THE FRACTIONAL D'ALEMBERT'S FORMULAS

CHENG-GANG LI, MIAO LI, SERGEY PISKAREV, AND MARK M. MEERSCHAERT

Abstract. In this paper we develop generalized d'Alembert's formulas for abstract fractional integro-differential equations and fractional differential equations on Banach spaces. Some examples are given to illustrate our abstract results, and the probability interpretation of these fractional d'Alembert's formulas are also given. Moreover, we also provide d'Alembert's formulas for abstract fractional telegraph equations.

1. Introduction

It is well-known that the solution of traditional wave equation on the line

$$\begin{cases} u_{tt}(t,x) = u_{xx}(t,x), & t>0,\, x\in\mathbb{R} \\ u(0,x)=\phi(x), & u_t(0,x)=\psi(x) \end{cases} \tag{1.1}$$

is given by d'Alembert's formula

$$u(t,x) = \frac{1}{2}[\phi(x+t)+\phi(x-t)] + \frac{1}{2}\int_{x-t}^{x+t}\psi(y)dy. \tag{1.2}$$

Including a forcing function, the solution of the wave equation on the line

$$\begin{cases} w_{tt}(t,x) = w_{xx}(t,x) + f(t,x), & t>0,\, x\in\mathbb{R} \\ w(0,x)=0, & w_t(0,x)=0 \end{cases} \tag{1.3}$$

is given by the Duhamel's principle formula

$$w(t,x) = \int_0^t r(t,x,\tau)d\tau,$$

where $r(t,x,\tau)$ is the solution of wave equation

$$\begin{cases} r_{tt}(t,x,\tau) = r_{xx}(t,x,\tau), & t>0,\, x\in\mathbb{R} \\ r(\tau,x,\tau)=0, & r_t(\tau,x,\tau)=f(\tau,x) \end{cases} \tag{1.4}$$

The fractional Duhamel's principle formula was obtained by [3, 34].

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It is also of interest to know a fractional version of the d'Alembert formula. Next we rewrite (1.1) as an integral equation, which is more easily generalized to the fractional case. By integrating the wave equation (1.1) twice with respect to t, we get the following integro-differential equation

$$u(t,x) = \phi(x) + t\psi(x) + \int_0^t (t-s)u_{xx}(s,x)ds.$$

One possible generalization of the above equation to fractional order $1 \le \alpha \le 2$ is

$$u(t,x) = \phi(x) + \frac{t^{\alpha/2}}{\Gamma(1+\frac{\alpha}{2})}\psi(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u_{xx}(s,x)ds. \tag{1.5}$$

Fujita studied the above equation in [10] and showed that the unique solution is given by

$$u(t,x) = \frac{1}{2}\mathbf{E}[\phi(x+Y_{\alpha/2}(t)) + \phi(x-Y_{\alpha/2}(t))] + \frac{1}{2}\mathbf{E}\int_{x-Y_{\alpha/2}(t)}^{x+Y_{\alpha/2}(t)} \psi(y)dy, \tag{1.6}$$

where $Y_{\alpha/2}(t) = \sup_{0\le s\le t} X_\alpha(s)$, and $X_\alpha(t)$ $(1 \le \alpha \le 2)$ is a *càdlàg* stable process with characteristic function $\mathbf{E}\exp\{i\xi X_\alpha(t)\} = \exp\{-t|\xi|^{2/\alpha}e^{-(\pi i/2)(2-2/\alpha)\mathrm{sgn}(\xi)}\}$. $Y_{\alpha/2}(t)$ can also be regarded as the inverse of an $\alpha/2$ stable subordinator [24, 26]. When $\alpha = 2$, the expression (1.6) reduces to d'Alembert's formula (1.2). Fujita also mentioned that $\mathbf{E}[\phi(x \pm Y_{\alpha/2}(t)) \pm \int_0^{x\pm Y_{\alpha/2}(t)} \psi(y)dy]$ are solutions for

$$u^{\pm}(t,x) = \phi(x) \pm \int_0^x \psi(y)dy \pm \frac{1}{\Gamma(\alpha/2)}\int_0^t (t-s)^{\alpha/2-1}u_x^{\pm}(s,x)ds, \tag{1.7}$$

respectively, and the solution of (1.5) can be decomposed as $u(t,x) = \frac{1}{2}(u^+(t,x) + u^-(t,x))$.

Next we convert the integro-differential equation (1.5) to a fractional differential equation. See Section 2 for the definition of fractional derivatives, fractional integrals, and the special functions $g_\alpha(t)$. We refer to [15, 25, 30, 32] for more details on fractional derivatives and fractional differential equations. Now if u satisfies the equation (1.5), then by differentiating it with respect to t for $\alpha/2$-times in the sense of Caputo fractional derivatives and by using the identity $D_t^\alpha 1 = 0$, we have

$$\begin{aligned} D_t^{\alpha/2}u(t,x) &= D_t^{\alpha/2}(u(t,x) - \phi(x)) \\ &= D_t^{\alpha/2}(g_{1+\alpha/2}(t)\psi(x) + (J_t^\alpha u_{xx})(t,x)) \\ &= \psi(x) + (J_t^{\alpha/2}u_{xx})(t,x), \end{aligned}$$

and next differentiating for $\alpha/2$-times again we get

$$\begin{aligned} D_t^{\alpha/2}(D_t^{\alpha/2}u(t,x)) &= D_t^{\alpha/2}(\psi(x) + (J_t^{\alpha/2}u_{xx})(t,x)) \\ &= D_t^{\alpha/2}(J_t^{\alpha/2}u_{xx})(t,x)) = u_{xx}(t,x). \end{aligned}$$

This suggests an α-order differential equation $D_t^{\alpha/2}D_t^{\alpha/2}u = u_{xx}$. It is also interesting to consider the integro-differential equation $D_t^\alpha u = u_{xx}$, because $D_t^\alpha = D_t^{\alpha/2}D_t^{\alpha/2}$ holds only under some special conditions.

Motivated by the above observations, we will first consider d'Alembert's formula for abstract fractional integro-differential equation in the form of

$$u(t) = \phi + \frac{t^{\alpha/2}}{\Gamma(1+\alpha/2)}\psi + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A^2u(s)ds, \quad t > 0 \tag{1.8}$$

on a Banach space. It is known that the well-posedness of equation (1.8) is equivalent to the existence of an α-times resolvent family for A^2. Thus the theory of fractional resolvent families will be our main tool. The notion of resolvent families was first introduced by Prüss [31] to study Volterra integral equations, and then developed systematically by Bajlekova [4] for fractional Cauchy problems. The fractional resolvent families can be considered as generalizations of C_0-semigroups and cosine operator functions [2, 8, 29]. The d'Alembert formula for wave equation is in fact the decomposition of a cosine operator function, see for example [16, Chapter III]. For its fractional analogue we will use the decomposition theorem for fractional resolvent families [22], i.e. our Lemma 2.6. Thanks to this lemma, we are able to give the solution of (1.8) in Theorem 2.8 and decompose the solution as $u = \frac{1}{2}(u^+ + u^-)$, where $u^\pm$ are solutions to

$$u^\pm(t,x) = \phi(x) \pm A^{-1}\psi \pm \frac{1}{\Gamma(\alpha/2)} \int_0^t (t-s)^{\alpha/2-1} A u^\pm(s,x)ds. \tag{1.9}$$

respectively, when ψ is in the range of the operator A. When $A = \frac{\partial}{\partial x}$, then (1.9) is the same as (1.7). The corresponding fractional differential equations for (1.8) and (1.9) are

$$\begin{cases} D_t^{\alpha/2} D_t^{\alpha/2} u(t) = A^2 u(t), \quad t > 0 \\ u(0) = \phi, \quad D_t^{\alpha/2} u(0) = \psi \end{cases}$$

and

$$\begin{cases} D_t^{\alpha/2} u^\pm(t) = \pm A u^\pm(t), \quad t > 0 \\ u^\pm(0) = \phi \pm A^{-1}\psi \end{cases}$$

respectively.

