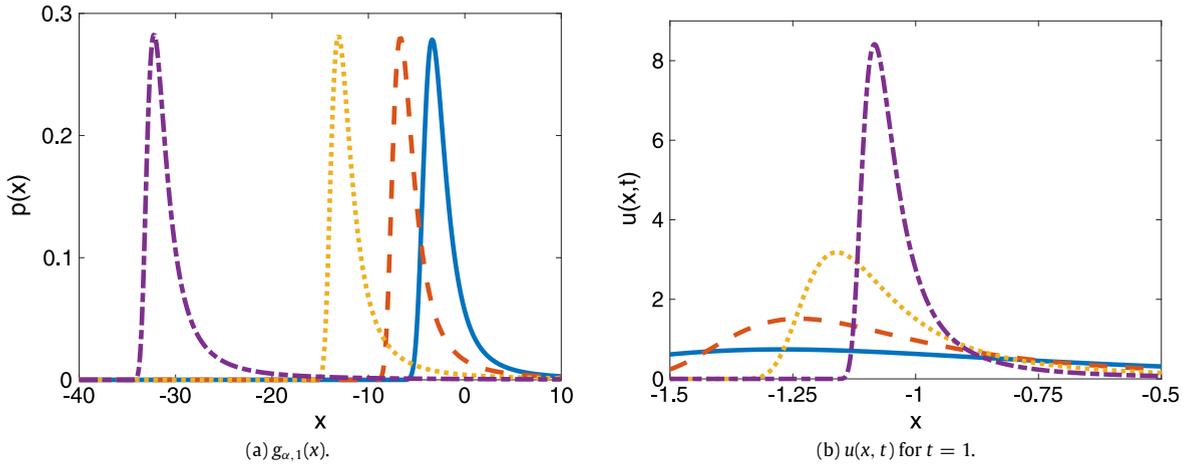


Fig. 1. Panel (a) displays the positively-skewed stable density $g_{\alpha,1}$ for $\alpha = 1.2$ (solid), 1.1 (dashed), 1.05 (dotted), and 1.02 (dash-dotted). Note that the peak tends towards $-\infty$ as $\alpha \downarrow 1$. Panel (b) shows the solution to the fractional diffusion Eq. (2.14) at time $t = 1$ for $\alpha = 1.2$ (solid), 1.1 (dashed), 1.05 (dotted), and 1.02 (dash-dotted). Since the scale $\sigma = |\cos(\pi\alpha/2)|^{1/\alpha}$ approaches zero as $\alpha \downarrow 1$, the solution $u(x, t)$ approaches $\delta(x + 1)$.

diffusion equations model super-diffusion of particles, since the particle spreading rate is faster than the $t^{1/2}$ rate associated with traditional diffusion [3]. Defining the Laplace transform via

$$\tilde{f}(s) = \mathcal{L}_t [f(t)] = \int_0^\infty f(t)e^{-st} dt, \tag{2.12}$$

we may calculate the point source solution of (2.11) by applying a Fourier transform followed by a Laplace transform to (2.11) and utilizing (2.3), yielding

$$\bar{u}(k, s) = \frac{1}{s - p(ik)^\alpha - q(-ik)^\alpha},$$

where $\bar{u}(k, s) = \mathcal{L}_t \mathcal{F}_x [u(x, t)]$. Calculating the inverse LT gives

$$\hat{u}(k, t) = \exp [tp(ik)^\alpha + tq(-ik)^\alpha], \tag{2.13}$$

for $t > 0$. Noting that $(\pm ik)^\alpha = |k|^\alpha [\cos(\pi\alpha/2) \pm i \operatorname{sgn}(k) \sin(\pi\alpha/2)]$, we write (2.13) as

$$\hat{u}(k, t) = \exp [t\sigma^\alpha \phi_{\alpha,\beta}(-k)]$$

where $\phi_{\alpha,\beta}(k)$ is given by (2.9), $\sigma = |\cos(\pi\alpha/2)|^{1/\alpha}$, and $\beta = p - q$. A little algebra along with (2.10) yields the solution [10]

$$u(x, t) = \frac{1}{t^{1/\alpha}\sigma} g_{\alpha,\beta} \left(\frac{x}{t^{1/\alpha}\sigma} \right), \tag{2.14}$$

a stable density with index α , skewness β , and a time-dependent scaling parameter. The case $0 < \alpha < 1$ is similar, and leads to the same solution (2.14), see Remark 5.10 in [9].

This point source solution, or Green’s function, is also the probability density function of a stable Lévy motion of order α and skew β . For example, if $\alpha < 1$ and $\beta = 1$ (positive skew), (2.14) is the density of a stable subordinator $D(t)$. Since the spreading rate is $t^{1/\alpha}$ and the peak decays at the same rate [9, p. 12], (2.14) models super-diffusion, which is faster than traditional diffusion ($\alpha = 2$). Clearly, the space-fractional diffusion equation (2.11) reduces to a transport equation (or one-way wave equation) when $\alpha = 1$, and the solution (2.14) is not valid in this case. However, there is a need to model ballistic diffusion, the case $\alpha = 1$. See, for example, Fig. 1 in [1]. Applications of anomalous diffusion with a linear spreading rate include: energy loss of fast particles (e.g. electrons) due to ionization [13,14], the propagation of short (wide-band) mechanical pulses through viscoelastic materials (e.g. polymers) [6,15], and turbulent dispersion of a contaminant plume [5,28].

Fig. 1(a) illustrates the discontinuity in the ST parameterization of the stable density at $\alpha = 1$. As α decreases to 1, the peak moves towards $-\infty$. Fig. 1(b) shows the behavior of the solution to the fractional diffusion equation (2.14) as α decreases to 1. The scale $\sigma = |\cos(\pi\alpha/2)|^{1/\alpha}$ in (2.14) approaches zero as $\alpha \downarrow 1$, hence the solution $u(x, t)$ approaches $\delta(x + 1)$, which is the solution to a transport equation.

Note that the solution (2.14) to the space-fractional diffusion equation (2.11) is also invalid for $\alpha = 1$, since the scale parameter $\sigma = |\cos(\pi\alpha/2)|^{1/\alpha}$ is zero for $\alpha = 1$. Hence, (2.11) cannot model super-diffusion that exhibits a linear spreading rate. This singularity is not surprising, since the governing equation (2.11) reduces to a transport equation if $\alpha = 1$.

Nevertheless, stable distributions are defined for $\alpha = 1$ and all β ranging from -1 to 1 using the LCF given by (2.9). A natural question arises: What is the governing equation of (2.14) for $\alpha = 1$? We are particularly interested in the case $(\alpha, \beta) = (1, 1)$, which has applications in both nuclear physics [13] and acoustics [20]. To our knowledge, this governing equation has not been reported in the literature. In the next section, we develop the mathematical machinery necessary to state the governing equation of Lévy motion of order $\alpha = 1$ with arbitrary skewness $-1 \leq \beta \leq 1$.

2.4. Zolotarev M parameterization

To analyze the important case of $\alpha = 1$, we must write stable densities and fractional derivatives in a parameterization that eliminates the singularity at $\alpha = 1$. Armed with these alternative parameterizations, we may consider the case $\alpha = 1$ in a natural manner.

Recall an alternative parameterization of stable random variables introduced by Zolotarev [23]. The motivation behind this parameterization is to eliminate the discontinuity at $\alpha = 1$.

