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Attenuated Fractional Wave Equations With Anisotropy

This paper develops new fractional calculus models for wave propagation. These models permit a different attenuation index in each coordinate to fully capture the anisotropic nature of wave propagation in complex media. Analytical expressions that describe power law attenuation and anomalous dispersion in each direction are derived for these fractional calculus models. [DOI: 10.1115/1.4025940]

1 Introduction

The attenuation coefficient $\alpha(\omega)$ for sound waves in a complex heterogeneous medium often follows a power law $\alpha(\omega) = \alpha_0|\omega|^y$ that depends on the frequency ω. The goal of fractional wave equations is to capture this power law attenuation in a convenient and realistic physical model. In practical applications, where the medium is not isotropic, the power law attenuation index y can vary with the coordinate. For example, a layered medium may exhibit a stronger anisotropy across layers than within layers. This paper develops space-fractional wave equations, where the attenuation index can vary with the coordinate, to model anisotropic sound propagation in complex media with independent power law attenuation and anomalous dispersion relations in each direction.

2 Fractional Calculus

Here we recall some relevant facts about fractional calculus, including fractional vector calculus. The fractional divergence $\nabla^{\vec{a}} \cdot \vec{u}$ is most simply defined in terms of its Fourier transform. Let

$$\vec{U}(\vec{k}) = \int e^{-i\vec{k}\cdot\vec{x}}\vec{u}(\vec{x})d\vec{x}$$

be the vector-valued Fourier transform of the d-dimensional vector field $\vec{u}(\vec{x})$. Then $\nabla^{\vec{a}} \cdot \vec{u}(\vec{x})$ is defined as the function whose Fourier transform is

$$\begin{pmatrix} (ik_1)^{a_1} \\ \vdots \\ (ik_d)^{a_d} \end{pmatrix} \cdot \vec{U}(\vec{k})$$

Then it follows that for any real-valued function $p(\vec{x})$ we have

$$\nabla^{\vec{a}} \cdot \nabla p(\vec{x}) = \Delta^{\vec{r}/2} p(\vec{x}) \tag{1}$$

where the vector order of fractional integration is $\vec{r} = \vec{a} + \vec{1}$ and the anisotropic fractional Laplacian $\Delta^{\vec{r}/2}p(\vec{x})$ has Fourier transform

$$-\sum_{j=1}^{d} (ik_j)^{r_j} P(\vec{k}) \tag{2}$$

If $\vec{a} = \vec{1}$ then $\vec{r}/2 = \vec{1}$ as well, and $\Delta^{\vec{r}/2}$ is the usual Laplacian. Here

$$P(\vec{k}) = \int e^{-i\vec{k}\cdot\vec{x}} p(\vec{x})d\vec{x}$$

is the scalar-valued Fourier transform of the d-dimensional scalar field $p(\vec{x})$. Note that $\nabla^{\vec{a}}p$ is a vector, while $\Delta^{\vec{r}/2}p = \nabla^{\vec{a}} \cdot \nabla p$ is a scalar. To see that (1) holds, recall that the gradient operator has the Fourier symbol $i\vec{k}$, so that for any scalar field $p(\vec{x})$, the vector $\nabla p(\vec{x})$ has Fourier transform $(i\vec{k})P(\vec{k})$, and then it follows that the left-hand side of (1) has Fourier transform

$$\begin{pmatrix} (ik_1)^{a_1} \\ \vdots \\ (ik_d)^{a_d} \end{pmatrix} \cdot \begin{pmatrix} ik_1 \\ \vdots \\ ik_d \end{pmatrix} P(\vec{k})$$

which reduces to the expression (2), which is the Fourier transform of the right-hand side of (1) after applying the dot product and simplifying. See [1] for more details on vector fractional calculus.

