Physica A 390 (2011) 505–511

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Extremal behavior of a coupled continuous time random walk

Rina Schumer [a,*], Boris Baeumer [b], Mark M. Meerschaert [c]

[a] Division of Hydrologic Sciences, Desert Research Institute, Reno, NV 89512, United States
[b] Department of Mathematics & Statistics, University of Otago, Dunedin, New Zealand
[c] Department of Statistics & Probability, Michigan State University, East Lansing, MI 48824, United States

ARTICLE INFO

Article history:
Received 8 April 2010
Received in revised form 16 September 2010
Available online 20 October 2010

Keywords:
Continuous time random walks
Extreme value theory
Power laws
Econophysics

ABSTRACT

Coupled continuous time random walks (CTRWs) model normal and anomalous diffusion of random walkers by taking the sum of random jump lengths dependent on the random waiting times immediately preceding each jump. They are used to simulate diffusion-like processes in econophysics such as stock market fluctuations, where jumps represent financial market microstructure like log returns. In this and many other applications, the magnitude of the largest observations (e.g. a stock market crash) is of considerable importance in quantifying risk. We use a stochastic process called a coupled continuous time random maxima (CTRM) to determine the density governing the maximum jump length of a particle undergoing a CTRW. CTRM are similar to continuous time random walks but track maxima instead of sums. The many ways in which observations can depend on waiting times can produce an equally large number of CTRM governing density shapes. We compare densities governing coupled CTRM with their uncoupled counterparts for three simple observation/wait dependence structures.

1. Introduction

Random walks have long been used to model behavior of random diffusive-type processes in finance and econophysics [1]. Price fluctuations are likened to particles undergoing random, though well-behaved, motions and long term market behavior is represented by sums of particle jumps. Continuous time random walks (CTRWs) generalize classical models by permitting particle jumps and waiting times between jumps to have arbitrary probability densities and allowing representation of anomalous (super- or sub-diffusive) behavior. Thus, CTRWs have been used to reproduce the behavior of price fluctuations in financial markets [2–6]. Recently, coupled CTRWs, in which the size of an observation is dependent on the length of the preceding waiting period, have been used to describe movement of stock or share prices [7–10], electricity markets [11], currency exchange [12], and other types of financial data behavior [13]. If the practitioner is interested in the largest price shock expected in a given interval rather than long term evolution of prices, then extreme value models are appropriate tools. Extreme events in financial time series have been characterized using the mean first-passage time and mean exit time of a CTRW [14]. We seek the density governing the maximum price fluctuation.

Classical extreme value models describe the density governing the largest of independent and identically distributed (i.i.d.) random observations with arbitrary probability density, independent of fixed-length or exponentially distributed interarrivals. If observations are particle jumps, then these models describe recurrence of the jumps of a particle undergoing a random walk. A generalization to accommodate an arbitrary waiting time between observations was recently accomplished in Ref. [15]. In this work, we extend that model to allow the distribution of an observation to depend on

* Corresponding author. Tel.: +1 775 673 7414.
E-mail addresses: rina@dri.edu (R. Schumer), bbaeumer@maths.otago.ac.nz (B. Baeumer), mcubed@stt.msu.edu (M.M. Meerschaert).

0378-4371/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2010.10.018

the waiting time before that observation. The purpose of this generalization is to estimate recurrence intervals for random phenomena that are well modeled by coupled CTRWs. Continuous time random maxima (CTRM) models are stochastic processes that track the largest observation in a series of events separated by random waiting times. They can be used to forecast the largest particle observation or price jump with time.

2. Coupled continuous time random maxima

Our goal is to estimate the density governing the maximum jump length that a particle undergoing a CTRW with i.i.d. jumps dependent on the i.i.d. waiting times that separate them. We call the space–time stochastic process $\{M_t\}_{t\geq 0}$ a *coupled continuous time random maxima* (CTRM) because its form is similar to a CTRW but it tracks jump maxima instead of sums. Let J_n = i.i.d. duration of interarrivals with density $\psi(t)$, X_n = i.i.d. event magnitudes, $T_n = J_1 + J_2 + \cdots + J_n$ = time of the nth event, and $N_t = \max\{n : T_n \leq t\}$ = number of events by time t. To be complete, we also set $T_0 = 0$ and $M_0 = -\infty$. Further let $M(n) = \max(X_1, X_2, \ldots, X_n)$ = maximum of the first n observed events and $M_t = M(N_t) = \max(X_1, X_2, \ldots, X_{N_t})$ = maximum observed event by time t. The joint density of the coupled waiting time/event magnitudes is denoted $(X_i, J_i) \sim f(x, t)$. The waiting time density is the time marginal of the transition density $\psi(t) = \int f(x, t)\,\mathrm{d}x$. Using the inverse relationship between the time until the nth event and the number of events by time $t\{N_t \geq n\} = \{T_n \leq t\}$ and the fact that the time between events n and $n+1$ is the $n+1$st wait J_{n+1} we find