Definition 2.5. A stable random variable X in the Zolotarev M parameterization [23, p. 11] has characteristic function

$$\mathbb{E} [e^{ikX}] = \exp (\psi(k; \alpha, \beta, \lambda, \gamma)) \tag{2.15}$$

where the LCF

$$\psi(k; \alpha, \beta, \lambda, \gamma) = \begin{cases} -\lambda|k|^\alpha \left[1 + i\beta \operatorname{sgn}(k) \tan \left(\frac{\pi\alpha}{2} \right) (|k|^{1-\alpha} - 1) \right] + ik\lambda\gamma & \alpha \neq 1 \\ -\lambda|k| \left[1 + \frac{2i\beta}{\pi} \operatorname{sgn}(k) \ln|k| \right] + ik\lambda\gamma & \alpha = 1 \end{cases} \tag{2.16}$$

has stable index $0 < \alpha \leq 2$, skewness $-1 \leq \beta \leq 1$, scale $\lambda > 0$ and location $\gamma \in \mathbb{R}$. The standard (in the Zolotarev M parameterization) stable LCF with $\lambda = 1$ and $\gamma = 0$ is then given by

$$\psi_{\alpha,\beta}(k) = \begin{cases} -|k|^\alpha \left[1 + i\beta \operatorname{sgn}(k) \tan (\theta) (|k|^{1-\alpha} - 1) \right] & \alpha \neq 1 \\ -|k| \left[1 + \frac{2i\beta}{\pi} \operatorname{sgn}(k) \ln|k| \right] & \alpha = 1 \end{cases} \tag{2.17}$$

where $\theta = \pi\alpha/2$. A standard (in the Zolotarev M parameterization) stable density with index α and skewness β is denoted by

$$\begin{aligned} f_{\alpha,\beta}(x) &= \mathcal{F}_x^{-1} [\exp (\psi_{\alpha,\beta}(-k))] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} [\exp (\psi_{\alpha,\beta}(-k))] dk. \end{aligned} \tag{2.18}$$

Remark 2.6. The various parameterizations of stable laws have been studied by Nolan [29]. The LCFs of the ST parameterization and the Zolotarev M-parameterization are related for $\alpha \neq 1$ via

$$\psi_{\alpha,\beta}(k) = \phi_{\alpha,\beta}(k) - ik\beta \tan \left(\frac{\pi\alpha}{2} \right). \tag{2.19}$$

Hence, the Zolotarev M parameterization subtracts a shift from $\phi_{\alpha,\beta}(k)$ that tends to infinity as $\alpha \rightarrow 1$. If $\alpha = 1$, then $\psi_{\alpha,\beta}(k) = \phi_{\alpha,\beta}(k)$.

Remark 2.7. Note that the ST parameterization (2.10) and the Zolotarev M parameterization (2.18) are identical for $\alpha = 1$. In particular

$$g_{1,0}(x) = f_{1,0}(x) = \frac{1}{\pi(1+x^2)} \tag{2.20a}$$

$$g_{1,1}(x) = f_{1,1}(x) = \frac{\pi}{2} f_L \left(\frac{\pi x}{2} + \ln \left(\frac{\pi}{2} \right) \right), \tag{2.20b}$$

where (2.20a) is the well-known Cauchy distribution and $f_L(x)$ is the Landau distribution [13, Equation (13)] defined by the contour integral

$$f_L(x) = \frac{1}{2\pi i} \int_B e^{s \ln s + xs} ds, \tag{2.21}$$

where B is the Bromwich contour. The connection between the Landau distribution and stable distributions was first made in the Soviet physics literature [14] and later summarized in [30, Section 13.4].

Proposition 2.8. The log characteristic function $\psi_{\alpha,\beta}(k)$ given by (2.17) is continuous with respect to index α for all $-1 \leq \beta \leq 1$.

Proof. It suffices to show $\psi_{\alpha,\beta}(k)$ is continuous at $\alpha = 1$. When $k = 0$, the result follows from $\psi_{1,\beta}(0) = 0$. Let $\alpha = 1 + \epsilon$ where $\epsilon > 0$. Then

$$\begin{aligned} \lim_{\alpha \rightarrow 1} \psi_{\alpha,\beta}(k) &= \lim_{\epsilon \rightarrow 0^+} -|k|^{1+\epsilon} \left[1 + i\beta \operatorname{sgn}(k) \tan\left(\frac{\pi(1+\epsilon)}{2}\right) (|k|^{-\epsilon} - 1) \right] \\ &= \lim_{\epsilon \rightarrow 0^+} -|k| \left[|k|^\epsilon + i\beta \operatorname{sgn}(k) \frac{|k|^\epsilon - 1}{\tan(\pi\epsilon/2)} \right] \\ &= -|k| \left[1 + \frac{2i\beta}{\pi} \operatorname{sgn}(k) \lim_{\epsilon \rightarrow 0^+} \frac{|k|^\epsilon - 1}{\epsilon} \right], \end{aligned}$$

where we used the identity $\tan(\pi/2 + \theta) = -\cot(\theta)$ in the second line and $\tan z \sim z$ for small z in the third line. Noting that

$$\lim_{\epsilon \downarrow 0} \frac{z^\epsilon - 1}{\epsilon} = \ln z \tag{2.22}$$

for $z > 0$ yields the $\alpha = 1$ case in (2.17). For $\alpha = 1 - \epsilon$, the proof is similar. \square

Corollary 2.9. By Proposition 2.8 and the Fourier continuity theorem [31, Theorem 1.3.6], it follows that $f_{\alpha,\beta}(x)$ is continuous with respect to α .

3. Zolotarev fractional derivative

In this section, we propose a new pseudo-differential operator [32] based on the standard stable LCF in the Zolotarev M parameterization, which we call the *Zolotarev fractional derivative*. To provide a setting for this operator, recall the space of *Bessel potentials*.

Definition 3.1. Let $\nu \geq 0$ be a real number. The space of *Bessel potentials* or the *Liouville space* H^ν is defined [32, Definition 1.14 ($p = 2$)]

$$H^\nu = \left\{ f \in L^2 : (1 + |k|^2)^{\nu/2} \hat{f}(k) \in L^2 \right\}. \tag{3.1}$$

If ν is an integer, then H^ν is a classical Sobolev space. Using [33, Corollary 7.18.1], Definition 3.1 is equivalent to fractional Sobolev space $W^{\nu,2}$ [7,34] for real ν equipped with the norm [34, Equation 2.2 with $p = 2$ and $n = 1$]

$$\|f(x)\|_{H^\nu} = \left(\int_{-\infty}^{\infty} |f(x)|^2 dx + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2\nu}} dx dy \right)^{1/2}. \tag{3.2}$$

Definition 3.2. The *Zolotarev fractional derivative* $D_x^{\alpha,\beta} f(x)$ of order $0 < \alpha \leq 2$ and skewness β of a function $f(x)$ is defined by

$$D_x^{\alpha,\beta} f(x) = \mathcal{F}_x^{-1} \left[\hat{f}(k) \psi_{\alpha,\beta}(-k) \right], \tag{3.3}$$

where $\psi_{\alpha,\beta}(k)$ is given by (2.17).

Remark 3.3. In the symmetric case $\beta = 0$, the Zolotarev fractional derivative

$$D_x^{\alpha,0} f(x) = \mathcal{F}_x^{-1} \left[-\hat{f}(k) |k|^\alpha \right] = \frac{\partial^\alpha}{\partial |x|^\alpha} f(x) \tag{3.4}$$

reduces to a Riesz fractional derivative (or the fractional Laplacian) [35] of order α . Unlike the RL or Caputo fractional derivatives, the Riesz fractional derivative is well-defined and continuous at $\alpha = 1$. In particular, if $\alpha = 1$ and $\beta = 0$, then

$$\begin{aligned} D_x^{1,0} f(x) &= \mathcal{F}_x^{-1} \left[-\hat{f}(k) |k| \right] \\ &= \mathcal{F}_x^{-1} \left[-\hat{f}(k) i \operatorname{sgn}(k) (ik) \right] \\ &= -\frac{d}{dx} \mathcal{H} [f(x)] \end{aligned} \tag{3.5}$$

where

$$\mathcal{H} [f(x)] = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|y| > \epsilon} \frac{f(y)}{x - y} dy \tag{3.6}$$

is the *Hilbert transform* on the real line [36, (1.2)] and the integral in (3.6) in the Cauchy principal value. The above relationship was noted in [37]. This computation utilized the Fourier transform relationship $\mathcal{F}_x [\mathcal{H} [f(x)]] = -i \operatorname{sgn}(k) \hat{f}(k)$.