The (positive) Riemann–Liouville fractional derivative $D_x^r f(x)$ of order $r > 0$ for a real-valued function $f(x)$ of the real variable x can be defined by

$$D_x^r f(x) = \frac{d^r f(x)}{dx^r} = \frac{1}{\Gamma(n-r)} \frac{d^n}{dx^n} \int_{-\infty}^{x} \frac{f(u)du}{(x-u)^{r+1-n}}$$

where n is an integer such that $n - 1 < r \le n$. The Fourier transform of $D_x^r f(x)$ is $(ik)^r \hat{f}(k)$, where

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx$$

is the one variable Fourier transform. See [2,3] for more details. Then we can write

$$\Delta^{\vec{r}/2} p(\vec{x}) = \sum_{j=1}^{d} \frac{\partial^{r_j}}{\partial x_j^{r_j}} p(x_1, \ldots, x_d) \tag{3}$$

a sum of Riemann–Liouville (partial) derivatives in each variable, where the order of fractional differentiation depends on the coordinate.

For a simple exponential function $f(x) = e^{bx}$ it is straightforward [2, example 2.6] to compute that

$$D_x^r \left[e^{bx}\right] = b^r e^{bx} \tag{4}$$

which extends the familiar integer-order derivative formula. Then it follows that

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$$\Delta^{\vec{r}/2}\left[e^{\vec{b}\cdot\vec{x}}\right]=\sum_{j=1}^{d} b_j^{r_j} e^{\vec{b}\cdot\vec{x}} \quad (5)$$

where $\vec{b}=(b_1,\ldots,b_d)'$. If we take $\vec{b}=(0,\ldots,0,b_j,0,\ldots,0)'$ pointing along the jth coordinate direction, then in this special case it follows that $\Delta^{\vec{r}/2}\left[e^{\vec{b}\cdot\vec{x}}\right]=b_j^{r_j}e^{\vec{b}\cdot\vec{x}}$, a formula we will use later in this paper.

The (positive) Riemann–Liouville fractional integral of order $r>0$ is defined by

$$I_x^r f(x)=\frac{1}{\Gamma(r)}\int_{-\infty}^{x}\frac{f(u)du}{(x-u)^{1-r}}$$

and the Fourier transform of $I_x^r f(x)$ is $(ik)^{-r}\hat{f}(k)$. This is the inverse of the Riemann–Liouville fractional derivative: $D_x^r I_x^r f(x)=f(x)$. One can also view the Riemann–Liouville fractional derivative $D_x^r f(x)$ as the integer derivative of a Riemann–Liouville fractional integral: $D_x^r f(x)=D_x^n I_x^{n-r} f(x)$.

3 Acoustical Variables

The total pressure P, equilibrium pressure P_0, and excess pressure (or acoustic pressure) p are related according to

$$P=P_0+p$$

which represents a first order Taylor expansion about the equilibrium pressure value. The total density ρ, equilibrium density ρ_0, and condensation s are related by

$$\rho=\rho_0+\rho_0 s$$

which is a first order Taylor expansion about the equilibrium value of the density. In nonlinear and linear acoustics, the wave equation is often expressed in terms of the acoustic pressure p, and the constitutive equations are written in terms of the total or acoustic pressure, the total/equilibrium density and/or the condensation, and the vector particle velocity $\vec{u}$.

4 Constitutive Relations of Nonlinear and Linear Acoustics

The nonlinear constitutive relations for acoustics describe the relationships between the total or acoustic pressure, the total density, and the particle velocity through the nonlinear equation of state, the nonlinear equation of motion, and the nonlinear equation of continuity. The linear constitutive relations likewise describe the relationships between the acoustic pressure, the condensation, and the particle velocity through the linear versions of these equations. In particular, the nonlinear equation of state relates the total pressure P to the total density ρ by the relation

$$P=f(\rho)$$

and the linearized equation of state relates the acoustic pressure p to the condensation s by

$$p=\rho_0 c^2 s \quad (6)$$

The nonlinear equation of motion, neglecting gravity, is

$$-\nabla P=\rho\left[\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}\right]$$

and the linearized equation of motion is

$$\rho_0\frac{\partial\vec{u}}{\partial t}=-\nabla p \quad (7)$$

The nonlinear equation of continuity is

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{u})=0$$

and the linearized equation of continuity is

$$\frac{\partial s}{\partial t}+\nabla\cdot\vec{u}=0 \quad (8)$$

The linear wave equation is obtained from the above linear constitutive relations, as shown in Kinsler et al. [4].