In Theorems 2.11 and 2.13 we will construct the d'Alembert formula for the more general equation

$$u(t) = \phi + \frac{t^\beta}{\Gamma(1+\beta)}\psi + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A^2 u(s)ds, \quad t > 0$$

or its corresponding fractional differential equation

$$\begin{cases} D_t^{\alpha-\beta} D_t^{\beta} u(t) = A^2 u(t), \quad t > 0 \\ u(0) = \phi, \quad D_t^{\beta} u(0) = \psi. \end{cases}$$

In particular, when $\beta = \alpha/2$, an alternative d'Alembert formula for (1.8) will be provided. More precisely, the solution of (1.8) can be decomposed as $u(t) = \frac{1}{2}(v^+(t) + v^-(t))$, where $v^\pm$ are solutions to

$$v^\pm(t) = \phi + \frac{t^{\alpha/2}}{\Gamma(1+\alpha/2)}\psi \pm \frac{1}{\Gamma(\alpha/2)} \int_0^t (t-s)^{\alpha/2-1} A v^\pm(s)ds, \quad t > 0,$$

and the corresponding fractional differential equations are

$$\begin{cases} D_t^{\alpha/2} v^\pm(t) = \pm A v^\pm(t) + \psi, \quad t > 0 \\ v^\pm(0) = \phi. \end{cases}$$

And the d'Alembert formula for fractional differential equation like

$$\begin{cases} D_t^{\alpha} u(t) = A^2 u(t), \quad t > 0 \\ u(0) = \phi, \quad u_t(0) = \psi \end{cases}$$

will also be considered.

Our papers is organized as follows: in Section 2 we will recall some results on fractional resolvent families and then derive the fractional d'Alembert's formula for abstract fractional integro-differential equations and fractional differential equations on Banach spaces; some concrete examples are given in Section 3 to illustrate our abstract results, and their probability interpretations are also given; finally in Section 4 we will give the fractional d'Alembert's formula for fractional telegraph equations.

2. d'Alembert's formula for abstract fractional equations

Let X be a Banach space and A be a closed linear densely defined operator on X. We begin with the definitions of fractional integrals and derivatives.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as

$$J_t^\alpha f(t) := (g_\alpha * f)(t) = \int_0^t g_\alpha(t-s)f(s)ds, \qquad f \in L^1([0,+\infty);X), \quad t>0$$

where

$$g_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t>0, \\ 0, & t \le 0. \end{cases}$$

Set moreover $J_t^0 f(t) = f(t)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined as

$$_{RL}D_t^\alpha f(t) := D_t^m J_t^{m-\alpha} f(t), \quad t \in (0,T)$$

for $m-1 < \alpha \le m$, m is an integer, $f \in L^1((0,T);X)$, and $g_{m-\alpha} * f \in W^{m,1}((0,T);X)$, where

$$W^{m,1}((0,T);X) \;:=\; \{f| \text{ there exists } \phi \in L^1((0,T);X) \text{ such that } f(t) = \sum_{k=0}^{m-1} c_k g_{k+1}(t) + (g_m * \phi)(t),\ t \in (0,T)\}.$$

The Caputo fractional derivative of order $\alpha > 0$ is defined as

$$D_t^\alpha f(t) := J_t^{m-\alpha} D_t^m f(t), \quad t \in (0,T)$$

if $f \in W^{m,1}((0,T);X)$. Moreover, we define

$$_{RL}D_t^\alpha f(0) := \lim_{t\to 0} {}_{RL}D_t^\alpha f(t), \quad D_t^\alpha f(0) := \lim_{t \to 0} D_t^\alpha f(t)$$

if the limits exist.

Now we consider the Volterra equation

$$u(t) = f(t) + \int_0^t g_\alpha(t-s)Au(s)ds, \quad t \ge 0 \tag{2.1}$$

where $f(t)$ is a continuous X-valued function.

Definition 2.3. Let $u(t) : \mathbb{R}_+ \to X$ be continuous.

(1) $u(t)$ is called a strong solution of (2.1) if $u(t) \in D(A)$ and (2.1) holds for $t \ge 0$;

(2) $u(t)$ is called a mild solution of (2.1) if $(g_\alpha * u)(t) \in D(A)$ and $u(t) = f(t) + A(g_\alpha * u)(t)$ for $t \ge 0$.

The solution family for (2.1) is defined by [4, 31].

Definition 2.4. A family $\{S_\alpha(t)\}_{t\geq 0} \subset B(X)$ is called an α-times resolvent family for the operator A (or generated by A) if the following conditions are satisfied:

(1) $S_\alpha(t)x : \mathbb{R}_+ \to X$ is continuous for every $x \in X$ and $S_\alpha(0) = I$;

(2) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;

(3) the resolvent equation

$$S_\alpha(t)x = x + (g_\alpha * S_\alpha)(t)Ax \tag{2.2}$$

holds for every $x \in D(A)$.

Remark 2.5. Since A is closed and densely defined, it is easy to show that for all $x \in X$, $(g_\alpha * S_\alpha)(t)x \in D(A)$ and

$$S_\alpha(t)x = x + A(g_\alpha * S_\alpha)(t)x. \tag{2.3}$$

It is shown in [31] that the Volterra equation (2.1) is well-posed if and only if the operator A generates an α-times resolvent family $S_\alpha(t)$, and the mild solution to (2.1) is given by

$$u(t) = \frac{d}{dt}\int_0^t S_\alpha(t-s)f(s)ds. \tag{2.4}$$

In particular, the mild solution to

$$u(t) = x + \int_0^t g_\alpha(t-s)Au(s)ds \tag{2.5}$$

is given by $u(t) = S_\alpha(t)x$; in addition, if $x \in D(A)$, then $u(t)$ is also a strong solution. By differentiating (2.5) α-times, we get a fractional equation of α-order

$$\begin{cases} D_t^\alpha u(t) = Au(t), \quad t > 0 \\ u(0) = x \quad (u_t(0) = 0 \text{ if } 1 < \alpha \leq 2). \end{cases} \tag{2.6}$$

It is also known that the well-posedness of (2.6) is equivalent to that of (2.1), and thus is equivalent to the existence of an α-times resolvent family $S_\alpha(t)$ for A. In this case, the unique mild solution of (2.6) is also given by $u(t) = S_\alpha(t)x$, For details we refer to [4].

Now we recall the following result on the generation of fractional resolvent families, which is crucial for our decomposition theorem.

Lemma 2.6. [22] *Let $0 < \alpha \leq 2$. Suppose that both A and $-A$ generate $\alpha/2$-times resolvent families $S^+_{\alpha/2}(t)$ and $S^-_{\alpha/2}(t)$, respectively. Then A^2 generates an α-times resolvent family $S_\alpha(t)$, which is given by $S_\alpha(t) = \frac{1}{2}[S^+_{\alpha/2}(t) + S^-_{\alpha/2}(t)]$.*

Remark 2.7. (1) If A generates a C_0-group then, by the subordination principle for fractional resolvent families [4, Theorem 3.1], both A and $-A$ generate $\alpha/2$-times resolvent families. Thus the generator of a C_0-group satisfies the assumptions in the above lemma.

(2) Let $1 < \alpha < 2$. Suppose that there is some $\theta > 0$ with $0 < \theta < \min\{\frac{\pi}{2}, \frac{\pi}{\alpha} - \frac{\pi}{2}\}$ such that

$$\sigma(A) \subset \{z \in \mathbb{C} : \frac{\alpha}{2}(\frac{\pi}{2} + \theta) \leq |\arg z| \leq \pi - \frac{\alpha}{2}(\frac{\pi}{2} + \theta)\} =: \Gamma_{\alpha,\theta}$$

and for every $\theta' > \theta$, there is a constant $M_{\theta'}$ such that

$$\|z(z-A)^{-1}\| \le M_{\theta'}, \quad z \in \mathbb{C} - \Gamma_{\alpha,\theta'}.$$

This is equivalent to saying that both A and $-A$ are sectorial operators with angle $\pi - \frac{\alpha}{2}(\frac{\pi}{2}+\theta)$. Then by [21, Lemma 2.7], both A and $-A$ generate bounded analytic $\alpha/2$-times resolvent families of angle θ, and A^2 also generates a bounded analytic α-times resolvent family of angle θ. The converse is also true by [6, Proposition 5.6]. For the case that $\alpha = 2$, we recall the fact that the generator of an analytic cosine function is always a bounded operator. It is interesting here to mention a result of Fattorini [9]: on a UMD space X, if A^2 generates a bounded cosine function, then A generates a C_0-group. It is not clear whether a similar result holds for the generator of a fractional resolvent family.

Let us begin with the d'Alembert formula for an abstract version of (1.5).