Remark 3.4. The relationship between $D_x^{1,0}$ and the Hilbert transform may be extended to $\beta \neq 0$. A direct calculation yields

$$\begin{aligned} D_x^{1,\beta}f(x) &= \mathcal{F}_x^{-1} \left[\hat{f}(k)\psi_{1,\beta}(-k) \right] \\ &= \mathcal{F}_x^{-1} \left[-\hat{f}(k)i \operatorname{sgn}(k)(ik) \right] + \mathcal{F}_x^{-1} \left[\hat{f}(k)\frac{2\beta}{\pi}(ik) \ln|k| \right] \\ &= -\frac{d}{dx} \mathcal{H}[f(x)] + \frac{2\beta}{\pi} \frac{d}{dx} \mathcal{F}_x^{-1} \left[\ln|k|\hat{f}(k) \right]. \end{aligned}$$

Note that [38, p. 72]

$$\mathcal{F}_x^{-1} [\ln|k|] = -\frac{1}{2|x|} - \gamma\delta(x) \tag{3.7}$$

where γ is the Euler–Mascheroni constant. Note that the inverse FT in (3.7) exists as a distribution. Applying (3.7) along with the convolution theorem [36, Equation (2.54)] yields

$$D_x^{1,\beta}f(x) = -\frac{2}{\pi} \frac{d}{dx} \left[\mathcal{H}[f(x)] + \beta\gamma f(x) + \beta \lim_{\epsilon \rightarrow 0} \int_{|y|>\epsilon} \frac{f(y)}{|x-y|} dy \right]. \tag{3.8}$$

Hence, the Zolotarev fractional derivative consists of three components: (1) a symmetric term involving the Hilbert transform, (2) a local term which is proportional to the skewness β , and (3) a nonlocal term also proportional to the skewness β .

We now identify an appropriate function space for the Zolotarev derivative.

Proposition 3.5. Let $\epsilon > 0$ and define

$$v = \begin{cases} \alpha & \text{for } \beta = 0 \text{ and } 0 < \alpha \leq 2 \\ 1 & \text{for } \alpha < 1 \text{ and } \beta \neq 0 \\ 1 + \epsilon & \text{for } \alpha = 1 \text{ and } \beta \neq 0 \\ \alpha & \text{for } 1 < \alpha \leq 2 \text{ and any } -1 \leq \beta \leq 1. \end{cases} \tag{3.9}$$

Suppose $f \in H^v$. Then $D_x^{\alpha,\beta}f(x) \in L^2$.

Proof. The proof will be divided into four cases. For $k \leq 1$, we have $|\psi_{\alpha,\beta}(k)| \leq 1$ for all α . We utilize this bound for all four cases.

- (i) If $\beta = 0$, then $D_x^{\alpha,0}$ is the Riesz derivative (3.4), and the result follows immediately from [33, Theorem 7.16 with $p = 2$ and $n = 1$].
- (ii) If $\alpha < 1$ then

$$|\psi_{\alpha,\beta}(k)| = \sqrt{|k|^{2\alpha} + \beta^2 \tan^2\theta (|k|^\alpha - |k|)^2}. \tag{3.10}$$

For $k \geq 1$, we have $|k|^\alpha \leq |k|$, implying $|\psi_{\alpha,\beta}(k)| \leq |k|$. Hence $|\psi_{\alpha,\beta}(k)| \leq (1 + |k|^2)^{1/2}$ for all $k \geq 0$. Taking $v = 1$ in (3.1) ensures that $\psi_{\alpha,\beta}(-k)\hat{f}(k)$ is in L^2 .

- (iii) Suppose $1 < \alpha \leq 2$. If $k > 1$, then $|\psi_{\alpha,\beta}(k)| \leq C_{\alpha,\beta}|k|^\alpha$ where $C_{\alpha,\beta} = \sqrt{1 + \beta^2 \tan^2\theta}$. Hence $|\psi_{\alpha,\beta}(k)| \leq C_{\alpha,\beta}(1 + |k|^2)^{\alpha/2}$.
- (iv) Suppose $\alpha = 1$ and $\beta \neq 0$. As in the previous cases, $|\psi_{1,\beta}(k)| \leq 1$ for $|k| \leq 1$. Also note that

$$\ln k \leq \frac{k^\epsilon - 1}{\epsilon}$$

for any $k > 0$ and $\epsilon > 0$. Hence, for $k > 1$, $\ln k \leq k^\epsilon/\epsilon$. Letting $C' = 2\beta/\pi$, we have

$$\begin{aligned} |\psi_{1,\beta}(k)| &= \sqrt{k^2 + \frac{4\beta^2}{\pi^2} k^2 \ln^2|k|} \\ &\leq |k| + C'|k| \ln|k| \\ &\leq |k| + C'|k| \frac{k^\epsilon}{\epsilon} \\ &\leq (1 + C'/\epsilon) |k|^{1+\epsilon}. \end{aligned}$$

Letting $C'' = (1 + C'/\epsilon)$, we have $|\psi_{1,\beta}(k)| \leq C''(1 + |k|^2)^{(1+\epsilon)/2}$. Setting $v = (1 + \epsilon)$ ensures that $\psi_{1,\beta}(-k)\hat{f}(k)$ is in L^2 .

This completes the proof. \square

Later, to prove continuity of $D_x^{\alpha,\beta}$ with respect to α , we will need a bound which was established in the proof of Proposition 3.5.

Lemma 3.6. For all $0 < \alpha \leq 2$ and all $-1 \leq \beta \leq 1$, $|\psi_{\alpha,\beta}(k)| \leq \sqrt{2}(1 + k^2)$.

Remark 3.7. From the previous proposition, we see that the Zolotarev derivative with $\beta \neq 0$ requires slightly more regularity than the corresponding Riesz derivative ($\beta = 0$) for $\alpha \leq 1$. For $\alpha < 1$, this requirement stems from the additional shift term in (2.17), while for $\alpha = 1$, the requirement stems from the logarithm term. For $1 < \alpha \leq 2$, the Riesz derivative and Zolotarev derivative with $\beta \neq 0$ have the same regularity requirements.

Remark 3.8. Proposition 3.5 specifies an appropriate fractional Sobolev space H^ν in which the Zolotarev derivative exists in the (weak) L^2 sense. For the Zolotarev derivative to exist pointwise, we may use a result in [39, Theorem 23.14.2]: Let $\mathcal{D} = \{f(x) \in L^1 : f(x), f'(x) \in AC \text{ and } f'(x), f''(x) \in L^1\}$, where AC is the space of absolutely continuous functions. If $f \in \mathcal{D}$, then $D_x^{\alpha,\beta}f(x)$ exists for all $x \in \mathbb{R}$, for all $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$. Since $f'' \in L^2$ implies $f \in H^2$, and $H^2 \subseteq H^\nu$ for $\nu \leq 2$ by [34, Corollary 2.3], it follows that $\mathcal{D} \subset H^\nu$. We have a proper subset since all functions in \mathcal{D} are continuous and differentiable.

Proposition 3.9. Let $\alpha \neq 1$, $n = \lceil \alpha \rceil$, and suppose $f^{(n)}(x)$ exists and $f(x), \dots, f^{(n)}(x) \in L^1$. Then the Zolotarev fractional derivative (3.3) can be written as

$$D_x^{\alpha,\beta}f(x) = \beta \tan(\theta) \frac{\partial f}{\partial x} - \frac{p}{\cos(\theta)} \frac{\partial^\alpha f}{\partial x^\alpha} - \frac{q}{\cos(\theta)} \frac{\partial^\alpha f}{\partial (-x)^\alpha}, \tag{3.11}$$

where $\beta = p - q$, $\theta = \pi\alpha/2$, and the second and third terms involve the Riemann–Liouville fractional derivatives defined by (2.1). In particular, if $\alpha = 2$, $D_x^{2,\beta}f(x) = f''(x)$.