5 Fractional Constitutive Relations

Here the equation of continuity is extended to the fractional case through application of the fractional divergence. The fractional nonlinear equation of continuity then becomes

$$\frac{\partial\rho}{\partial t}+\nabla^{\vec{a}}\cdot(\rho\vec{u})=0 \quad (9)$$

and the corresponding fractional linear equation of continuity is

$$\frac{\partial s}{\partial t}+\nabla^{\vec{a}}\cdot\vec{u}=0 \quad (10)$$

The fractional linear equation of continuity (10) enables the derivation of the following fractional linear wave equation.

6 Fractional Anisotropic Wave Equation

Combine the linearized state equation (6) with the linearized fractional continuity equation (10) to obtain

$$\frac{1}{\rho_0 c^2}\frac{\partial p}{\partial t}+\nabla^{\vec{a}}\cdot\vec{u}=0$$

Then, take the partial with respect to time of both sides

$$\frac{1}{\rho_0 c^2}\frac{\partial^2 p}{\partial t^2}+\frac{\partial}{\partial t}\nabla^{\vec{a}}\cdot\vec{u}=0 \quad (11)$$

Next, apply the fractional divergence operator to the equation of motion (7) to obtain

$$\rho_0\nabla^{\vec{a}}\cdot\frac{\partial\vec{u}}{\partial t}=-\nabla^{\vec{a}}\cdot\nabla p=-\Delta^{\vec{r}/2}p$$

using (1). Then, combine with (11), which yields

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}=\Delta^{\vec{r}/2}p \quad (12)$$

This anisotropic fractional wave equation extends the isotropic forms considered in Mainardi [5] to allow a different fractional space derivative in each coordinate.

6.1 Plane Wave Solution. Assume a plane wave solution $p(\vec{x},t)=e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ with complex $\vec{k}$ and note that a general steady state solution can be approximated by a linear combination of plane waves. From (5) we have

$$\Delta^{\vec{r}/2}p(\vec{x},t)=\sum_{j=1}^{d}(ik_j)^{r_j}p(\vec{x},t)$$

Substitute into (12) and divide by $p(\vec{x},t)$ to arrive at the characteristic equation

$$\frac{1}{c^2}(-i\omega)^2 = \sum_{j=1}^{d}(ik_j)^{r_j}$$

for a general wave vector $\vec{k} = (k_1,\ldots,k_d)'$. For a plane wave in the j coordinate direction, where $\vec{k} = (0,\ldots,0,k_j,0,\ldots,0)'$, it follows that $(ik)^r = (-i\omega)^2/c^2$, where we write $k = k_j$ and $r = r_j$ for simplicity. Then the complex wave number k in this coordinate satisfies

$$\begin{aligned} ik &= \frac{(-i\omega)^{2/r}}{c^{2/r}} \\ k &= -i\frac{|\omega|^{2/r}}{c^{2/r}}\left[\cos\left(\frac{\pi}{r}\right) - i\,\mathrm{sgn}(\omega)\sin\left(\frac{\pi}{r}\right)\right] \\ k &= \frac{|\omega|^y}{c^y}\left[-\mathrm{sgn}(\omega)\sin\left(\frac{\pi y}{2}\right) - i\cos\left(\frac{\pi y}{2}\right)\right] \end{aligned} \tag{13}$$

where the attenuation index $y = 2/r$. Since $r = a+1$ where $0 < a < 1$, it follows that $1 < y < 2$. Writing $k = k_j(\omega) = \omega/c_j(\omega) + i\alpha_j(\omega)$ for an outward-going wave, and noting that $\cos(\pi y/2) < 0$ for $1 < y < 2$, the attenuation in the jth coordinate direction is