Theorem 2.8. *Let $1 < \alpha \le 2$. Suppose that both A and $-A$ generate $\alpha/2$-times resolvent families $S^+_{\alpha/2}(t)$ and $S^-_{\alpha/2}(t)$ on X, respectively. Suppose also that $\phi \in X$, $\psi \in R(A)$ and $\psi = A\Psi$ for some $\Psi \in D(A)$. Then the unique mild solution of the following integro-differential equation*

$$u(t) = \phi + \frac{t^{\alpha/2}}{\Gamma(1+\frac{\alpha}{2})}\psi + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A^2u(s)ds. \tag{2.7}$$

is given by

$$u(t) = \frac{1}{2}[S^+_{\alpha/2}(t)\phi + S^-_{\alpha/2}(t)\phi] + \frac{1}{2}[S^+_{\alpha/2}(t)\Psi - S^-_{\alpha/2}(t)\Psi]. \tag{2.8}$$

And the solution u can be decomposed as $u = \frac{1}{2}(u^+ + u^-)$, where u^+ and u^- are mild solutions to

$$u^+(t) = \phi + \Psi + \frac{1}{\Gamma(\alpha/2)}\int_0^t (t-s)^{\alpha/2-1}Au^+(s)ds, \quad t > 0, \tag{2.9}$$

and

$$u^-(t) = \phi - \Psi - \frac{1}{\Gamma(\alpha/2)}\int_0^t (t-s)^{\alpha/2-1}Au^-(s)ds, \quad t > 0, \tag{2.10}$$

respectively. Moreover, the corresponding fractional differential equation for (2.8) is

$$\begin{cases} D_t^{\alpha/2}D_t^{\alpha/2}u(t) = A^2u(t), & t > 0 \\ u(0) = \phi, \quad D_t^{\alpha/2}u(0) = \psi, \end{cases} \tag{2.11}$$

and the fractional differential equation corresponding to (2.9) and (2.10) are

$$\begin{cases} D_t^{\alpha/2}u^+(t) = Au^+(t), & t > 0 \\ u^+(0) = \phi + \Psi. \end{cases} \tag{2.12}$$

and

$$\begin{cases} D_t^{\alpha/2}u^-(t) = Au^-(t), & t > 0 \\ u^-(0) = \phi - \Psi. \end{cases} \tag{2.13}$$

respectively.

Proof. Suppose that u^+ and u^- are mild solutions of (2.9) and (2.10) respectively. Then $J_t^{\alpha/2}u^+, J_t^{\alpha/2}u^- \in D(A)$ and the integral equation

$$u^+ = \phi + \Psi + AJ_t^{\alpha/2}u^+$$

and

$$u^- = \phi - \Psi - AJ_t^{\alpha/2}u^-$$

hold. Thus,

$$\begin{aligned} u^+ + u^- &= 2\phi + AJ_t^{\alpha/2}(u^+ - u^-) \\ &= 2\phi + AJ_t^{\alpha/2}[2\Psi + AJ_t^{\alpha/2}(u^+ + u^-)]; \end{aligned}$$

since $\Psi \in D(A)$, this implies that $J_t^\alpha(u^+ + u^-) \in D(A^2)$ and

$$u^+ + u^- = 2\phi + 2J_t^{\alpha/2}\psi + A^2J_t^\alpha(u^+ + u^-).$$

Therefore, the function $u := \frac{1}{2}(u^+ + u^-)$ is the mild solution for (2.7). Moreover, since both A and $-A$ generate $\alpha/2$-times resolvent families,

$$u^+(t) = S^+_{\alpha/2}(t)(\phi + \Psi), \quad u^-(t) = S^-_{\alpha/2}(t)(\phi - \Psi),$$

this gives the representation (2.8). The uniqueness of the mild solution follows from the well-posedness of the integro-differential equation

$$u(t) = y + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A^2u(s)ds.$$

where $y \in X$, which is guaranteed by Lemma 2.6. The corresponding fractional differential equations can be derived by differentiation for fractional times. □

Remark 2.9. (1) When $\alpha = 2$ and $A = d/dx$ on the line, then

$$S_1^+(t)\phi(x) = \phi(x+t), \quad S_1^-(t)\phi(x) = \phi(x-t),$$

$\Psi(x) = \int_0^x \psi(y)dy$, and

$$S^+_{\alpha/2}(t)\Psi(x) - S^-_{\alpha/2}(t)\Psi(x) = \int_0^{x+t}\psi(y)dy - \int_0^{x-t}\psi(y)dy = \int_{x-t}^{x+t}\psi(y)dy.$$

Hence the formula (2.8) is exactly the classical d'Alembert formula (1.2). Thus (2.8) can be considered as the d'Alembert formula for the abstract integro-differential equation (2.7).

(2) When $A = d/dx$ on the line, the equations (2.9) and (2.10) are the same as Fujita's equation (1.7), so our decomposition can be viewed as the abstract version of Fujita's decomposition.

(3) When $\alpha = 2$, our expression (2.8) reduces to the formula (1.13) in [16, Chapter 3] given by Krein.

By Lemma 2.6 we can also derive the following result for more general fractional integro-differential equations.

Theorem 2.10. *Let $0 < \alpha \le 2$, f be a continuous function on X which is in $W^{1,1}([0,T], X)$ for every $T > 0$, and A a densely defined closed operator on X. If the two Volterra equations*

$$u^+_{\alpha/2}(t) = f(t) + \frac{1}{\Gamma(\alpha/2)}\int_0^t (t-s)^{\alpha/2-1}Au^+_{\alpha/2}(s)ds, \quad t \ge 0 \tag{2.14}$$

and

$$u^-_{\alpha/2}(t) = f(t) - \frac{1}{\Gamma(\alpha/2)} \int_0^t (t-s)^{\alpha/2-1} A u^-_{\alpha/2}(s)ds, \quad t \geq 0 \tag{2.15}$$

are well-posed, then the Volterra equation

$$u_\alpha(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A^2 u_\alpha(s)ds, \quad t \geq 0 \tag{2.16}$$

is also well-posed. Moreover, the unique mild solution of (2.16) *is given by*

$$u_\alpha(t) = \frac{1}{2}(u^+_{\alpha/2}(t) + u^-_{\alpha/2}(t)),$$

where $u^+_{\alpha/2}(t)$ and $u^-_{\alpha/2}(t)$ are mild solutions of (2.14) and (2.15), respectively.

Proof. The well-posedness of (2.14) and (2.15) implies that both A and $-A$ generate $\alpha/2$-times resolvent families on X, and we then denote them by $S^+_{\alpha/2}(t)$ and $S^-_{\alpha/2}(t)$ respectively. By Lemma 2.6 A^2 generates α-times resolvent family $S_\alpha(t)$, and therefore the Volterra equation (2.16) is also well-posed. By (2.4) we have the mild solution for (2.14) and (2.15) are given by

$$u^+_{\alpha/2}(t) = S^+_{\alpha/2}(t)f(0) + \int_0^t S^+_{\alpha/2}(t-s)f'(s)ds$$

and

$$u^-_{\alpha/2}(t) = S^-_{\alpha/2}(t)f(0) + \int_0^t S^-_{\alpha/2}(t-s)f'(s)ds,$$

respectively, and the mild solution for (2.16) is given by

$$u_\alpha(t) = S_\alpha(t)f(0) + \int_0^t S_\alpha(t-s)f'(s)ds,$$

therefore our claim follows from the fact that $S_\alpha(t) = (S^+_{\alpha/2}(t) + S^-_{\alpha/2}(t))/2$. □

As a consequence of Theorem 2.10, we have the next decomposition of the solutions to the following integro-differential equations mentioned in Introduction, and also for their corresponding fractional equations with Caputo derivatives.