Proof. Write $k = |k| \operatorname{sgn}(k)$ and $(ik)^\alpha = |k|^\alpha (\cos \theta + i \operatorname{sgn} k \sin \theta)$, so that we also have $(-ik)^\alpha = |k|^\alpha (\cos \theta - i \operatorname{sgn} k \sin \theta)$. Apply a FT to (3.11) utilizing (2.3) to get

$$\begin{aligned} \mathcal{F}_x [D_x^{\alpha,\beta}f(x)] &= \hat{f}(k) \left[\tan \theta \beta ik - \frac{p}{\cos \theta} (ik)^\alpha - \frac{q}{\cos \theta} (-ik)^\alpha \right] \\ &= \hat{f}(k) [-(p + q)|k|^\alpha + i \operatorname{sgn}(k) \tan \theta (\beta|k| - p|k|^\alpha + q|k|^\alpha)] \\ &= -\hat{f}(k)|k|^\alpha [1 - \beta i \operatorname{sgn}(k) \tan \theta (|k|^{1-\alpha} - 1)], \end{aligned}$$

where we used $p + q = 1$ and $p - q = \beta$ in the second line. Since $|-k| = |k|$ and $\operatorname{sgn}(-k) = -\operatorname{sgn}(k)$, the bracketed expression is $\psi_{\alpha,\beta}(-k)$. \square

Remark 3.10. Unlike the Riemann–Liouville fractional derivative, the Zolotarev fractional derivative does not possess a semi-group property if $\beta \neq 0$. That is

$$D_x^{\alpha_1,\beta} D_x^{\alpha_2,\beta} f(x) \neq D_x^{\alpha_1+\alpha_2,\beta} f(x). \tag{3.12}$$

In particular, $D_x^{1,\beta} D_x^{1,\beta} \neq \partial^2 / \partial x^2$ if $\beta \neq 0$.

We now prove that the Zolotarev derivative is continuous with respect to the index α , thereby removing the “barrier” at $\alpha = 1$.

Theorem 3.11. Let $f \in H^2$ and let $\alpha_n \rightarrow \alpha$. Then

$$D_x^{\alpha_n,\beta} f(x) \rightarrow D_x^{\alpha,\beta} f(x) \tag{3.13}$$

with respect to the L^2 norm.

Proof. Since $f \in H^2$, it follows that $\psi_{\alpha_n,\beta}(-k)\hat{f}(k) \in L^2$ and $\psi_{\alpha,\beta}(-k)\hat{f}(k) \in L^2$ for all $0 \leq \alpha \leq 2$. Then

$$\begin{aligned} \|D_x^{\alpha_n,\beta} f(x) - D_x^{\alpha,\beta} f(x)\|_{L^2}^2 &= \left\| \mathcal{F}_x^{-1} \left[(\psi_{\alpha_n,\beta}(-k) - \psi_{\alpha,\beta}(-k)) \hat{f}(k) \right] \right\|_{L^2}^2 \\ &= \frac{1}{2\pi} \| (\psi_{\alpha_n,\beta}(-k) - \psi_{\alpha,\beta}(-k)) \hat{f}(k) \|_{L^2}^2 \end{aligned}$$

where we used Plancherel’s Theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk$$

in the second line. Let $G_n(k) = |(\psi_{\alpha_n,\beta}(-k) - \psi_{\alpha,\beta}(-k))\hat{f}(k)|^2$. By Lemma 3.6, we have

$$G_n(k) \leq 2(1 + |k|^2)^2 |\hat{f}(k)|^2$$

which is integrable. Since $\psi_{\alpha_n,\beta}(-k) \rightarrow \psi_{\alpha,\beta}(-k)$ by Proposition 2.8, we apply the Dominated Convergence Theorem and Theorem 3.11 follows. \square

4. Explicit forms of Zolotarev derivative of order one

In this section, we develop an explicit formula for (3.3) for $\alpha = 1$ and all β . Generator, Caputo, and Riemann–Liouville forms of $D_x^{1,\beta}$ are computed. To establish the generator form, we recall the Lévy–Khintchine representation of a stable law with index $\alpha = 1$ [31, Lemma 7.3.9] or [40, Eq. (6.8)]. We write the (positively skewed) Zolotarev derivative of order one $D_x^{1,\beta}$ using the LCFs $\psi_+(-k) = \psi_{1,1}(-k)$ and the negatively skewed Zolotarev derivative of order one as $D_x^{1,-1}$ with Fourier multiplier $\psi_-(-k) = \psi_{1,-1}(-k)$.

Theorem 4.1. *The Fourier symbol $\psi_{1,\beta}(k)$ of the Zolotarev fractional derivative of order one can be written in the form*

$$\psi_{1,\beta}(k) = p\psi_+(k) + q\psi_-(k) \tag{4.1}$$

where

$$\psi_+(k) = \frac{2}{\pi} \int_0^\infty (e^{iky} - 1 - ik \sin y) y^{-2} dy, \tag{4.2a}$$

$$\psi_-(k) = \frac{2}{\pi} \int_{-\infty}^0 (e^{iky} - 1 - ik \sin y) |y|^{-2} dy \tag{4.2b}$$

and $\beta = p - q$.

Proof. The calculation of (4.2) is an application of the Lévy–Khintchine formula (2.6) for infinitely divisible laws. Eq. (4.2a) is calculated on [31, p. 268] and then (4.2b) follows by a simple change of variable. Noting that $\beta = p - q$ and $p + q = 1$, (4.1) follows immediately from (4.2). \square

We now write the generator forms of the Zolotarev derivatives $D_x^{1,1}$ and $D_{-x}^{1,-1}$ which form the basis for the remainder of the paper.

Proposition 4.2. *The generator forms of $D_x^{1,1}$ and $D_{-x}^{1,-1}$ are*

$${}_GD_x^1 f(x) = \frac{2}{\pi} \int_0^\infty (f(x - y) - f(x) + f'(x) \sin y) y^{-2} dy \tag{4.3a}$$

$${}_GD_{-x}^1 f(x) = \frac{2}{\pi} \int_0^\infty (f(x + y) - f(x) - f'(x) \sin y) y^{-2} dy. \tag{4.3b}$$

Proof. Let $\hat{f}(k)$ denote the FT of $f(x)$ and use (4.2a) to obtain

$${}_GD_x^1 f(x) = \mathcal{F}_x^{-1} \left[\psi_+(-k) \hat{f}(k) \right] = \frac{2}{\pi} \mathcal{F}_x^{-1} \left[\int_0^\infty (e^{-iky} - 1 + ik \sin y) y^{-2} dy \hat{f}(k) \right].$$

The inverse FT and integration over y may be interchanged using Fubini’s theorem. Apply the shifting property $\mathcal{F}_x^{-1}[e^{-iky} \hat{f}(k)] = f(x - y)$ and (2.3a) with $\alpha = 1$, yielding (4.3a). A similar calculation utilizing (4.2b) yields (4.3b). \square

Suppose $f(x) \in C^1$ and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$. Integrate by parts with $u = f(x - y) - f(x) + f'(x) \sin y$ and $dv = y^{-2} dy$, and note that the boundary terms at $y = 0$ and $y = \infty$ both vanish. The boundary term at ∞ vanishes since $f(x - y) \rightarrow 0$ as $y \rightarrow \infty$ for any x , while the boundary term at zero vanishes by performing a Taylor series expansion. This calculation yields the Caputo form on the real axis

$${}_cD_x^1 f(x) = \frac{2}{\pi} \int_0^\infty [f'(x) \cos y - f'(x - y)] y^{-1} dy. \tag{4.4}$$

Note that (4.4) involves differentiation followed by an integration, which is analogous to the definition of Caputo fractional derivative [41]. Thus, the following definition is motivated.