$$\alpha_j(\omega) = \frac{|\omega|^{y_j}}{c^{y_j}}\left|\cos\left(\frac{\pi y_j}{2}\right)\right|$$

as a function of frequency ω. This shows that the anisotropic fractional wave equation (12) exhibits power law attenuation with attenuation index $y_j = 2/r_j$ in the jth coordinate direction, with $1 < y_j < 2$ depending on the coordinate. Noting that $\sin(\pi y/2) > 0$ for $1 < y < 2$, the phase speed (also called the dispersion) as a function of frequency $\omega > 0$ in the jth coordinate direction is

$$c_j(\omega) = \frac{c^{y_j}}{\omega^{y_j-1}}\left|\csc\left(\frac{\pi y_j}{2}\right)\right|$$

which also varies with the coordinate. The phase speed decreases in each direction as the frequency increases, indicating anomalous (or negative) dispersion. Anomalous dispersion frequently occurs in bone [6], which is also anisotropic [7]. Anomalous dispersion has also been demonstrated in plates of Lexan with a step discontinuity, and in a Lexan plate bonded to a Plexiglas plate [8]. Examples of models that predict anomalous dispersion in bone include a model that superposes fast and slow waves [9] and a multiple scattering model that also incorporates absorption [10].

The solution (13) is not unique, because for any $z = e^{in\pi y}$ we have $z^r = 1$, and hence $(ikz)^r = (-i\omega)^2/c^2$ as well. It follows that the general solution is

$$k = \frac{|\omega|^y}{c^y}\left[\sin\left(n\pi y - \mathrm{sgn}(\omega)\frac{\pi y}{2}\right) - i\cos\left(n\pi y - \mathrm{sgn}(\omega)\frac{\pi y}{2}\right)\right]$$

where n is any integer. The power law scaling of attenuation and dispersion are the same for any solution.

Following the same steps with (10) replaced by the traditional continuity equation

$$\frac{\partial s}{\partial t} + \nabla\cdot\vec{u} = 0$$

leads to the traditional isotropic wave equation

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \nabla\cdot\nabla p = \Delta p \tag{14}$$

The anisotropic version of this equation is

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \nabla\cdot J\nabla p \tag{15}$$

where the two-tensor J accounts for anisotropy. For example, if

$$J = \begin{pmatrix} D_1^2 & 0 & \cdots & 0 \\ 0 & D_2^2 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & D_d^2 \end{pmatrix}$$

then the speed of sound cD_i is different in each coordinate. The three operators on the right-hand side of (15), $\nabla\cdot J\nabla p$, account for (reading from right to left) motion, anisotropy, and continuity.

The fractional divergence can also be written in the form $\nabla^{\vec{a}}\vec{u} = \nabla\cdot J^{\vec{a}-\vec{1}}\vec{u}$, where the fractional integration tensor $J^{\vec{a}-\vec{1}}$ applies a fractional integral in each coordinate. The fractional integration tensor is defined so that the formula

$$\begin{pmatrix} (ik_1)^{a_1-1} & 0 & \cdots & 0 \\ 0 & (ik_2)^{a_2-1} & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & (ik_d)^{a_d-1} \end{pmatrix}\vec{U}(\vec{k})$$

equals the Fourier transform of $J^{\vec{a}-\vec{1}}\vec{u}(\vec{x})$. This operator applies a Riemann–Liouville fractional integral of order $1-a_j$ in each coordinate direction $j = 1,\ldots,d$. Then we can also write

$$\nabla^{\vec{a}}\cdot\nabla p(\vec{x}) = \nabla\cdot J^{\vec{a}-\vec{1}}\nabla p(\vec{x}) = \Delta^{\vec{r}/2}p(\vec{x})$$

which extends the traditional anisotropic diffusion operator. With this notation, the anisotropic fractional wave equation (12) can be rewritten in the form

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \nabla\cdot J^{\vec{a}-\vec{1}}\nabla p \tag{16}$$

in which the two-tensor J in (15) is replaced by a fractional integration tensor. Both model anisotropy. The anisotropic wave equation (15) models *mild anisotropy* where the wave propagation is essentially the same in each coordinate, just at a different speed. The anisotropic fractional wave equation (16) models *strong anisotropy* where the power law attenuation index varies with the coordinate. An even more general model is