Theorem 2.11. *Let $1 < \alpha \leq 2$, $\alpha/2 \leq \beta \leq \alpha$. Suppose that both A and $-A$ generate $\alpha/2$-times resolvent families $S^+_{\alpha/2}(t)$ and $S^-_{\alpha/2}(t)$ on X, respectively. Then for $\phi, \psi \in X$, the integro-differential equation*

$$u(t) = \phi + \frac{t^\beta}{\Gamma(1+\beta)}\psi + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A^2 u(s)ds, \quad t > 0 \tag{2.17}$$

has a unique mild solution which is given by

$$u(t) = \frac{1}{2}(S^+_{\alpha/2}(t)\phi + S^-_{\alpha/2}(t)\phi) + \frac{1}{2}(J^\beta_t S^+_{\alpha/2}(t)\psi + J^\beta_t S^-_{\alpha/2}(t)\psi), \tag{2.18}$$

and $u(t)$ can be decomposed into

$$u(t) = \frac{1}{2}(u^+(t) + u^-(t)),$$

where $u^+(t)$ and $u^-(t)$ are mild solutions to

$$u^+(t) = \phi + \frac{t^\beta}{\Gamma(1+\beta)}\psi + \frac{1}{\Gamma(\alpha/2)} \int_0^t (t-s)^{\alpha/2-1} A u^+(s)ds \tag{2.19}$$

and

$$u^-(t) = \phi + \frac{t^\beta}{\Gamma(1+\beta)}\psi - \frac{1}{\Gamma(\alpha/2)}\int_0^t (t-s)^{\alpha/2-1}Au^-(s)ds \tag{2.20}$$

respectively. Moreover, when $\beta = 1$, (2.18) is

$$u(t) = \frac{1}{2}[S^+_{\alpha/2}(t)\phi + S^-_{\alpha/2}(t)\phi] + \frac{1}{2}\int_0^t [S^+_{\alpha/2}(s)\psi + S^-_{\alpha/2}(s)\psi]ds, \tag{2.21}$$

which gives the unique mild solution to the fractional equation

$$\begin{cases} D_t^\alpha u(t) = A^2 u(t), \quad t > 0 \\ u(0) = \phi, \quad u_t'(0) = \psi; \end{cases} \tag{2.22}$$

if $\beta \neq 1$, (2.18) gives the unique mild solution to the fractional equation

$$\begin{cases} D_t^\alpha u(t) = A^2 u(t) + \frac{t^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}\psi, \quad t > 0 \\ u(0) = \phi, \quad D_t^\beta u(0) = \psi. \end{cases} \tag{2.23}$$

In both cases $u^+(t)$ and $u^-(t)$ given by (2.19) and (2.20), respctively, are mild solutions to

$$\begin{cases} D_t^{\alpha/2} u^+(t) = Au^+(t) + \frac{t^{\beta-\alpha/2}}{\Gamma(\beta-\alpha/2+1)}\psi, \quad t > 0 \\ u^+(0) = \phi, \end{cases} \tag{2.24}$$

and

$$\begin{cases} D_t^{\alpha/2} u^-(t) = -Au^-(t) + \frac{t^{\beta-\alpha/2}}{\Gamma(\beta-\frac{\alpha}{2}+1)}\psi, \quad t > 0 \\ u^-(0) = \phi, \end{cases} \tag{2.25}$$

respectively. If $\phi \in D(A^2)$ and $\psi \in D(A)$, then both the above mild solutions are strong solutions.

Proof. By taking $f(t) = \phi + \frac{t^\beta}{\Gamma(1+\beta)}\psi$ in Theorem 2.10, we have that the mild solution to (2.17) is given by

$$u(t) = S_\alpha(t)\phi + \int_0^t S_\alpha(t-s)\frac{s^{\beta-1}}{\Gamma(\beta)}\psi ds = S_\alpha(t)\phi + (J_t^\beta S_\alpha)(t)\psi.$$

This gives (2.18) since $S_\alpha(t) = \frac{1}{2}(S^+_{\alpha/2}(t)\phi + S^-_{\alpha/2}(t))$. Equations (2.22) and (2.23) follow by differentiating (2.17) α-times, and (2.24), (2.25) hold by differentiating (2.19), (2.20) $\alpha/2$-times, respectively.

Now for $\phi \in D(A^2)$ and $\psi \in D(A)$, we need only show that $u(t)$ is a strong solution of (2.17). Indeed, by the resolvent equation (2.2) and Remark 2.5 we have

$$\begin{aligned} u(t) &= S_\alpha(t)\phi + (J_t^\beta S_\alpha)(t)\psi \\ &= \phi + (g_\alpha * S_\alpha)(t)A^2\phi + J_t^\beta(\psi + A^2(g_\alpha * S_\alpha)(t)\psi) \\ &= \phi + \frac{t^\beta}{\Gamma(\beta+1)}\psi + (g_\alpha * A^2 S_\alpha)(t)\phi + J_t^\beta A^2(g_\alpha * S_\alpha)(t)\psi. \end{aligned}$$

To show that u is a strong solution it remains to show that $(J_t^\beta S_\alpha)(t)\psi \in D(A^2)$ and then $J_t^\beta A^2 J_t^\alpha S_\alpha(t)\psi = J_t^\alpha A^2 J_t^\beta S_\alpha(t)\phi$ by the closedness of the operator A.

Since $\beta \geq \alpha/2$, by the semigroup properties of Riemann-Liouville integrals and the resolvent equation,

$$\begin{aligned}(J_t^\beta S_\alpha)(t)A\psi &= \frac{1}{2}(J_t^\beta S_{\alpha/2}^+(t)A\psi + J_t^\beta S_{\alpha/2}^-(t)A\psi)\\ &= \frac{1}{2}J_t^{\beta-\alpha/2}(J_t^{\alpha/2}S_{\alpha/2}^+(t)A\psi + J_t^{\alpha/2}S_{\alpha/2}^-(t)A\psi)\\ &= \frac{1}{2}J_t^{\beta-\alpha/2}(S_{\alpha/2}^+(t)\psi - \psi + \psi - S_{\alpha/2}^-(t)\psi)\\ &= \frac{1}{2}J_t^{\beta-\alpha/2}(S_{\alpha/2}^+(t)\psi - S_{\alpha/2}^-(t)\psi),\end{aligned}$$

and this implies that $(J_t^\beta S_\alpha)(t)\psi \in D(A^2)$. □

Remark 2.12. (1) When $\alpha = 2$, $\beta = 1$ and $A = d/dx$ on the line, then

$$S_1^+(t)\phi(x) = \phi(x+t), \quad S_1^-(t)\phi(x) = \phi(x-t),$$

and

$$\int_0^t [S_{\alpha/2}^+(s)\psi(x) + S_{\alpha/2}^-(s)\psi(x)]ds = \int_0^t [\psi(x+s) + \psi(x-s)]ds = \int_{x-t}^{x+t} \psi(y)dy,$$

so the formula (2.18) is exactly the classical d'Alembert's formula (1.2). It is therefore reasonable to call (2.18) the d'Alembert formula for (2.17).

(2) When $\beta = 1$, the equation (2.17) becomes

$$u(t) = \phi + t\psi + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A^2u(s)ds. \tag{2.26}$$

The identity for u in (2.21) could be considered as the d'Alembert formula for Volterra equation (2.26).

(3) When $\beta = \alpha/2$, then (2.17) is in fact (2.7). The formula

$$u(t) = \frac{1}{2}(S_{\alpha/2}^+(t)\phi + S_{\alpha/2}^-(t)\phi) + \frac{1}{2}(J_t^{\alpha/2}S_{\alpha/2}^+(t)\psi + J_t^{\alpha/2}S_{\alpha/2}^-(t)\psi)$$

is then an alternative d'Alembert formula for (2.7), and a generalization of (1.6). Their difference will be clarified in Remark 2.14.

(4) The α-order equation

$$\begin{cases} D_t^\alpha u(t) = A^2u(t), \quad t > 0\\ u(0) = \phi, \quad u_t'(0) = 0\end{cases}$$

is equivalent to the sequential α-order equation

$$\begin{cases} D_t^{\alpha/2}D_t^{\alpha/2}u(t) = A^2u(t), \quad t > 0\\ u(0) = \phi, \quad D_t^{\alpha/2}(0) = 0;\end{cases}$$

their corresponding integral equation is

$$u(t) = \phi + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A^2u(s)ds.$$

The solution is given by $u(t) = \frac{1}{2}(S_{\alpha/2}^+(t)\phi + S_{\alpha/2}^-(t)\phi)$.

On the other hand, motivated by the analysis in the Introduction, we connect the integro-differential equation (2.17) with a fractional differential equation of sequential Caputo derivatives [30].