Definition 4.3. *The positive and negative Zolotarev integrals are defined by*

$$\mathcal{Z}_+ f(x) = \frac{2}{\pi} \int_0^\infty y^{-1} [f(x) \cos y - f(x - y)] dy \tag{4.5a}$$

$$\mathcal{Z}_- f(x) = \frac{2}{\pi} \int_0^\infty y^{-1} [f(x + y) - f(x) \cos y] dy. \tag{4.5b}$$

Definition 4.4. The positive and negative Zolotarev–Caputo derivatives of order one are defined by

$${}_c D_x^1 f(x) = \mathcal{Z}_+ f'(x) = \frac{2}{\pi} \int_0^\infty y^{-1} [f'(x) \cos y - f'(x - y)] dy \tag{4.6a}$$

$${}_c D_{-x}^1 f(x) = \mathcal{Z}_- f'(x) = \frac{2}{\pi} \int_0^\infty y^{-1} [f'(x + y) - f'(x) \cos y] dy. \tag{4.6b}$$

If $f(x) \in C^1$ and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$, then ${}_c D_x^1 f(x) = {}_c D_x^1 f(x)$. Similarly, if $f(x) \in C^1$ and $f(x) \rightarrow 0$ as $x \rightarrow \infty$, then ${}_c D_{-x}^1 f(x) = {}_c D_{-x}^1 f(x)$. We may also define a Riemann–Liouville form

Definition 4.5. The positive and negative Zolotarev–Riemann–Liouville derivatives of order one are defined by

$${}_{RL} D_x^1 f(x) = \frac{d}{dx} \mathcal{Z}_+ f(x) = \frac{2}{\pi} \frac{d}{dx} \int_0^\infty y^{-1} [f(x) \cos y - f(x - y)] dy \tag{4.7a}$$

$${}_{RL} D_{-x}^1 f(x) = \frac{d}{dx} \mathcal{Z}_- f(x) = -\frac{2}{\pi} \frac{d}{dx} \int_0^\infty y^{-1} [f(x) \cos y - f(x + y)] dy. \tag{4.7b}$$

In general, we do not expect the Zolotarev–Riemann–Liouville and Zolotarev–Caputo derivatives to be equal. Later, we will establish a relationship between the two forms on the half line.

Remark 4.6. The fractional derivative given by (3.4) is related to the RL fractional derivatives (2.1) via [35, (5.70)] by

$$\frac{d^\alpha}{d|x|^\alpha} f(x) = \frac{1}{2|\cos(\pi\alpha/2)|} \left(\frac{d^\alpha}{dx^\alpha} f(x) + \frac{d^\alpha}{d(-x)^\alpha} f(x) \right)$$

for $\alpha \neq 1$. Since $\psi_{\alpha,1}(-k) + \psi_{\alpha,-1}(-k) = 2\psi_{\alpha,0}(-k) = -2|k|^\alpha$, it follows from the definition of the Zolotarev fractional derivative (3.3) that

$$\frac{d^\alpha}{d|x|^\alpha} f(x) = \frac{1}{2} (D_x^{\alpha-1} f(x) + D_x^{\alpha-1} f(x)),$$

which is valid for all $0 < \alpha \leq 2$. In particular, for $\alpha = 1$, we have

$$\frac{d}{d|x|} f(x) = \frac{1}{2} (D_x^{1-1} f(x) + D_x^{1-1} f(x)), \tag{4.8}$$

which follows from (2.17). Hence, (4.3a) is the positive component of the Riesz derivative, while (4.3b) is the negative component.

Proposition 4.7. Let $f, g \in H^v \cap L^1$ where v is given by (3.9). Then the Zolotarev derivatives $D_x^{\alpha,\beta}$ and $D_x^{\alpha,-\beta}$ form an adjoint pair:

$$\int_{-\infty}^\infty D_x^{\alpha,\beta} f(x) g^*(x) dx = \int_{-\infty}^\infty f(x) D_x^{\alpha,-\beta} g^*(x) dx, \tag{4.9}$$

where “*” denotes complex conjugate. In addition, (4.7a) and (4.7b) form an adjoint pair.

Proof. Let $f(x) \leftrightarrow \hat{f}(k)$ and $g(x) \leftrightarrow \hat{g}(k)$ be FT pairs. Since both $f, g \in L^1 \cap L^2$, we apply Plancherel’s theorem [39, Eq. (22.5.4)],

$$\begin{aligned} \int_{-\infty}^\infty D_x^{\alpha,\beta} f(x) g^*(x) dx &= \frac{1}{2\pi} \int_{-\infty}^\infty \psi_{\alpha,\beta}(-k) \hat{f}(k) \hat{g}^*(k) dk \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \psi_{\alpha,-\beta}^*(-k) \hat{g}^*(k) \hat{f}(k) dk \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty (\psi_{\alpha,-\beta}(-k) \hat{g}(k))^* \hat{f}(k) dk, \end{aligned}$$

where we observed that $\psi_{\alpha,\beta}^*(k) = \psi_{\alpha,-\beta}(k)$ in the second line. (4.9) follows by application of Plancherel’s theorem. To show (4.7a) and (4.7b) are adjoints, note the following FTs

$$\mathcal{F}_x [\mathcal{Z}_+ f(x)] = \frac{2}{\pi} \ln(ik) \hat{f}(k) \tag{4.10a}$$

$$\mathcal{F}_x [\mathcal{Z}_- f(x)] = -\frac{2}{\pi} \ln(-ik) \hat{f}(k), \tag{4.10b}$$

which follow from (6.2). Multiplying (4.10a) and (4.10b) by ik yields the FT of (4.7a) and (4.7b) respectively, and since these FT are complex conjugates, it follows as in the first part of the proof that (4.7a) and (4.7b) form an adjoint pair. \square

If $\beta = 0$, Proposition 4.7 restates that the Riesz derivative is self-adjoint. The adjoint operator may be used to solve inverse problems, such as contaminant source prediction in hydrology [42] or parameter identification [43] for fractional diffusion problems.

5. Governing equation for anomalous diffusion with ballistic scaling

Armed with the Zolotarev derivative, we may now state and prove the central result of this paper.

Theorem 5.1. Let $D_x^{\alpha,\beta}$ be the Zolotarev derivative defined by (3.3). The point source solution of the fractional advection diffusion equation

$$\frac{\partial}{\partial t} u(x, t) + v \frac{\partial}{\partial x} u(x, t) = a D_x^{\alpha,\beta} u(x, t) \tag{5.1}$$

with velocity $v > 0$, fractional dispersion $a > 0$, and initial condition $u(x, 0) = \delta(x)$ is

$$u(x, t) = \begin{cases} \frac{1}{(at)^{1/\alpha} f_{\alpha,\beta}} \left(\frac{x - vt + \beta \tan \theta (at - (at)^{1/\alpha})}{(at)^{1/\alpha}} \right) & \alpha \neq 1; \\ \frac{1}{at} f_{1,\beta} \left(\frac{x - vt}{at} - \frac{2\beta}{\pi} \ln(at) \right) & \alpha = 1, \end{cases} \tag{5.2}$$

where $f_{\alpha,\beta}(x)$ is the standard stable density in the Zolotarev M parameterization given by (2.18).

Proof. Apply a Fourier transform \mathcal{F}_x followed by a Laplace transform \mathcal{L}_t to (5.1), yielding

$$\bar{u}(k, s) = \frac{1}{s + ivk - a\psi_{\alpha,\beta}(-k)}.$$

Apply an inverse LT, yielding

$$\hat{u}(k, t) = \exp(at\psi_{\alpha,\beta}(-k) - ivtk).$$

Put $\tau = at$. First, assume $\alpha \neq 1$ and put $\theta = \pi\alpha/2$. Since $\tau > 0$, we have

$$\begin{aligned} \tau \psi_{\alpha,\beta}(-k) &= -\tau |k|^\alpha [1 - i\beta \operatorname{sgn}(k) \tan(\theta) (|k|^{1-\alpha} - 1)] \\ &= -|k\tau|^{1/\alpha} [1 + i\beta \operatorname{sgn}(k) \tan \theta] + ik\tau \beta \tan \theta \\ &= -|w|^\alpha [1 + i\beta \operatorname{sgn}(w) \tan \theta] - iw\beta \tan \theta + iw\beta \tan \theta + \beta \tan \theta iw\tau^{1-1/\alpha}, \end{aligned}$$

where $w = k\tau^{1/\alpha}$. Rewrite as

$$\tau \psi_{\alpha,\beta}(-k) = \psi_{\alpha,\beta}(-w) - iw\tau^{-1/\alpha} \beta \tan \theta (\tau^{1/\alpha} - \tau)$$

and apply an inverse FT using the substitution above, yielding

$$u(x, t) = \frac{1}{2\pi \tau^{1/\alpha}} \int_{-\infty}^{\infty} \exp(\psi_{\alpha,\beta}(-w)) \exp(iw\tau^{-1/\alpha}(x - vt - \beta \tan \theta (\tau^{1/\alpha} - \tau))) dw.$$

Now (5.2) for $\alpha \neq 1$ follows from (2.18).