$$\begin{aligned} P(M(N_t) \leq x) &= \sum_{n=0}^{\infty} P(M(n) \leq x, N_t = n) \\ &= \sum_{n=0}^{\infty} [P(M(n) \leq x, N_t \geq n) - P(M(n) \leq x, N_t \geq n+1)] \\ &= \sum_{n=0}^{\infty} [P(M(n) \leq x, T_n \leq t) - P(M(n) \leq x, T_{n+1} \leq t)] \\ &= \sum_{n=0}^{\infty} [P(M(n) \leq x, T_n \leq t) - P(M(n) \leq x, T_n + J_{n+1} \leq t)] \\ &= \sum_{n=0}^{\infty} \left[P(M(n) \leq x, T_n \leq t) - \int_0^{\infty} P(M(n) \leq x, T_n \leq t - \tau)\psi(\tau)\,\mathrm{d}\tau \right], \end{aligned} \tag{1}$$

where the integral in the last line of (1) includes the probability of all possible waiting times between jumps n and $n+1$. Note that the distribution of $M(N_t)$ has an atom at $x = -\infty$, equal to the probability $P(N_t = 0) = P(J_1 > t)$ that the first observation has not yet occurred by time t. Let $F(x, t) = \int_{-\infty}^{x} f(u, t)\,\mathrm{d}u$ and define the CDF–Laplace (C–L) transform of f to be

$$\bar{f}(x, s) = \int_0^{\infty} \mathrm{e}^{-st} F(x, t)\,\mathrm{d}t = E[\mathrm{e}^{-sJ} I(X \leq x)],$$

which at $x = \infty$ is the Laplace transform of ψ and at $s = 0$ is the CDF of X. While the density of a sum is the convolution (product in transform space) of the individual summand densities, the cumulative distribution function (cdf) governing the maxima of a set of i.i.d. random variables is the product of the individual cdfs. This is reflected by the fact that the density of $(M(n), T_n)$ has C–L transform

$$E[\mathrm{e}^{-sT_n} I(M_n \leq x)] = E\left[\prod_{i=1}^{n} \mathrm{e}^{-sT_i} I(M_i \leq x)\right] = \prod_{i=1}^{n} E[\mathrm{e}^{-sT_i} I(M_i \leq x)] = \bar{f}(x, s)^n$$

which is also true for $n = 0$.

Next, integrate by parts to get

$$\begin{aligned} \int_0^{\infty} \mathrm{e}^{-st} P(M(n) \leq x, T_n \leq t)\,\mathrm{d}t &= E\left[\int_0^{\infty} \mathrm{e}^{-st} I(M(n) \leq x, T_n \leq t)\,\mathrm{d}t\right] \\ &= E\left[s^{-1} \int_0^{\infty} \mathrm{e}^{-st} I(M(n) \leq x | T_n = t)\psi(t)\,\mathrm{d}t\right] \\ &= s^{-1} \int_0^{\infty} E\left[I(M(n) \leq x)\mathrm{e}^{-sT_n} | T_n = t\right] \psi(t)\,\mathrm{d}t \\ &= s^{-1} E\left[I(M(n) \leq x)\mathrm{e}^{-sT_n}\right] \\ &= s^{-1} \bar{f}(x, s)^n \end{aligned} \tag{2}$$

by conditioning on $T_n = t$. Let $M(x, t) = P(M(N_t) \le x)$ and take Laplace transforms in (1) to get

$$\tilde{M}(x, s) = \sum_{n=0}^{\infty} \left[s^{-1}\bar{f}(x, s)^n - s^{-1}\bar{f}(x, s)^n \tilde{\psi}(s) \right]$$
$$= \frac{1 - \tilde{\psi}(s)}{s} \frac{1}{1 - \bar{f}(x, s)}, \tag{3}$$

since $\sum_{i=0}^{\infty} r^i = 1/(1 - r)$, provided that $|r| < 1$. Multiplying both sides of (3) by $1 - \bar{f}(x, s)$ and inverting the Laplace transform yields that

$$M(x, t) - \int_0^t F(x, t - \tau) M(x, \tau)\, \mathrm{d}\tau = \int_t^{\infty} \psi(\tau)\, \mathrm{d}\tau, \tag{4}$$

and therefore for each x, $M(x, t)$ is the solution of a Volterra integral equation (4) of the second kind [16].