Theorem 2.13. *Let $1 < \alpha \leq 2$ and $\alpha/2 \leq \beta \leq \alpha$. Suppose that both A and $-A$ generate $\alpha/2$-times resolvent families $S^+_{\alpha/2}(t)$ and $S^-_{\alpha/2}(t)$ on X, respectively. Suppose also that $\phi \in D(A^2)$ and $\psi \in D(A)$. Then (2.18) gives a strong solution to the sequential α-order Cauchy problem*

$$\begin{cases} D_t^{\alpha-\beta} D_t^{\beta} u(t) = A^2 u(t), \quad t > 0 \\ u(0) = \phi, \quad D_t^{\beta} u(0) = \psi. \end{cases} \tag{2.27}$$

Proof. Note that for $\phi \in D(A^2)$ by applying the resolvent equation (2.2) twice we have

$$\begin{aligned} S^{\pm}_{\alpha/2}(t)\phi &= \phi \pm (g_{\alpha/2} * S^{\pm}_{\alpha/2})(t)A\phi \\ &= \phi \pm (g_{\alpha/2} * 1)(t)A\phi + (g_{\alpha/2} * g_{\alpha/2} * S^{\pm}_{\alpha/2})(t)A^2\phi \\ &= \phi \pm g_{\alpha/2+1}(t)A\phi + (g_{\alpha} * S^{\pm}_{\alpha/2})(t)A^2\phi, \end{aligned}$$

thus $S^{\pm}_{\alpha/2}(t)\phi$ are β-order differentiable and

$$D_t^{\beta} S^{\pm}_{\alpha/2}(t)\phi = \pm g_{\alpha/2+1-\beta}(t)A\phi + (g_{\alpha-\beta} * S^{\pm}_{\alpha/2})(t)A^2\phi.$$

By summing the above two identities we get

$$D_t^{\beta}(S^+_{\alpha/2}(t)\phi + S^-_{\alpha/2}(t)\phi) = g_{\alpha-\beta} * (S^+_{\alpha/2} + S^-_{\alpha/2})(t)A^2\phi,$$

from which it follows that $D_t^{\beta}(S^+_{\alpha/2}(t)\phi + S^-_{\alpha/2}(t)\phi)$ is $(\alpha-\beta)$-order differentiable and

$$\begin{aligned} D_t^{\alpha-\beta} D_t^{\beta}(S^+_{\alpha/2}(t)\phi + S^-_{\alpha/2}(t)\phi) &= (S^+_{\alpha/2} + S^-_{\alpha/2})(t)A^2\phi \\ &= A^2(S^+_{\alpha/2}(t)\phi + S^-_{\alpha/2}(t)\phi), \end{aligned} \tag{2.28}$$

since A commutes with $S^{\pm}_{\alpha/2}(t)$. For $\psi \in D(A)$, as in the proof of Theorem 2.11 we have $(J_t^{\beta} S^+_{\alpha/2}(t)\psi + J_t^{\beta} S^-_{\alpha/2}(t)\psi) \in D(A^2)$ and

$$A^2(J_t^{\beta} S^+_{\alpha/2}(t)\psi + J_t^{\beta} S^-_{\alpha/2}(t)\psi) = J_t^{\beta-\alpha/2}(S^+_{\alpha/2}(t)A\psi - S^-_{\alpha/2}(t)A\psi). \tag{2.29}$$

On the other hand, since

$$J_t^{\beta} S^{\pm}_{\alpha/2}(t)\psi = g_{\beta+1}(t)\psi \pm J_t^{\beta+\alpha/2} S^+_{\alpha/2}(t)A\psi,$$

we have

$$D_t^{\beta} J_t^{\beta} S^{\pm}_{\alpha/2}(t)\psi = \psi \pm J_t^{\alpha/2} S^+_{\alpha/2}(t)A\psi = S^{\pm}_{\alpha/2}\psi,$$

and thus

$$D_t^{\alpha-\beta} D_t^{\beta}(J_t^{\beta} S^+_{\alpha/2}(t)\psi + J_t^{\beta} S^-_{\alpha/2}(t)\psi) = J_t^{\beta-\alpha/2}(S^+_{\alpha/2}(t)A\psi - S^-_{\alpha/2}(t)A\psi). \tag{2.30}$$

Combining (2.28), (2.29) and (2.30) we have proven that $u(t)$ defined by (2.18) is a strong solution of (2.27), where the initial conditions are easy to check. □

Remark 2.14. As a direct consequence, we have the following sequential fractional differential equation corresponding to (2.7):

$$\begin{cases} D_t^{\alpha/2} D_t^{\alpha/2} u(t) = A^2 u(t), \quad t > 0 \\ u(0) = \phi, \quad D_t^{\alpha/2} u(0) = \psi. \end{cases}$$

The solution can be decomposed as half of the sum of the solutions to

$$\begin{cases} D_t^{\alpha/2} u^+(t) = Au^+(t) + \psi, \quad t > 0 \\ u^+(0) = \phi, \end{cases}$$

and

$$\begin{cases} D_t^{\alpha/2} u^-(t) = -Au^-(t) + \psi, \quad t > 0 \\ u^-(0) = \phi. \end{cases}$$

In case of $A = \partial/\partial x$ and $\alpha = 2$, this means that we can decompose the solution of the traditional wave equation (1.1) into the half sum of solutions to the following two first-order nonhomogeneous equation

$$\begin{cases} u_t^+(t,x) = u_x^+(t,x) + \psi(x) \\ u_0^+(t,x) = \phi(x) \end{cases} \quad \text{and} \quad \begin{cases} u_t^-(t,x) = -u_x^-(t,x) + \psi(x) \\ u_0^-(t,x) = \phi(x). \end{cases}$$

In other words, Fujita's decomposition means that one can decompose the solution of the wave equation into the half sum of

$$u^+(x,t) = \phi(x+t) + \int_0^{x+t} \psi(y)dy \quad \text{and} \quad u^-(x,t) = \phi(x-t) - \int_0^{x-t} \psi(y)dy;$$

while our decomposition is

$$u^+(x,t) = \phi(x+t) + \int_x^{x+t} \psi(y)dy \quad \text{and} \quad u^-(x,t) = \phi(x-t) + \int_{x-t}^{x} \psi(y)dy.$$

Finally in this section, let us consider the Riemann-Liouville fractional differential equation

$$\begin{cases} {}_{RL}D_t^{\alpha} u(t) = A^2 u(t), \quad t > 0 \\ (g_{2-\alpha} * u)(0) = 0, \quad (g_{2-\alpha} * u)'(0) = \psi \end{cases} \tag{2.31}$$

with $1 < \alpha \leq 2$. By integration with respect to t for α-times, we get the corresponding integro-differential equation as follows

$$u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\psi + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A^2 u(s)ds.$$

If A^2 generates an α-times resolvent family $S_\alpha(t)$, then the solution u is given by $J_t^{\alpha-1} S_\alpha(t)\psi$ [19]. Indeed, it is easy to verify that

$$\begin{aligned} u(t) &= J_t^{\alpha-1} S_\alpha(t)\psi \\ &= J_t^{\alpha-1}(\psi + A^2(g_\alpha * S_\alpha)\psi) \\ &= g_\alpha(t)\psi + A^2(g_\alpha * J_t^{\alpha-1} S_\alpha)\psi \\ &= g_\alpha(t)\psi + A^2(g_\alpha * u)(t). \end{aligned}$$

On the other hand,

$$u(t) = J_t^{\alpha-1} S_\alpha(t)\psi = \frac{1}{2} J_t^{\alpha-1}(S_{\alpha/2}^+(t) + S_{\alpha/2}^-(t))\psi$$

where $J_t^{\alpha-1} S_{\alpha/2}^+(t)\psi$ and $J_t^{\alpha-1} S_{\alpha/2}^-(t)\psi$ are solutions of

$$v(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\psi + \frac{1}{\Gamma(\alpha/2)} \int_0^t (t-s)^{\alpha/2-1} Av(s)ds.$$

and

$$v(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\psi - \frac{1}{\Gamma(\alpha/2)}\int_0^t (t-s)^{\alpha/2-1}Av(s)ds.$$

respectively, and the corresponding Riemann-Liouville fractional equations are

$$\begin{cases} {}_{RL}D_t^{\alpha/2}v(t) = Av(t) + \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)}\psi, \quad t > 0 \\ (g_{1-\alpha/2} * v)(0) = 0 \end{cases} \tag{2.32}$$

and

$$\begin{cases} {}_{RL}D_t^{\alpha/2}v(t) = -Av(t) + \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)}\psi, \quad t > 0 \\ (g_{1-\alpha/2} * v)(0) = 0 \end{cases} \tag{2.33}$$

respectively.