Now consider $\alpha = 1$:

$$\begin{aligned} \tau \psi_{1,\beta}(-k) &= -\tau |k| \left(1 - \frac{2i\beta}{\pi} \operatorname{sgn}(k) \ln |k| \right) \\ &= -|w| \left(1 - \frac{2i\beta}{\pi} \operatorname{sgn}(w) \ln \left| \frac{w}{k} \right| \right) \\ &= \psi_{1,\beta}(-w) - \frac{2\beta iw}{\pi} \ln \tau, \end{aligned}$$

where $w = \tau k$. Apply an inverse FT utilizing (2.18), yielding (5.2) for $\alpha = 1$. \square

Fig. 2 evaluates (5.2) for $\beta = 1$ (positively skewed) and $\alpha = 0.8, 1, 1.2,$ and 2 using MATLAB’s Statistics and Machine Learning Toolbox (r2016a) with $a = 1$ and $v = 0$. The left panel is evaluated at $t = 1$ and the right panel at $t = 3$. Note that the solutions smoothly transition between $\alpha > 1$ and $\alpha < 1$.

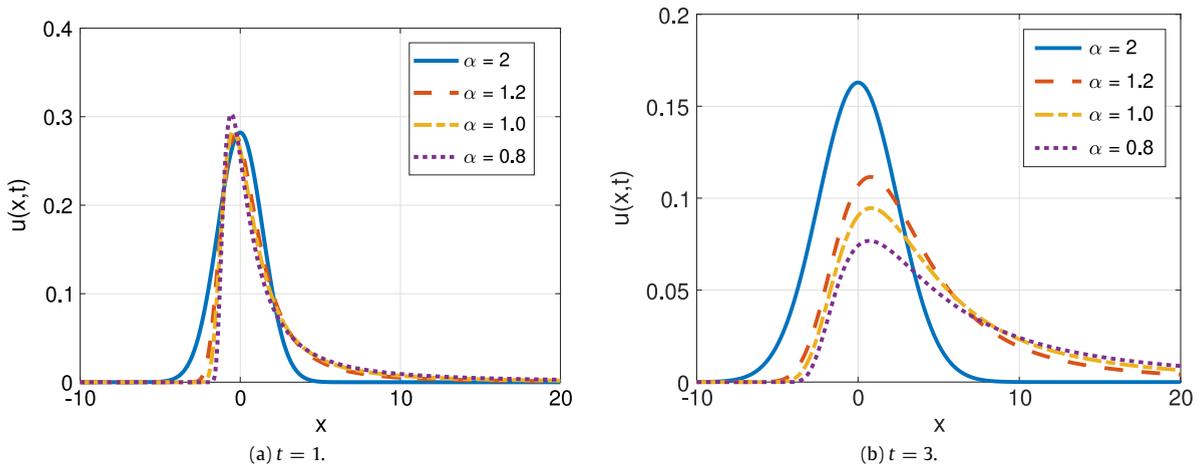


Fig. 2. Plot of (5.2) for $\beta = 1$ (positively skewed) and $\alpha = 0.8$ (dotted), 1 (dash-dotted), 1.2 (dashed), and 2 (solid). The left panel is evaluated at $t = 1$ and the right panel at $t = 3$. In both plots, $a = 1$ and $v = 0$.

Remark 5.2. In particular, if $\alpha = 1$, we have

$$\frac{\partial}{\partial t}u(x, t) + v \frac{\partial}{\partial x}u(x, t) = aD_x^{1,1}u(x, t) \tag{5.3}$$

which is the governing equation for Lévy motion of order $\alpha = 1$. If $\beta = 1$ and $a = \pi/2$, the point source solution is

$$u(x, t) = \frac{1}{t} f_L \left(\frac{x - vt}{t} - \ln t \right), \tag{5.4}$$

where $f_L(z)$ is the Landau distribution [13,14]. Because of the logarithmic correction, the mode of the plume is given by

$$x_c = tm_1 + vt + t \ln t, \tag{5.5}$$

where $m_1 = -0.22278$ is the mode of the Landau distribution. From a physical point of view, the “effective” velocity, which governs the location of the plume peak, is $v_{eff} = v + \ln t$. For $t < 1$, the effective velocity is less than v , while for $t > 1$, the effective velocity is greater than v .

Fig. 3 displays snapshots of (5.4) for times $t = 0.3, 0.6, 1.0$, and 1.5 for $v = 0$ (left) and $v = 2$ (right). In the case of no drift ($v = 0$), the peak (mode) moves slightly in the negative x direction for $t = 0.3, 0.6, 1.0$. For the choice of $v = 2$, the peak moves rightward for $t = 0.3, 0.6, 1.0$ since the drift term overpowers the logarithmic correction term.

By [44, (14.37)], $f_L(z) \sim z^{-2}$ for $z \gg 1$, yielding

$$u(x, t) \sim t^{-1} \left(\frac{x - vt}{t} - \ln t \right)^{-2} \tag{5.6}$$

for $x \gg xv + t \ln t$.

Remark 5.3. Space-fractional diffusion equation equations are often used in hydrology to model groundwater flows. In this application [45], the fractional exponent is often near one. Hence, a governing equation which is continuous with respect to α may be useful for parameter estimation [46] or source identification [42] in hydrology. In particular, (5.3) models anomalous super-diffusion with a ballistic spreading rate, and skewness.

Remark 5.4. This paper was motivated by a problem in biomedical acoustics: modeling the attenuation of acoustic energy in a biological tissue. Experimentally, the attenuation coefficient in most biological tissue is a power law with respect to angular frequency ω by $\alpha(\omega) = \alpha_0 |\omega|^y$, where the exponent y may vary from zero to two [22]. We modeled this wave propagation in 3D by a time-fractional PDE called the PLWE in [20]; however, the PLWE is not valid for $y = 1$. A simplified PLWE for acoustic pressure $p(z, t)$, valid for one-way propagation in one spatial dimension is given by [47]

$$\frac{1}{c_0} \frac{\partial}{\partial t} p(z, t) + \frac{\alpha_0}{\cos \theta} \frac{\partial^y}{\partial t^y} p(z, t) + \frac{\partial}{\partial z} p(z, t) = b(z, t), \tag{5.7}$$

where $\theta = \pi y/2$, c_0 is a reference speed of sound, and $b(z, t)$ is a boundary term given in [47, Section 5]. Like the 3D PLWE, (5.7) is not valid for $y = 1$. As mentioned in the introduction, the power-law exponent is very close to one in many

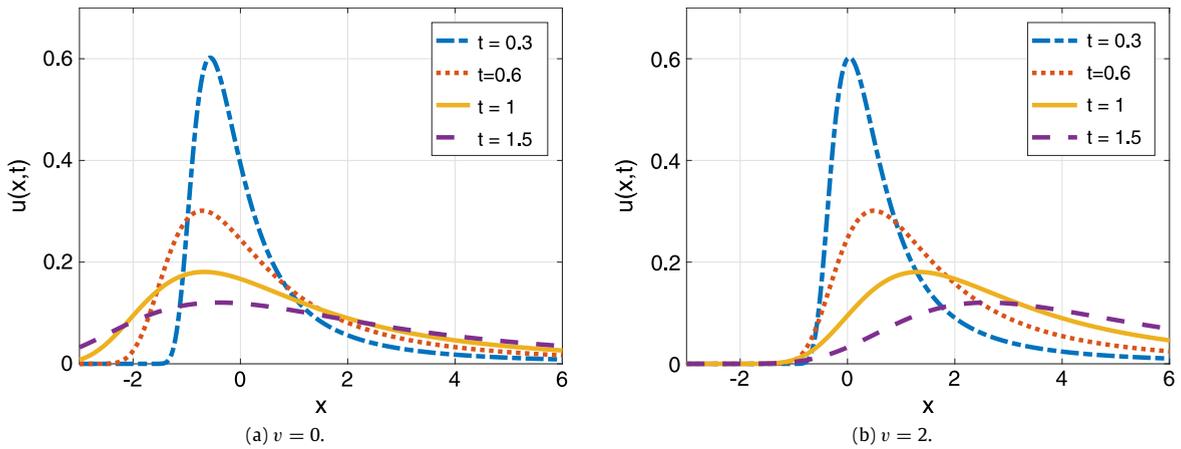


Fig. 3. Snapshots of (5.4) for (a) $v = 0$ and (b) $v = 2$ for four successive times: $t = 0.3$ (dash-dotted), $t = 0.6$ (dotted), $t = 1$ (solid), and $t = 1.5$ (dashed). In the case of no drift ($v = 0$), the peak (mode) moves in the negative x direction. For the choice of $v = 2$, the peak moves rightward.