This master equation for coupled CTRM reduces to the master equation for uncoupled CTRM when event magnitudes with distribution function $F_e(x)$ and interarrivals are uncorrelated so that $\bar{F}(x, s) = F_e(x)\tilde{\psi}(s)$ [15]:

$$\mathcal{L}[P(M(N_t) \le x)] = \frac{1 - \tilde{\psi}(s)}{s} \frac{1}{1 - F_e(x)\tilde{\psi}(s)} \tag{5}$$

or

$$M(x, t) - F_e(x) \int_0^t \psi(t - \tau) M(x, \tau)\, \mathrm{d}\tau = \int_t^{\infty} \psi(\tau)\, \mathrm{d}\tau. \tag{6}$$

Solutions to uncoupled CTRM with Poissonian interarrivals are exactly solvable, regardless of event magnitude distribution $F_e(x)$. In these cases, classical extreme value distributions (Gumbel and Frechet) arise in the scaling limit depending on the tail behavior of $F_e(x)$. If interarrivals are heavy-tailed and independent of event magnitude, the densities governing the largest events can be calculated by transforming classical extreme value densities with a subordination integral [15].

3. Examples

Here we demonstrate the effect of a coupled wait/observation density on the probability distribution function (pdf) of the maxima predicted by the CTRM model, where observations may represent financial returns or actuarial claims. Coupling combinations are unlimited. If the Laplace transform in (3) is known exactly one can numerically invert the Laplace transform directly. In general, the densities governing the maximum value (Eq. (4)) do not exist in closed form and have to be computed numerically. Numerical techniques for solving Volterra integral equations will give good approximations to the solution. Here, we use a basic stepping method to generate densities (Appendix); for more refined numerical methods, see for example Ref. [17].

3.1. Exponential waiting times and jump distributions

Example 1. Here we assign an exponential distribution with mean $1/\lambda_t = 10$ to the waiting times J_i between observations. We also assign the same distribution, exponential with mean $1/\lambda_x = 10$, to the observations Y_i that occur at the end of each waiting time. We compare the pdf of the maximum observation in two very different cases: (a) observations Y_i are independent of waiting times J_i; and (b) observations are coupled to length of the waiting times by $Y_i = J_i$. Since J_i are exponential, N_t has a Poisson distribution with mean $\lambda_t t$ in this example. Then case (a) represents the maximum observation, $M(N_t)$, for a compound Poisson process with exponential jumps. Since Y_i has exponential tails, the pdf of $M(N_t)$ is governed at late time by a Gumbel density [18]. In fact, for $x \ge 0$ we have

$$\begin{aligned} P(M(N_t) \le x) &= E(P(M(N_t) \le x | N_t)) \\ &= \sum_{n=0}^{\infty} P(M(N_t) \le x | N_t = n) P(N_t = n) \\ &= \sum_{n=0}^{\infty} P(M_n \le x) P(N_t = n) \\ &= \sum_{n=0}^{\infty} (1 - \mathrm{e}^{-\lambda_x x})^n \mathrm{e}^{-\lambda_t t} \frac{(\lambda_t t)^n}{n!} \\ &= \mathrm{e}^{-\lambda_t t} \sum_{n=0}^{\infty} \frac{[(\lambda_t t)(1 - \mathrm{e}^{-\lambda_x x})]^n}{n!} \\ &= \mathrm{e}^{-\lambda_t t} \mathrm{e}^{(\lambda_t t)(1 - \mathrm{e}^{-\lambda_x x})} \\ &= \mathrm{e}^{-\lambda_t t \mathrm{e}^{-\lambda_x x}} \end{aligned}$$