In summary, we have

Theorem 2.15. *Let $1 < \alpha \le 2$. If both A and $-A$ generate $\alpha/2$-times resolvent families $S^+_{\alpha/2}(t)$ and $S^-_{\alpha/2}(t)$ on X, respectively, then the unique mild solution of the α-order Riemann-Liouville fractional Cauchy problem (2.31) with $\psi \in X$ is given by*

$$u_\alpha(t) = \frac{1}{2}(J_t^{\alpha-1}S^+_{\alpha/2}(t)\psi + J_t^{\alpha-1}S^-_{\alpha/2}(t)\psi) =: \frac{1}{2}(u^+_{\alpha/2}(t) + u^-_{\alpha/2}(t)), \tag{2.34}$$

where $u^+_{\alpha/2}(t)$ and $u^-_{\alpha/2}(t)$ are mild solution to (2.32) and (2.33), respectively.

3. Examples

In this section, we will illustrate the interpretation and application of our abstract setting with several examples. The inverse of a standard stable subordinator will play an important role in our examples, therefore we will give a brief of the connection with time-fractional differential equation for reader's convenience. See [25, 26] for more details.

Throughout this section, we define S_t as the following processes on $\{x : x \ge 0\}$ for $0 < \alpha \le 2$:

- (i) When $0 < \alpha < 2$, S_t is a standard $\alpha/2$-stable subordinator, a Lévy process such that $\mathbf{E}\exp\{-\lambda S_t\} = \exp\{-t\lambda^{\alpha/2}\}$;
- (ii) When $\alpha = 2$, S_t is a deterministic continuous process with uniform velocity 1, starting from 0. Indeed, S_t in this case can be regarded as a degenerate stable subordinator with density function $\delta(x-t)$ and $\mathbf{E}\exp\{-\lambda S_t\} = \exp\{-t\lambda\}$. For unity, we will use 1-stable subordinator to denote S_t for $\alpha = 2$.

Then the inverse $\alpha/2$-stable subordinator $Y_{\alpha/2}(t) = \inf\{u > 0 : S_u > t\}$ is a continuous, nondecreasing and nonnegative process with $Y_{\alpha/2}(0) = 0$, and

$$\mathbf{E}\exp\{-sY_{\alpha/2}(t)\} = E_{\alpha/2}(-st^{\alpha/2}),$$

where $E_\alpha(t) = \sum_{n=0}^\infty \frac{t^n}{\Gamma(\alpha n+1)}$ is the Mittag-Leffler function. Denote the probability density function of $Y_{\alpha/2}(t)$ by $\varphi_{\alpha/2}(t,\cdot)$, then we have

$$\int_0^\infty e^{-\lambda t}\varphi_{\alpha/2}(t,s)dt = \lambda^{\frac{\alpha}{2}-1}e^{-s\lambda^{\alpha/2}}. \tag{3.1}$$

From the above identity, it is not hard to show that for a suitable function f the following holds for λ large enough:

$$\int_0^\infty e^{-\lambda t}\mathbf{E}[f(Y_{\alpha/2}(t))]dt = \lambda^{\frac{\alpha}{2}-1}\int_0^\infty f(s)e^{-s\lambda^{\alpha/2}}ds. \tag{3.2}$$

On the other hand, since the first-order derivative operator $\partial_x f(t,x) = f_x(t,x)$ generates a C_0-semigroup given by $T(t)\phi(x) = \phi(x+t)$, by the subordination principle for fractional resolvent family [4], for $1<\alpha<2$, the $\alpha/2$-times resolvent family $S^+_{\alpha/2}(t)$ is given by

$$S^+_{\alpha/2}(t)\phi(x) = \int_0^\infty \varphi_{\alpha/2}(t,s)T(s)\phi(x)ds = \int_0^\infty \varphi_{\alpha/2}(t,s)\phi(s+x)ds,$$

It follows that

$$\int_0^\infty e^{-\lambda t}S^+_{\alpha/2}(t)\phi(x)dt = \lambda^{\frac{\alpha}{2}-1}\int_0^\infty \phi(s+x)e^{-s\lambda^{\alpha/2}}ds$$

from (3.1).

Comparing with (3.2) and using the uniqueness of the Laplace transform, we have

$$S^+_{\alpha/2}(t)\phi(x) = \mathbf{E}[\phi(x+Y_{\alpha/2}(t))]. \tag{3.3}$$

This is to say that $\mathbf{E}[\phi(x+Y_{\alpha/2}(t))]$ gives the unique solution to the fractional differential equation

$$\begin{cases} D_t^{\alpha/2}u(t,x) = u_x(t,x), \quad t>0, \quad x\in\mathbb{R} \\ u(0,x)=\phi(x). \end{cases} \tag{3.4}$$

Similarly, the unique solution to the fractional differential equation

$$\begin{cases} D_t^{\alpha/2}u(t,x) = -u_x(t,x), \quad t>0, \quad x\in\mathbb{R} \\ u(0,x)=\phi(x) \end{cases} \tag{3.5}$$

is given by

$$S^-_{\alpha/2}(t)\phi(x) = \mathbf{E}[\phi(x-Y_{\alpha/2}(t))]. \tag{3.6}$$

The fundamental solution to equation (3.5) is the probability density function of an inverse $\alpha/2$-stable subordinator. Meanwhile, the fundamental solution to equation (3.4) is the reflected density ($x\mapsto -x$) of an inverse $\alpha/2$-stable subordinator.

Example 3.1. *Let $1<\alpha\le 2$. By Theorem 2.8 and the above analysis, the solution of*

$$u(t,x) = \phi(x) + \frac{t^{\alpha/2}}{\Gamma(1+\alpha/2)}\psi(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u_{xx}(s,x)ds \tag{3.7}$$

or

$$\begin{cases} D_t^{\alpha/2}D_t^{\alpha/2}u(t,x) = u_{xx}(t,x), \quad t>0, \quad x\in\mathbb{R} \\ u(0,x)=\phi(x), \quad D_t^{\alpha/2}u(0,x)=\psi(x) \end{cases} \tag{3.8}$$

can be represented as

$$u(t,x) = \frac{1}{2}\mathbf{E}[\phi(x+Y_{\alpha/2}(t)) + \phi(x-Y_{\alpha/2}(t))] + \frac{1}{2}\mathbf{E}\int_{x-Y_{\alpha/2}(t)}^{x+Y_{\alpha/2}(t)}\psi(y)dy. \tag{3.9}$$

This is the same as Fujita's formula (1.6). Thus the motion (or the evolution) of $u(t,x)$ governed by equation (3.8) is mixed with two components: positive direction and negative direction on line $\mathbb{R}$. The solution of

$$\begin{cases} D_t^{\alpha/2}u(t,x) = u_x(t,x), \quad t>0, \quad x\in\mathbb{R} \\ u(0,x) = \phi(x) + \int_0^x \psi(y)dy \end{cases}$$

gives the motion on positive direction, while the solution of equation

$$\begin{cases} D_t^{\alpha/2}u(t,x) = -u_x(t,x), \quad t>0, \quad x\in\mathbb{R} \\ u(0,x) = \phi(x) - \int_0^x \psi(y)dy \end{cases}$$

gives the motion on negative direction.

The above interpretation, adapted from [24], is consistent with the classical d'Alembert formula for the one-dimensional wave equation on line: decompose the wave equation into two transport equations with positive direction and negative direction. Moreover, the fractional d'Alembert formula indicates the 'random alternative of direction' or 'wander in positive or negative direction' for the motion governed by one dimensional fractional wave equation [11].

Example 3.2. *Here we consider the following integro-differential equation*

$$u(t,x) = \phi(x) + \frac{t^\beta}{\Gamma(1+\beta)}\psi(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u_{xx}(s,x)ds \tag{3.10}$$

and the corresponding fractional Cauchy problem

$$\begin{cases} D_t^{\alpha-\beta}D_t^\beta u(t,x) = u_{xx}(t,x), \quad t>0 \\ u(0,x) = \phi(x), \quad D_t^\beta u(0,x) = \psi(x) \end{cases} \tag{3.11}$$

where $\alpha/2 \le \beta \le \alpha$. We will explore the connection between their fractional d'Alembert solution and its stochastic interpretation.