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \nabla\cdot J^{\vec{a}-\vec{1}}J\nabla p \tag{17}$$

where the speed of sound and the power law attenuation index both vary with the coordinate. In fact, it is not hard to show, following the same steps as before, that the attenuation

$$\alpha_j(\omega) = \frac{|\omega|^{y_j}}{(cD_j)^{y_j}}\left|\cos\left(\frac{\pi y_j}{2}\right)\right|$$

and the phase speed

$$c_j(\omega) = \frac{(cD_j)^{y_j}}{\omega^{y_j-1}}\left|\csc\left(\frac{\pi y_j}{2}\right)\right|$$

for the general model (17).

7 Fractional Stokes Wave Equation

A related fractional wave equation can be derived by combining the fractional continuity equation (10) with a more general equation of state that also contains a first partial derivative of the condensation with respect to time,

$$p = \rho_0 c^2 \left(s + \tau \frac{\partial s}{\partial t} \right) \tag{18}$$

to obtain

$$p = \rho_0 c^2 \left(s - \tau \nabla^{\vec{a}} \cdot \vec{u} \right)$$

Take the partial with respect to time of both sides

$$\frac{\partial p}{\partial t} = \rho_0 c^2 \left(\frac{\partial s}{\partial t} - \tau \frac{\partial}{\partial t} \nabla^{\vec{a}} \cdot \vec{u} \right)$$

and again insert the fractional continuity equation (10), yielding

$$\frac{\partial p}{\partial t} = \rho_0 c^2 \left(-\nabla^{\vec{a}} \cdot \vec{u} - \tau \frac{\partial}{\partial t} \nabla^{\vec{a}} \cdot \vec{u} \right) \tag{19}$$

Take one more derivative in time to get

$$\frac{\partial^2 p}{\partial t^2} = \rho_0 c^2 \left(-\nabla^{\vec{a}} \cdot \frac{\partial}{\partial t} \vec{u} - \tau \frac{\partial}{\partial t} \nabla^{\vec{a}} \cdot \frac{\partial}{\partial t} \vec{u} \right) \tag{20}$$

Now evaluate the fractional divergence of both sides of the linearized equation of motion (7) and then apply (1) to get

$$\rho_0 \nabla^{\vec{a}} \cdot \frac{\partial}{\partial t} \vec{u} = -\nabla^{\vec{a}} \cdot \nabla p = -\Delta^{\vec{r}/2} p \tag{21}$$

and finally, combine with the previous equation (20) to obtain

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \Delta^{\vec{r}/2} p + \tau \frac{\partial}{\partial t} \Delta^{\vec{r}/2} p \tag{22}$$

This expression is recognized as the Stokes wave equation with both Laplacian operators replaced by anisotropic fractional Laplacians. Following the derivation for the Stokes wave equation given in [11], the expression (22) is also obtained for compressional waves when the linearized Navier–Stokes equation is combined with the equation of state (6) and the fractional continuity equation (10).

7.1 Plane Wave Solution. As in Sec. 6.1, we assume a plane wave solution $p(\vec{x}, t) = e^{i(\vec{k} \cdot \vec{x} - \omega t)}$, substitute into (22), and cancel the common p term on both sides to get the characteristic equation

$$\frac{1}{c^2} (-i\omega)^2 = [1 - i\omega\tau] \sum_{j=1}^{d} (ik_j)^{r_j}$$

For a plane wave in the j coordinate, this reduces to

$$(ik)^r = \frac{1}{c^2} \frac{(-i\omega)^2}{1 - i\omega\tau}$$

where $r = r_j$, and $k = k_j$ is the wave number in this coordinate. In the low frequency limit $\omega\tau \ll 1$, we can apply the approximation $(1 - i\omega\tau)^{-r} \approx 1 + ir\omega\tau$ to see that