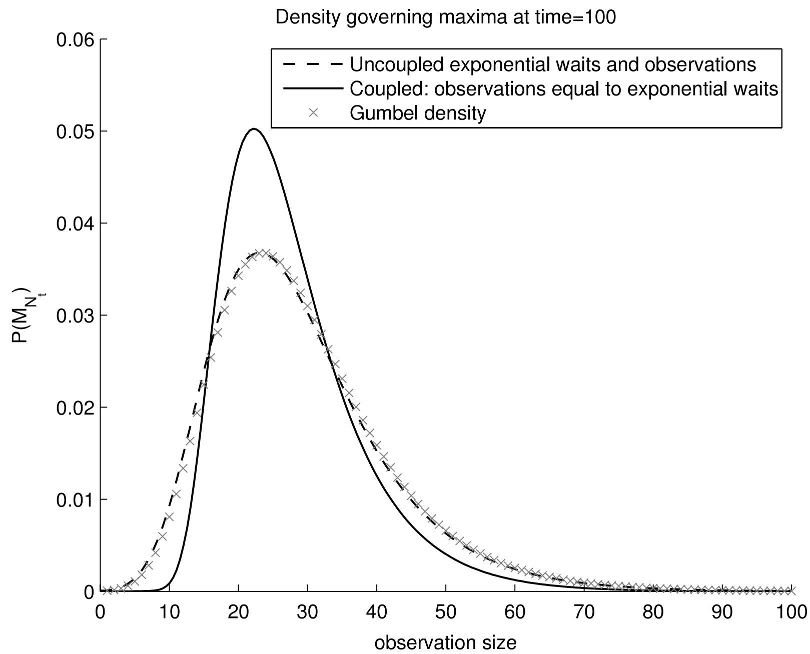

Fig. 1. The density governing maximum observation size at time = 100 for a CTRM with exponential observation length equal to the preceding independent exponential ($\lambda = 1/10$) waiting times (solid line) has thinner tails and a higher peak than the density for the analogous uncoupled model (dashed line).

which is exactly the Gumbel formula. If $x \leq 0$ then we have $P(M(N_t) \leq x) = e^{-\lambda_t t}$ since $M(N_t) = -\infty$ when $N_t = 0$, which happens with probability $P(J_1 > t) = e^{-\lambda_t t}$ in this example. The only difference between the Gumbel and the pdf of $M(N_t)$ in this example is that the probability assigned to $x < 0$ under the Gumbel pdf is concentrated at $x = -\infty$ in the pdf of $M(N_t)$. Since this probability tends to zero as t increases, the asymptotic distribution of $M(N_t)$ is exactly Gumbel.

Fig. 1 shows that pdf of the largest observation at $t = 100$. The coupled pdf has thinner tails and higher peak than the uncoupled model, reflecting the decreased probability of large or small observations when event magnitude is tied to the preceding waiting time. The atom $P(J_1 > 100) = e^{-10}$ is negligible in this case.

Example 2. Now suppose that the waiting times J_i between observations have a standard Pareto distribution with tail parameter $\beta = 0.7$: $P(J_i > t) = (1 + t)^{-0.7}$ for $t > 0$. As in Example 1, we compare pdfs that arise from the CTRM model if: (a) observations Y_i have Pareto ($\beta = 0.7$) distribution independent from that of the waiting times; and (b) observations Y_i are equal to the preceding waiting time J_i. Fig. 2 compares the pdf for each case at time $t = 100$. Both densities have a peak at zero. The density for the uncoupled CTRM decreases monotonically from the peak at zero, following the decay of the observation density. In comparison, the coupled CTRM density is bimodal.

Example 3. The evolution of LIFFE bond future prices was modeled in Ref. [7] using a coupled CTRW for the tick-by-tick data: the early waiting times between trades can be fit a Pareto distribution with $\beta = 0.74$ and $C = 0.061$, using the Hill estimator [19]. The corresponding log returns can be modeled by

$$Y_i = J_i^{1/2} Z_i \tag{7}$$

where Z_i are normal with mean zero and standard deviation 0.000040, independent of the waits J_i. This implies that the log returns are drawn from a mean zero Gaussian distribution with variance proportional to the preceding waiting time. This coupling, originally proposed by Shlesinger [20], provides that large observations tend to follow long waiting times, which also controls the rate Y_i/J_i of price change. Fig. 3 compares the density governing the maxima under this model (solid line) to the uncoupled model (dotted line) with returns of the form $Y_i = P_i^{1/2} Z_i$, where P_i is independent of J_i, with the same Pareto distribution. The independence of P_i and J_i, leading to price returns that are uncoupled from waiting times, is the only change from the coupled model. Although the graphs appear similar, the probability of a log return greater than 0.001 during the first 500 s of trading is 1.8% in the coupled model, versus 11.2% in the uncoupled model.