measurements. Using the Zolotarev fractional derivative given by (3.11), Eq. (5.7) may be rewritten as

$$\frac{1}{c_1} \frac{\partial}{\partial t} p(z, t) - \alpha_0 D_t^{\gamma, 1} p(z, t) + \frac{\partial}{\partial z} p(z, t) = b(z, t), \tag{5.8}$$

where c_1 is an alternative reference frequency. Unlike the one-way PLWE (5.7), (5.8) is continuous for all $0 < y \leq 2$ and valid for the physically important case of $y = 1$.

Remark 5.5. In many practical applications, the coefficients of the governing FPDE (e.g. (5.1)) are not constant, or complicated boundary conditions are involved. Since analytical solutions are not available, these problems require stable, accurate, and efficient numerical methods to discretize the fractional derivative. The discretization of the Zolotarev fractional derivative is an interesting open problem. One approach is to use a shifted Grünwald approximation [48], which is conditionally stable and applicable to the advection–dispersion equation in hydrology. However, the Grünwald approximation for $\alpha = 1$ reduces to the first derivative. Alternatively, one may develop spectral methods based on eigenfunction expansions [49] or discontinuous spectral element approaches [50], which reformulate the governing equation in a weak form. The adjoint formula for the Zolotarev derivative given by (4.9) may be interpreted as an integration by parts formula, which is an ingredient for writing a weak form. A third common approach, used to discretize the Caputo derivative, applies a Riemann sum approximation to the fractional integral [51], such as (6.5) in the next section. Emerging approaches for nonlinear FPDEs include the variational iteration method and Adomain decomposition method [52].

6. Zolotarev–Caputo and Zolotarev–Riemann–Liouville derivatives on the half-axis

Initial-value problems (IVPs) require functions and operators defined on the half-axis. The fractional Caputo derivative is useful for capturing nonlocal temporal phenomena, such as fractional relaxation in viscoelasticity [53] and long waiting times in hydrology [54]. Therefore, we investigate the Zolotarev derivatives of order one on the half-line in this section.

A function $f(t)$ is termed *causal* if $f(t) = 0$ if $t < 0$. In this section, we restrict our attention to causal functions.

Proposition 6.1. Given a causal function $f(t)$ with Laplace transform $\tilde{f}(s)$ defined by (2.12)

$$\mathcal{L}_t [\mathcal{Z}_+ f(t)] = \frac{2}{\pi} \ln s \tilde{f}(s). \tag{6.1}$$

Proof. Recall the shifting property of Laplace transforms: $\mathcal{L}_t [f(t - y)] = e^{-sy} \tilde{f}(s)$. Interchange the Laplace transform with the integral, yielding

$$\mathcal{L}_t [\mathcal{Z}_+ f(t)] = \frac{2}{\pi} \int_0^\infty y^{-1} [\cos y - e^{-sy}] dy \tilde{f}(s) = \frac{2}{\pi} \ln s \tilde{f}(s),$$

where the identity

$$\ln s = \int_0^\infty y^{-1} [\cos y - e^{-sy}] dy \tag{6.2}$$

is invoked (see Appendix). □

Corollary 6.2. Given a causal function $f(t)$ with Laplace transform $\tilde{f}(s)$,

$$\mathcal{L}_t [{}_C D_t^1 f(t)] = \frac{2}{\pi} \tilde{f}(s) \ln s - \frac{2}{\pi} f(0) \ln s, \tag{6.3}$$

and

$$\mathcal{L}_t [{}_{RL} D_t^1 f(t)] = \frac{2}{\pi} \tilde{f}(s) \ln s - \mathcal{Z}_+ f(0^+). \tag{6.4}$$

Proof. Recall that $\mathcal{L}_t [f'(t)] = s\tilde{f}(s) - f(0)$. Apply Proposition 6.1, yielding (6.3). To establish (6.4), integrate by parts, yielding

$$\begin{aligned} \mathcal{L}_t [{}_{RL} D_t^1 f(t)] &= \int_0^\infty e^{-st} \frac{\partial}{\partial t} \mathcal{Z}_+ f(t) dt \\ &= s \int_0^\infty e^{-st} \mathcal{Z}_+ f(t) dt \\ &= \frac{2}{\pi} s \ln \tilde{f}(s) - \mathcal{Z}_+ f(0^+). \quad \square \end{aligned}$$

With Corollary 6.2, we can relate the Caputo and Riemann–Liouville derivatives of order one.

Proposition 6.3. Given a causal function $f(t)$

$${}_C D_t^1 f(t) = {}_{RL} D_t^1 f(t) - \Phi(t)f(0) \tag{6.5}$$

where

$$\Phi(t) = -\frac{2}{\pi} \left(\frac{H(t)}{t} + \gamma \delta(t) \right) + \mathcal{Z}_+ f(0^+) \delta(t). \tag{6.6}$$

Proof. Subtract (6.4) from (6.1), yielding

$$\mathcal{L}_t [{}_C D_t^1 f(t)] - \mathcal{L}_t [{}_{RL} D_t^1 f(t)] = -\frac{2}{\pi} f(0) \ln s + \mathcal{Z}_+ f(0^+).$$

By the Laplace transform pair [55, Appendix 4]

$$\mathcal{L}_t \left(\frac{H(t)}{t} \right) = -\gamma - \ln s, \tag{6.7}$$

where γ is again the Euler–Mascheroni constant, and then (6.5) follows. \square

Remark 6.4. Note that (6.6) takes the role of the Gel'fand–Shilov function [53,56] $H(t)t^{\alpha-1}/\Gamma(\alpha)$ that relates the standard Caputo and Riemann–Liouville derivatives. Unlike the Gel'fand–Shilov function, (6.6) consists of both a global component and a local (Dirac) component.

We close this section with an explicit formula for the Caputo derivative of order one for causal functions.

Proposition 6.5. Let $f(t)$ be a differentiable causal function. Then

$${}_C D_t^1 [f(t)] = \frac{2}{\pi} \left[(-\gamma - \ln t) f'(t) + \int_0^t y^{-1} (f'(t) - f'(t-y)) dy \right]. \tag{6.8}$$

Proof. Letting $f(t) = H(t)g(t)$, we have

$$\begin{aligned} \mathcal{Z}_+ [H(t)g(t)] &= \frac{2}{\pi} H(t) \int_0^\infty y^{-1} (g(t) \cos y - H(t-y)g(t-y)) dy \\ &= \frac{2}{\pi} H(t) \left[\int_0^t y^{-1} (g(t) \cos y - g(t-y)) dy + g(t) \int_t^\infty \frac{\cos y}{y} dy \right] \\ &= \frac{2}{\pi} H(t) \left[\int_0^t y^{-1} (g(t) \cos y - g(t-y)) dy \right. \\ &\quad \left. - g(t) \left(\gamma + \ln t + \int_0^t \frac{\cos y - 1}{y} dy \right) \right] \\ &= \frac{2}{\pi} H(t) \left[(-\gamma - \ln t) g(t) + \int_0^t y^{-1} (g(t) - g(t-y)) dy \right], \end{aligned}$$

where we used the following identity involving the Cosine integral $Ci(z)$ [57]

$$Ci(z) = - \int_z^\infty \frac{\cos y}{y} dy = \gamma + \ln z + \int_0^z \frac{\cos y - 1}{y} dy. \tag{6.9}$$

Proposition 6.5 follows by (4.6a). \square

Remark 6.6. By the previous proposition, ${}_c D_t^1$ has two components: a local component, consisting of differentiation multiplied by a logarithmic weight, and a nonlocal component consisting of differentiation followed by a regularized convolution against $1/y$.