$$\begin{aligned} ik &= \frac{(-i\omega)^{2/r}}{c^{2/r}} (1 - i\omega\tau)^{-r} \\ k &\approx -i \frac{|\omega|^{2/r}}{c^{2/r}} [-i\text{sgn}(\omega)]^{2/r} (1 + ir\omega\tau) \\ k &\approx \frac{|\omega|^y}{c^y} \left[-\text{sgn}(\omega) \sin\left(\frac{\pi y}{2}\right) + \frac{2\omega\tau}{y} \cos\left(\frac{\pi y}{2}\right) \right] \\ &\quad + i \frac{|\omega|^y}{c^y} \left[-\cos\left(\frac{\pi y}{2}\right) - \frac{2\omega\tau}{y} \text{sgn}(\omega) \sin\left(\frac{\pi y}{2}\right) \right] \end{aligned} \tag{23}$$

where again $y = 2/r$ is a real number between 1 and 2. Writing $k = k_j(\omega) = \omega / c_j(\omega) + i\alpha_j(\omega)$ for an outward-propagating wave, the attenuation at frequencies $1/\tau \gg \omega > 0$ is

$$\alpha_j(\omega) \approx \frac{|\omega|^{y_j}}{c^{y_j}} \left| \cos\left(\frac{\pi y_j}{2}\right) + \frac{2\omega\tau}{y_j} \sin\left(\frac{\pi y_j}{2}\right) \right|$$

This shows that the fractional wave equation (12) exhibits power law attenuation at low frequencies with attenuation index $y_j = 2/r_j$ in the jth coordinate direction, where the attenuation index $1 < y_j < 2$ depends on the coordinate. The phase speed or dispersion as a function of frequency $1/\tau \gg \omega > 0$ in the jth coordinate direction is

$$c_j(\omega) = \frac{c^{y_j}}{\omega^{y_j - 1}} \left| \sin\left(\frac{\pi y_j}{2}\right) - \frac{2\omega\tau}{y_j} \cos\left(\frac{\pi y_j}{2}\right) \right|^{-1}$$

so that the phase speed also varies with the coordinate in the attenuated model (22). For small frequencies $\omega\tau \ll 1$, the phase speed in each direction decreases with increasing frequency, so anomalous dispersion is also predicted by this model. As in Sec. 6.1, the solution (23) is not unique, but every solution exhibits the same power scaling of attenuation and dispersion at low frequencies.

If we apply the traditional dispersion operator in the s term in (18) instead of the fractional dispersion operator, Eq. (22) reduces to

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \Delta p + \tau \frac{\partial}{\partial t} \Delta^{\vec{r}/2} p \tag{24}$$

In the isotropic case where $r_1 = \cdots = r_d$, Eq. (24) is very similar to the fractional wave equation in Chen and Holm [12, Eq. (23)]. The difference is that, instead of the operator $\Delta^{(r/2)\vec{1}}$ with symbol $[(ik_1)^r, \ldots, (ik_d)^r]'$, they employ a different operator (let us call it $\bar{\Delta}^{y/2}$) with symbol $(-k_1^y, \ldots, -k_d^y)'$ [12, Eq. (25)]. The Chen and Holm model

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \Delta p + \tau \frac{\partial}{\partial t} \bar{\Delta}^{y/2} p$$

exhibits power law attenuation with index y in every coordinate. By extending their model to any coordinate direction and using the anisotropic operator $\bar{\Delta}^{\vec{y}/2}$ with symbol $(-k_1^{y_1}, \ldots, -k_d^{y_d})'$, one can show that the anisotropic version of their model

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \Delta p + \tau \frac{\partial}{\partial t} \bar{\Delta}^{\vec{y}/2} p$$

exhibits power law attenuation with $\alpha_j(\omega) \approx \alpha_0 |\omega|^{y_j}$ in the jth coordinate direction. The Chen and Holm model was extended by Treeby and Cox [13] with an additional space-fractional term that accounts for positive dispersion. Since the resulting phase speed increases as the frequency increases, the Treeby and Cox model is applicable only to normal (positive) dispersion.