4. Discussion

Coupling of jump and waiting time densities in a CTRW significantly alters the extremal behavior. The coupled CTRM framework developed in this paper describes the maximum observation in a coupled CTRW. This coupling can significantly affect the pdf of the maximum jump over a given time interval. An example concerning LIFFE bond futures illustrates the importance of coupling. In applications to finance, it is sometimes true that the coupling disappears in the long-time limit.

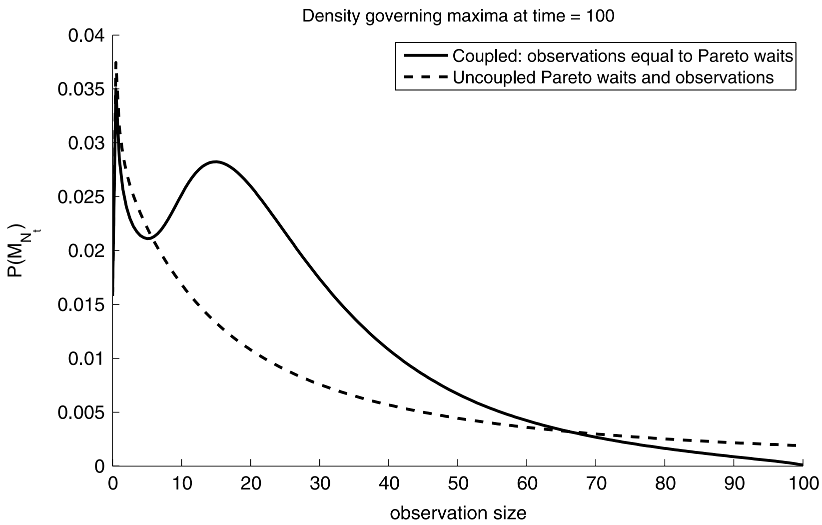

Fig. 2. The density governing maximum observation size at time = 100 for a CTRM with Pareto observations equal to the preceding independent Pareto ($\beta = 0.7$) waiting times (solid line) is bimodal and has a thinner tail than the density for the analogous uncoupled model (dashed line).

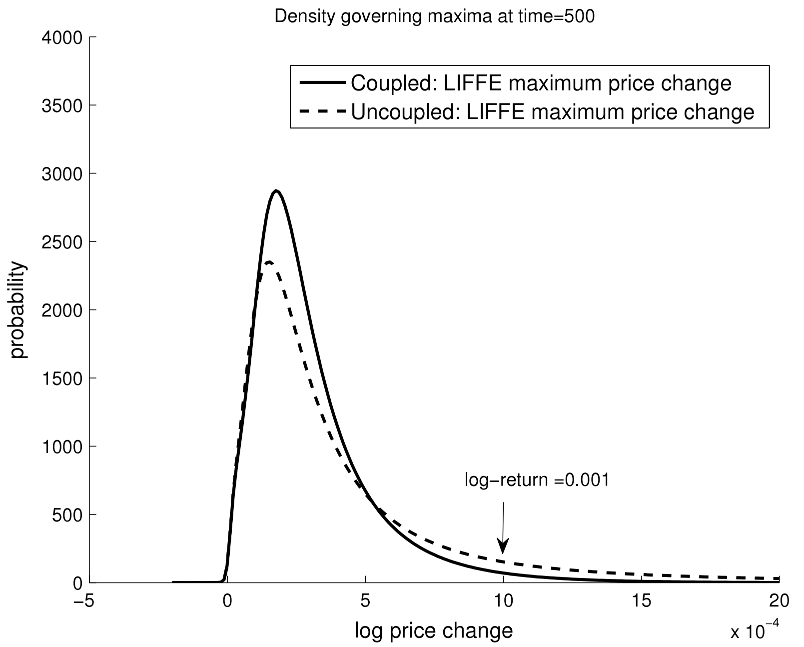

Fig. 3. Comparison of models describing the evolution of LIFFE bond future prices: density of maximum log return by time $t = 500$ s for a coupled CTRM with Pareto waits J and log returns $Y = J^{1/2}Z$ where Z is normal vs. the same density for an uncoupled CTRM with $Y = P^{1/2}Z$ where P is independent of J with the same Pareto distribution. The probability of experiencing a log return greater than 0.001 during the first 500 s of trading is given by the area under the curve to the right of the arrow in each case.