7. Examples

We close with several explicit examples of Zolotarev–Riemann–Liouville and Zolotarev–Caputo derivatives of order one.

Example 7.1. Let $f(x) = e^{\lambda x}$, where $\lambda > 0$. Then

$$\begin{aligned} \mathcal{Z}_+ [e^{\lambda x}] &= \frac{2}{\pi} \int_0^\infty y^{-1} [e^{\lambda x} \cos y - e^{\lambda(x-y)}] dy \\ &= \frac{2}{\pi} e^{\lambda x} \int_0^\infty y^{-1} [\cos y - e^{-\lambda y}] dy \\ &= \frac{2}{\pi} e^{\lambda x} \ln \lambda \end{aligned}$$

where (6.2) is again used. Taking a derivative yields

$${}_{RL} D_x^1 [e^{\lambda x}] = \frac{2}{\pi} \lambda \ln \lambda e^{\lambda x}. \tag{7.1}$$

The Zolotarev–Caputo derivative of order one of $e^{\lambda x}$ is identical. If $\lambda < 1$, this derivative is negative; if $\lambda = 1$, the derivative is zero.

Now apply (3.11) to $e^{\lambda x}$ with $\beta = 1$. Using [9, Ex. 2.6] yields

$$\begin{aligned} D_x^{\alpha,1} [e^{\lambda x}] &= \tan(\pi\alpha/2) \lambda e^{\lambda x} - \frac{1}{\cos(\pi\alpha/2)} \lambda^\alpha e^{\lambda x} \\ &= \frac{\lambda e^{\lambda x}}{\cos(\pi\alpha/2)} (\sin(\pi\alpha/2) - \lambda^{\alpha-1}). \end{aligned}$$

Take the limit as $\alpha \downarrow 1$. Let $\epsilon = \alpha - 1$ and note $\cos(\pi\alpha/2) = -\sin(\pi\epsilon/2) \sim -\pi\epsilon/2$ for $\epsilon \ll 1$. Letting $\epsilon \rightarrow 0$ yields

$$\begin{aligned} \lim_{\alpha \downarrow 1} D_x^{\alpha,1} [e^{\lambda x}] &= \lambda e^{\lambda x} \lim_{\epsilon \downarrow 0} \frac{\lambda^\epsilon - 1}{\pi\epsilon/2} \\ &= \frac{2}{\pi} \lambda \ln \lambda e^{\lambda x} \end{aligned}$$

where (2.22) is invoked in the second line. Since the Zolotarev fractional derivative is continuous in α , by Theorem 3.11, we conclude that the Zolotarev derivative agrees with the Zolotarev–Riemann–Liouville in this case, for $\alpha = 1$.

Example 7.2. Let $f(x) = H(x)x^\alpha$ where $\alpha > 0$. Put $g(x) = x^\alpha$ and apply (6.8). Then

$$\begin{aligned} {}_c D_x^1 [H(x)x^\alpha] &= H(x) \frac{2}{\pi} \left[(-\gamma - \ln x) \alpha x^{\alpha-1} + \alpha \int_0^x y^{-1} (x^{\alpha-1} - (x-y)^{\alpha-1}) dy \right] \\ &= H(x) \frac{2\alpha}{\pi} [(-\gamma - \ln x) x^{\alpha-1} + x^{\alpha-1} (\gamma + \psi(\alpha))] \\ &= H(x) \frac{2\alpha}{\pi} x^{\alpha-1} [\psi(\alpha) - \ln x] \end{aligned} \tag{7.2}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function [57]. In this example, the Zolotarev–Caputo derivative of order one consists of the power rule result $\alpha x^{\alpha-1}$ multiplied by a logarithmic term. If $\alpha = 1$, then $\psi(1) = \gamma$, and we have

$${}_c D_x^1 [H(x)x] = -H(x) \frac{2}{\pi} (\gamma + \ln x).$$

Note that the Zolotarev–Caputo derivative of order one of a power function has a singularity of at $x = 0$ for $\alpha \leq 1$.

Fig. 4 displays plots of (7.2) for $\alpha = 2, 1, 0.5$, and 0.05 . We remark that for $\alpha = 0.05$ the plot in Fig. 4 eventually tends to $+\infty$, but that is not visible at this scale.

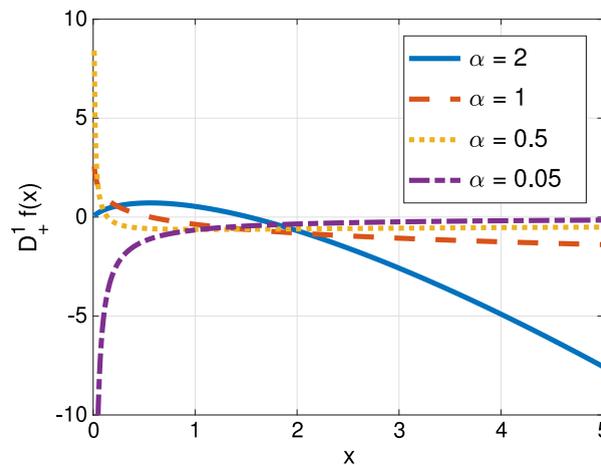


Fig. 4. Plot of (7.2) for $\alpha = 2$ (solid), 1 (dashed), 0.5 (dotted), and 0.05 (dash-dotted).

8. Conclusion

This paper has presented a fractional diffusion equation valid for all orders $0 < \alpha \leq 2$, including the important but often neglected case of $\alpha = 1$. Eq. (5.3) models anomalous diffusion with arbitrary skewness and ballistic (linear) scaling, using a new operator that we term the Zolotarev fractional derivative. The point source solution of this equation spreads at rate $t^{1/\alpha}$ away from its center of mass, including the case $\alpha = 1$ of ballistic scaling. The Fourier symbol of this operator is the log-characteristic function of a stable law in the Zolotarev M parameterization. For the special case of positive skewness ($\beta = 1$), the point source solution of (5.3) can be written using the Landau distribution from nuclear physics. The Zolotarev fractional derivative, defined by (3.3), is a nonlocal operator. The generator, Riemann–Liouville, and Caputo forms of the Zolotarev derivative of order one were investigated on both the real line and the positive half-line. We proved that the Zolotarev fractional derivative is continuous at $\alpha = 1$. For problems on the half-line, the Caputo form (6.8) illustrates how this new fractional derivative consists of both a local component involving the natural logarithm, and a global component. Two concrete examples were given, illustrating the behavior of this new derivative.

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Appendix

Let $x \geq 0$ and define

$$f(x; s) = \int_0^\infty e^{-xy} y^{-1} (\cos y - e^{-sy}) dy$$

where $f(x; s) \rightarrow 0$ as $x \rightarrow \infty$. Then

$$\begin{aligned} f'(x; s) &= - \int_0^\infty e^{-xy} \cos y dy + \int_0^\infty e^{-(x+s)y} dy \\ &= - \frac{x}{1+x^2} + \frac{1}{x+s}. \end{aligned}$$

Integrate, yielding

$$\begin{aligned} f(x; s) &= - \frac{1}{2} \ln(1+x^2) + \ln(x+s) + C \\ &= \ln \left(\frac{x+s}{\sqrt{1+x^2}} \right) + C. \end{aligned}$$

By letting $x \rightarrow \infty$, we determine $C = 0$, implying

$$\int_0^{\infty} e^{-xy} y^{-1} (\cos y - e^{-sy}) dy = \ln \left(\frac{x+s}{\sqrt{1+x^2}} \right). \quad (\text{A.1})$$

Setting $x = 0$ in (A.1) yields (6.2).

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