A third possibility is to use the traditional isotropic fractional Laplacian $\Delta^{r/2}p(\vec{x})$ with Fourier transform $-\|\vec{k}\|^r P(\vec{k})$. This is different from both the operator $\Delta^{\vec{r}/2}$ we use in this paper, and the operator $\bar{\Delta}^{y/2}$ used by Chen and Holm. Yet another possibility $\breve{\Delta}^{\vec{r}/2}$ uses a fractional Laplacian in each coordinate, so that its Fourier symbol is $(-|k_1|^{r_1}, \ldots, -|k_d|^{r_d})'$. The isotropic version with every $r_j = r$ was employed by Magin et al. [14] to develop a fractional calculus model for diffusion tensor imaging. The anisotropic version was recently applied by GadElkarim et al. [15] to extend the model of Magin et al. to applications in which the diffusion scales at a different rate in each coordinate. Both $\Delta^{r/2}p$ and $\breve{\Delta}^{\vec{r}/2}p$ produce the same result for plane waves in a coordinate direction. Neither lead to a simple attenuation relation for the fractional Stokes wave equation (22), since the corresponding characteristic equation $-\omega^2/c^2 = -|k|^r[1 - i\omega\tau]$ has no solution when the symbol $-|k|^r$ is a real number.

To clearly understand the rich variety of fractional Laplacian operators, it is helpful to consider the random walk models behind these operators. In [2, Chap. 6] it is shown that the isotropic fractional diffusion equation $\partial p(\vec{x}, t)/\partial t = \Delta^{r/2}p(\vec{x}, t)$ governs the long time limit of a random walk with heavy tailed symmetric particle jumps $X = R\Theta$, where the jump length $P(R > x) \approx x^{-r}$ and the jump direction Θ is a random unit vector, uniformly distributed around the unit sphere. In the application to wave equations, the random walk models the progression of acoustic energy. If the medium is isotropic, then the wave motion should also be isotropic. The fractional derivative codes the power law jumps, see [2]. Physically, the power law jumps are the result of medium heterogeneity. In a homogeneous medium, a traditional wave equation would apply. In a heterogeneous medium, significant variations in the local speed of sound may occur, leading to a power law distribution of effective speed, modeled by the fractional derivative [16]. The anisotropic fractional diffusion equation $\partial p/\partial t = \Delta^{\vec{r}/2}p$ governs the case where the jump direction Θ selects a positive coordinate j at random, and then there is a positive jump satisfying $P(R > x) \approx x^{-r_j}$ in that coordinate direction. The variation $\partial p/\partial t = \breve{\Delta}^{r/2}p$ from Magin et al. [14] governs the case where the jump can be either positive or negative, and in the generalized form $\partial p/\partial t = \breve{\Delta}^{\vec{r}/2}p$ from GadElkarim et al. [15], the power law index r_j of the random jump varies with the coordinate. Such models may be appropriate in layered media. We could not connect the operator $\bar{\Delta}^{y/2}$ from Chen and Holm [12] to any random walk model, since the Fourier symbol $(-k_1^y, \ldots, -k_d^y)'$ does not seem to be related to any infinitely divisible stochastic process. All of these operators, including the one in Chen and Holm [12], reduce to the traditional Laplacian when $r = 2$, reflecting the fact that all random walk models have a Gaussian diffusion limit when the jumps lengths have a finite variance.

8 Conclusion

Two anisotropic wave equations with space-fractional derivatives were derived by combining the traditional linear equations of state and motion with a novel anisotropic fractional continuity equation. Analytical expressions for the attenuation and dispersion of plane wave solutions were obtained from the characteristic equation for each space-fractional wave equation. Both of these fractional anisotropic wave equations exhibit power law attenuation and anomalous dispersion in each direction, with a power law index that depends on the direction. Different forms of the fractional Laplacian are evaluated, and a random walk model with power law jumps provides a physical explanation for the fractional Laplacian in these space-fractional wave equations.

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