To start with, we first claim the following formula holds:

$$(g_\beta * f_{\alpha/2})(t,y) = \int_y^{+\infty} (g_{\beta-\alpha/2} * f_{\alpha/2})(t,z)dz, \quad t,y>0. \tag{3.12}$$

where $f_{\alpha/2}(t,y)$ is the probability density function of the inverse $\alpha/2$-stable subordinator $Y_{\alpha/2}(t)$. By the properties of the inverse stable subordinator [25]*, the Laplace transform on variable t of the left side is*

$$\widehat{g_\beta}(\lambda)\widehat{f_{\alpha/2}}(\lambda,y) = \lambda^{-\beta}\cdot\lambda^{\alpha/2-1}e^{-\lambda^{\alpha/2}y};$$

on the other hand, the Laplace transform on variable t of the right side is

$$\begin{aligned} & \int_0^{+\infty} e^{-\lambda t}dt \int_y^{+\infty} (g_{\beta-\alpha/2} * f_{\alpha/2})(t,z)dz \\ = & \int_y^{+\infty} dz \int_0^{+\infty} e^{-\lambda t}(g_{\beta-\alpha/2} * f_{\alpha/2})(t,z)dt \\ = & \int_y^{+\infty} \lambda^{-\beta+\alpha/2}\cdot\lambda^{\alpha/2-1}e^{-\lambda^{\alpha/2}z}dz \\ = & \lambda^{-\beta}\cdot\lambda^{\alpha/2-1}e^{-\lambda^{\alpha/2}y}. \end{aligned}$$

Therefore we draw the conclusion by the uniqueness of the Laplace transform.

Also, the equation (3.12) is formally equivalent to

$$D_t^{\alpha/2}(g_{\beta-\alpha/2} * f_{\alpha/2})(t,y) = -\frac{d}{dy}(g_{\beta-\alpha/2} * f_{\alpha/2})(t,y),$$

which reduces to the governing equation for the inverse $\alpha/2$-stable subordinator when $\beta = \alpha/2$. The term $(g_{\beta-\alpha/2} * f_{\alpha/2})(t,y)$ is not probability density if $\beta \neq \alpha/2$. However, for any $\alpha/2 \leq \beta \leq \alpha$, it is easy to see $(g_{\beta-\alpha/2} * f_{\alpha/2})(t,y)$ is positive and could be normalized as $(g_{1+\beta-\alpha/2}(t))^{-1}(g_{\beta-\alpha/2} * f_{\alpha/2})(t,y)$, i.e., $\int_0^\infty (g_{1+\beta-\alpha/2}(t))^{-1}(g_{\beta-\alpha/2} * f_{\alpha/2})(t,y)dy = 1$. We denote the corresponding random variables by $H_\beta(t)$, and note that $H_\beta(0) = 0$ almost surely.

Let $S^+_{\alpha/2}(t)$ and $S^-_{\alpha/2}(t)$ be given by (3.3) and (3.6), respectively. By Theorem 2.11, the solution of (3.10) can be represented as

$$u(t,x) = \frac{1}{2}(S^+_{\alpha/2}(t) + S^-_{\alpha/2}(t))\phi(x) + \frac{1}{2}(J_t^\beta S^+_{\alpha/2}(t)\psi(x) + J_t^\beta S^-_{\alpha/2}(t)\psi(x)).$$

By formula (3.12) we have

$$\begin{aligned}
J_t^\beta S^+_{\alpha/2}(t)\psi(x) &= \int_0^t g_\beta(t-s)\mathbf{E}[\psi(x+Y_{\alpha/2}(s))]ds \\
&= \int_0^t g_\beta(t-s)\int_0^\infty \psi(x+y) f_{\alpha/2}(s,y)dyds \\
&= \int_0^\infty \psi(x+y)\int_0^t g_\beta(t-s) f_{\alpha/2}(s,y)dsdy \\
&= \int_0^\infty \psi(x+y)\int_y^\infty (g_{\beta-\alpha/2} * f_{\alpha/2})(t,z)dzdy \\
&= \int_x^\infty \psi(y)dy\int_{y-x}^\infty (g_{\beta-\alpha/2} * f_{\alpha/2})(t,z)dz \\
&= \int_0^\infty (g_{\beta-\alpha/2} * f_{\alpha/2})(t,z)dz\int_x^{x+z} \psi(y)dy \\
&= g_{1+\beta-\alpha/2}(t)\mathbf{E}[\Psi(x+H_\beta(t))] - \int_0^\infty \Psi(y)(g_{\beta-\alpha/2} * f_{\alpha/2})(t,z)dz
\end{aligned}$$

where $\Psi(x) = \int_0^x \psi(y)dy$; similarly we have

$$J_t^\beta S^-_{\alpha/2}(t)\psi(x) = -g_{1+\beta-\alpha/2}(t)\mathbf{E}[\Psi(x-H_\beta(t))] + \int_0^\infty \Psi(x)(g_{\beta-\alpha/2} * f_{\alpha/2})(t,z)dz.$$

Therefore,

$$\begin{aligned}
u(t,x) &= \frac{1}{2}(S^+_{\alpha/2}(t) + S^-_{\alpha/2}(t))\phi(x) \\
&\quad + \frac{1}{2}g_{1+\beta-\alpha/2}(t)(\mathbf{E}[\Psi(x+H_\beta(t))] - \mathbf{E}[\Psi(x-H_\beta(t))]) \\
&= \frac{1}{2}\mathbf{E}[\phi(x+Y_{\alpha/2}(t)) + \phi(x-Y_{\alpha/2}(t))] + \frac{1}{2}g_{1+\beta-\alpha/2}(t)\mathbf{E}\int_{x-H_\beta(t)}^{x+H_\beta(t)} \psi(y)dy.
\end{aligned}$$

Roughly speaking, the fractional d'Alembert's formula is essentially based on the properties of the square root of $A^2 f(x) = f''(x)$, rather than the order of the initial values. The formula and stochastic interpretation are clean and elegant when $\beta = \alpha/2$, because $H_\beta(t) = Y_{\alpha/2}(t)$ and $g_{1+\beta-\alpha/2}(t) = 1$ in this case. The above formula *is then consistent with the formula (1.6) of Fujita. However, we are unable to clarify the interpretation for the other $H_\beta(t)$.*

As a continuation of Example 3.2, we shall compare two special cases: $\beta = \alpha/2$ and $\beta = 1$.

Example 3.3. *The solution to (3.7) could be represented in two different ways:*

$$u(t,x) = \frac{1}{2}\mathbf{E}[\phi(x+Y_{\alpha/2}(t)) + \phi(x-Y_{\alpha/2}(t))] + \frac{1}{2}\mathbf{E}\int_{x-Y_{\alpha/2}(t)}^{x+Y_{\alpha/2}(t)} \psi(y)dy$$

and

$$\begin{aligned} u(t,x) &= \frac{1}{2}\mathbf{E}[\phi(x+Y_{\alpha/2}(t)) + \phi(x-Y_{\alpha/2}(t))] \\ &\quad + \frac{1}{2}\Big\{\frac{1}{\Gamma(\alpha/2)}\int_0^t (t-s)^{\alpha/2-1}\mathbf{E}[\psi(x+Y_{\alpha/2}(s)) + \psi(x-Y_{\alpha/2}(s))]ds\Big\}. \end{aligned}$$

The solution to the integro-differential equation

$$v(t,x) = \phi(x) + t\psi(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v_{xx}(s,x)ds. \tag{3.13}$$

could be represented in two different ways:

$$v(t,x) = \frac{1}{2}\mathbf{E}[\phi(x+Y_{\alpha/2}(t)) + \phi(x-Y_{\alpha/2}(t))] + \frac{1}{2}g_{2-\alpha/2}(t)\mathbf{E}\int_{x-H_1(t)}^{x+H_1(t)} \psi(y)dy$$

and

$$\begin{aligned} v(t,x) &= \frac{1}{2}\mathbf{E}[\phi(x+Y_{\alpha/2}(t)) + \phi(x-Y_{\alpha/2}(t))] \\ &\quad + \frac{1}{2}\int_0^t \mathbf{E}[\psi(x+Y_{\alpha/2}(s)) + \psi(x-Y_{\alpha/2}(s))]ds. \end{aligned}$$

If a more compact form for d'Alembert's solution is needed, we should choose the first representation in the case $\beta = \alpha/2$, and the second kind representation in the case $\beta = 1$.

We will study the d'Alembert solution to differential-difference equations in the next example.