This was the case with tick-by-tick data on General Electric stock examined [7]. Although the log returns Y_i and waiting times J_i show significant dependence, the coupling disappears in the long-time limit. This can only happen when large returns are correlated with short waiting times, and vice versa. In this case, the maximum can be characterized using traditional extreme value theory [21–23]. But for strongly coupled CTRW (e.g., when large jumps are correlated with long waiting times) the coupled CTRM presents a significant improvement over traditional methods. We note that some underlying theory for coupled CTRM was already present in the work of Silvestrov and Teugels [24] and Pancheva et al. [25]. The CTRM framework parallels that of the well known CTRW. Thus, further development for models that use alternate methods of coupling (e.g. copulas, see Ref. [26]) should be straightforward.

As a specific example of the many, diverse applications of the CTRM framework, this paper presents an application to finance, based on a model from Ref. [7]. The jumps Y_i are log returns of a LIFFE bond future, and the waiting times J_i are the intervals between trades. Then N_t is the number of trades by time $t > 0$, $M(N_t)$ is the largest positive log return by time t, and the CTRM quantifies the risk (or opportunity) of a large price jump in the interval $(0, t)$. It is clear from Fig. 3 that coupling can significantly decrease this risk. For this reason, ignoring the coupling can lead to a seriously deficient strategy for assessing the chance of a large price changes, which is in fact considerably less than the uncoupled model suggests.

5. Conclusions

In many applications, magnitude of the largest event and estimates of their recurrence are central to risk analysis. Coupled CTRMs are random walks in space–time that quantify likelihood of the largest observations in a given interval for processes whose evolutions are well represented by coupled CTRWs. This paper computes the exact pdf of the CTRM in terms of Laplace transforms, and develops limit theory to describe the approximate late-time pdf.

Acknowledgements

The first author was partially supported by NSF grants EAR-0120914 and EAR-0817073. The third author was partially supported by NSF grants DMS-0125486, DMS-0803360, EAR-0823965 and NIH grant R01-EB012079-01.

Appendix. Numerical approximation of Volterra equations

In order to approximate the solution to (4), we approximated the convolution integral using Simpson's rule; i.e. for a given time step $\mathrm{d}t$ and $t \approx n\,\mathrm{d}t$,

$$\int_0^t F(x,t-\tau)M(x,\tau)\,\mathrm{d}\tau \approx \mathrm{d}t \sum_{i=1}^{n-1} F(x,(n-i)\,\mathrm{d}t)M(x,i\,\mathrm{d}t) + \frac{\mathrm{d}t}{2}F(x,n\,\mathrm{d}t)M(x,0) + \frac{\mathrm{d}t}{2}F(x,0)M(x,n\,\mathrm{d}t). \tag{8}$$

Hence we step by approximating $M(x,(n+1)\,\mathrm{d}t)$ with $M_{n+1}(x)$ satisfying

$$M_{n+1}(x) = \mathrm{d}t \sum_{i=1}^{n} F(x,(n-i)\,\mathrm{d}t)M_i(x) + \frac{\mathrm{d}t}{2}F(x,n\,\mathrm{d}t)M_0(x) + \frac{\mathrm{d}t}{2}F(x,0)M_{n+1}(x) + \int_{(n+1)\mathrm{d}t}^{\infty} \psi(\tau)\,\mathrm{d}\tau, \tag{9}$$

or

$$M_{n+1}(x) = \left(\mathrm{d}t \sum_{i=1}^{n} F(x,(n-i)\,\mathrm{d}t)M_i(x) + \frac{\mathrm{d}t}{2}F(x,n\,\mathrm{d}t)M_0(x) + \int_{(n+1)\mathrm{d}t}^{\infty} \psi(\tau)\,\mathrm{d}\tau\right) \Bigg/ \left(1 - \frac{\mathrm{d}t}{2}F(x,0)\right). \tag{10}$$

The density of M with respect to x needed for the figures can then be obtained by (central) differencing in x.

References

[1] L. Bachelier, Theory of Speculation, MIT Press, Cambridge, 1969.
[2] F. Mainardi, M. Raberto, R. Gorenflo, E. Scalas, Fractional calculus and continuous-time finance II: the waiting-time distribution, Physica A 287 (3–4) (2000) 468–481.
[3] E. Scalas, R. Gorenflo, F. Mainardi, Fractional calculus and continuous-time finance, Physica A 284 (1–4) (2000) 376–384.
[4] E. Scalas, Five Years of Continuous-Time Random Walks in Econophysics, vol. 567, Springer, Berlin, 2006, pp. 3–16.
[5] L. Sabatelli, S. Keating, J. Dudley, P. Richmond, Waiting time distributions in financial markets, European Physical Journal B 27 (2) (2002) 273–275.
[6] G. Germano, M. Politi, E. Scalas, R.L. Schilling, Stochastic calculus for uncoupled continuous-time random walks, Physical Review E 79 (6) (2009) 6102. doi:10.1103/PhysRevE.79.066102.
[7] M.M. Meerschaert, E. Scalas, Coupled continuous time random walks in finance, Physica A—Statistical Mechanics and its Applications 370 (1) (2006) 114–118.
[8] M. Raberto, E. Scalas, F. Mainardi, Waiting-times and returns in high-frequency financial data: an empirical study, Physica A—Statistical Mechanics and its Applications 314 (1–4) (2002) 749–755.
[9] J. Masoliver, M. Montero, G.H. Weiss, Continuous-time random-walk model for financial distributions, Physical Review E 67 (2) (2003) 1112. doi:10.1103/PhysRevE.67.021112.
[10] J. Masoliver, M. Montero, J. Perello, G.H. Weiss, The continuous time random walk formalism in financial markets, Journal of Economic Behavior & Organization 61 (4) (2006) 577–598.
[11] E. Broszkiewicz-Suwaj, A. Jurlewicz, Pricing on electricity market based on coupled-continuous-time-random-walk concept, Physica A—Statistical Mechanics and its Applications 387 (22) (2008) 5503–5510.
[12] P. Repetowicz, P. Richmond, Modeling of waiting times and price changes in currency exchange data, Physica A—Statistical Mechanics and its Applications 343 (2004) 677–693.
[13] A. Jurlewicz, A. Wylomanska, P. Zebrowski, Financial data analysis by means of coupled continuous-time random walk in Rachev–Ruschendorf model, Acta Physica Polonica A 114 (3) (2008) 629–635.
[14] J. Masoliver, M. Montero, J. Perello, Extreme times in financial markets, Physical Review E 71 (5) (2005) 056130.
[15] D. Benson, R. Schumer, M. Meerschaert, Recurrence of extreme events with power-law interarrival times, Geophysical Research Letters 34 (2007) l16404. doi:10.1029/2007GL030767.
[16] G. Arfken, H. Weber, Mathematical Methods for Physicists, 4th ed., Academic Press, San Diego, 1995.
[17] C.T.H. Baker, A perspective on the numerical treatment of Volterra equations, Journal of Computational and Applied Mathematics 125 (1–2) (2000) 217–249.
[18] R.-D. Reiss, M. Thomas, Statistical Analysis of Extreme Values: with Applications to Insurance, Finance, Hydrology and Other Fields, 2nd ed., Birkhäuser, Basel, 2001.
[19] B. Hill, A simple general approach to inference about the tail of a distribution, Annals of Statistics 3 (5) (1975) 1163–1173.
[20] M.F. Shlesinger, J. Klafter, Y. Wong, Random walks with infinite spatial and temporal moments, Journal of Statistical Physics 27 (3) (1982) 499–512.
[21] R. Fisher, L. Tippett, Limiting forms of the freqeuency distribution of the largest or smallest member of a sample, Proceedings of the Cambridge Philosophical Society 24 (1928) 180–190.
[22] M.M. Meerschaert, S.A. Stoev, Extremal limit theorems for observations separated by random power law waiting times, Journal of Statistical Planning and Inference 139 (7) (2009) 2175–2188.

[23] D.S. Silvestrov, J.L. Teugels, Limit theorems for extremes with random sample size, Advances in Applied Probability 30 (3) (1998) 777–806.
[24] D.S. Silvestrov, J.L. Teugels, Limit theorems for mixed max-sum processes with renewal stopping, Annals of Applied Probability 14 (4) (2004) 1838–1868.
[25] E. Pancheva, I.K. Mitov, K.V. Mitov, Limit theorems for extremal processes generated by a point process with correlated time and space components, Statistics & Probability Letters 79 (3) (2009) 390–395.
[26] J. Masoliver, M. Montero, J. Perello, G.H. Weiss, The CTRW in finance: direct and inverse problems with some generalizations and extensions, Physica A—Statistical Mechanics and its Applications 379 (2007) 151–167.