Example 3.4. *Consider the first-order backward difference operator*

$$A\phi(x) = \phi(x) - \phi(x-1), \quad \phi \in C_0(\mathbb{R}),\ x \in \mathbb{R}. \tag{3.14}$$

It is clear that A is a bounded operator with $\|A\| = 2$, and A generates a bounded C_0-semigroup e^{tA} which is represented as $e^{tA}\phi(x) = e^t \sum_{k=0}^\infty \frac{(-t)^k\phi(x-k)}{k!}$. Therefore, A generates an $\alpha/2$-times resolvent family $S^+_{\alpha/2}(t)$ by the subordination principle:

$$S^+_{\alpha/2}(t)\phi(x) = \int_0^\infty \varphi_{\alpha/2}(t,s)e^{sA}\phi(x)ds, \quad 0 < \alpha \le 2 \tag{3.15}$$

where $\varphi_{\alpha/2}(t,\cdot)$ is given by (3.1). Analogously, the operator $(-A)$ generates a bounded C_0-semigroup $e^{-tA}\phi(x) = e^{-t}\sum_{k=0}^\infty \frac{t^k\phi(x-k)}{k!}$ and $\alpha/2$-times resolvent family

$$S^-_{\alpha/2}(t)\phi(x) = \int_0^\infty \varphi_{\alpha/2}(t,s)e^{-sA}\phi(x)ds. \tag{3.16}$$

In other words, $S^+_{\alpha/2}(t)\phi(x)$ and $S^-_{\alpha/2}(t)\phi(x)$ satisfy the equations

$$\begin{cases} D_t^{\alpha/2}u^+(t,x) = Au^+(t,x), \quad t>0, \quad x\in\mathbb{R} \\ u^+(0,x)=\phi(x) \end{cases} \tag{3.17}$$

and

$$\begin{cases} D_t^{\alpha/2}u^-(t,x) = -Au^-(t,x), \quad t>0, \quad x\in\mathbb{R} \\ u^-(0,x)=\phi(x) \end{cases} \tag{3.18}$$

respectively.

It is easy to see $A^2\phi(x)=\phi(x)-2\phi(x-1)+\phi(x-2)$. Next we introduce the following wave-type differential-difference equation

$$\begin{cases} D_t^{\alpha}u(t,x) = A^2u(t,x), \quad t>0, \quad x\in\mathbb{R} \\ u(0,x)=\phi(x), \quad u_x(0,x)=0 \end{cases} \tag{3.19}$$

in which $1<\alpha\le 2$. By Theorem 2.11 the solution of equation (3.19) can be represented as

$$u(t,x)=\frac{1}{2}[S^+_{\alpha/2}(t)\phi(x)+S^-_{\alpha/2}(t)\phi(x)]. \tag{3.20}$$

Next we consider the discrete version of Example 3.4, which is related to the time-fractional Poisson process [18, 23, 27].

Example 3.5. *For $1<\alpha\le 2$, consider a wave-type differential-difference equation on the integers:*

$$\begin{cases} D_t^{\alpha/2}D_t^{\alpha/2}p(t,k) = p(t,k)-2p(t,k-1)+p(t,k-2), \quad t>0, \quad k\in\mathbb{Z} \\ p(0,0)=1 \\ p(0,k)=0, \quad k\neq 0 \\ D_t^{\alpha/2}p(0,k)=0. \end{cases} \tag{3.21}$$

The solution for (3.21) can be represented as

$$p_\alpha(t,k)=\frac{1}{2}(p^+_{\alpha/2}(t,k)+p^-_{\alpha/2}(t,k)), \tag{3.22}$$

in which $p^+_{\alpha/2}(t,k)$ and $p^-_{\alpha/2}(t,k)$ are the solutions for

$$\begin{cases} D_t^{\alpha/2}p^+_{\alpha/2}(t,k) = p^+_{\alpha/2}(t,k)-p^+_{\alpha/2}(t,k-1), \quad t>0, \quad k\in\mathbb{Z} \\ p^+_{\alpha/2}(0,0)=1 \\ p^+_{\alpha/2}(0,k)=0, \quad k\neq 0 \end{cases} \tag{3.23}$$

and

$$\begin{cases} D_t^{\alpha/2}p^-_{\alpha/2}(t,k) = -\big[p^-_{\alpha/2}(t,k)-p^-_{\alpha/2}(t,k-1)\big], \quad t>0, \quad k\in\mathbb{Z} \\ p^-_{\alpha/2}(0,0)=1 \\ p^-_{\alpha/2}(0,k)=0, \quad k\neq 0 \end{cases} \tag{3.24}$$

respectively.

If k is restricted to non-negative integers, then equation (3.24) governs the time fractional Poisson process, see [18, 23, 27] and references therein. The time fractional Poisson process governed by (3.24) is a non-decreasing counting process $N^-(t)$, and $p^-_{\alpha/2}(t,k)$ is the probability $p^-_{\alpha/2}(t,k)=P(N^-(t)=k)$. Similarly, *we can regard equation (3.23) as the governing equation for the (negative) counting process* $N^+(t)$ *which is a non-increasing process taking values in the non-positive integers.*

Therefore, the d'Alembert solution (3.22) indicates that the stochastic process $N(t)$ *governed by (3.21) is a sum of the (positive and negative) time fractional Poisson processes* $N^+(t)$ *and* $N^-(t)$*, and we may consider* $N(t)$ *as a wave-type fractional Poisson process.*

The d'Alembert formula solution to evolutionary differential equations can also be realized in higher dimensions, as we will illustrate in the next example. The decomposition is related to the fractional Schrödinger equation.

Example 3.6. *Consider the following time fractional diffusion-wave equation*

$$\begin{cases} D_t^\alpha u(t,x) = \Delta u(t,x), \quad t>0,\, x\in\mathbb{R}^n, \\ u(0,x)=\phi(x), \quad u_t(0,x)=0, \\ \lim\limits_{|x|\to\infty} u(t,x) = 0 \end{cases} \tag{3.25}$$

where $1<\alpha\le 2$ *and* $\phi\in L^2(\mathbb{R}^n)$*. For simplicity, we will only study the mild solution on* $L^2(\mathbb{R}^n)$*.*

Since Δ *generates cosine function* $C(t)$ *on* $L^2(\mathbb{R}^n)$ *([2]), it also generates an* α*-times fractional resolvent family* $S_\alpha(t)$ *on* $L^2(\mathbb{R}^n)$ *by subordination principle and*

$$S_\alpha(t) = \int_0^\infty \varphi_{\alpha/2}(t,s)C(s)ds, \qquad 1<\alpha\le 2$$

where $\varphi_{\alpha/2}(t,\cdot)$ *is given by (3.1). Then the mild solution to (3.25) can be represented as*

$$u(t,x) = S_\alpha(t)\phi(x).$$

Especially, $u(t,x)=C(t)\phi(x)$ *when* $\alpha=2$*.*

Define $-(-\Delta)^{\alpha/2}$ *as the usual Riesz fractional Laplacian operator, i.e.,*

$$-(-\Delta)^{\alpha/2}f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}|\xi|^\alpha \hat{f}(\xi)d\xi,$$

where $\hat{f}(\xi)=\int_{\mathbb{R}^n} e^{-i\xi\cdot x}f(x)dx$*.*

Let $A=i[-(-\Delta)^{1/2}]$ *and* $-A=-i[-(-\Delta)^{1/2}]$*. Based on the Theorem 3.16.7 in [2], we obtain the following conclusion:* $(\pm A)^2=\Delta$*,* A *and* $-A$ *generate* C_0*-group* $U(t)$ *and* $U(-t)$ *on* $L^2(\mathbb{R}^n)$ *respectively, and the group reduction formula*

$$C(t) = \frac{1}{2}(U(t)+U(-t)).$$

Therefore the mild solution to (3.25) can also be represented as

$$u(t,x) = \frac{1}{2}\Big(\int_0^\infty \varphi_{\alpha/2}(t,s)U(s)\phi(x)ds + \int_0^\infty \varphi_{\alpha/2}(t,s)U(-s)\phi(x)ds\Big) \tag{3.26}$$

which can be regarded as the d'Alembert formula solution in higher dimensions. Here $\int_0^\infty \varphi_{\alpha/2}(t,s)U(s)\phi(x)ds$ *and* $\int_0^\infty \varphi_{\alpha/2}(t,s)U(-s)\phi(x)ds$ *are solutions to*

$$\begin{cases} D_t^{\alpha/2}u^+(t,x) = Au^+(t,x) = i[-(-\Delta)^{1/2}]u^+(t,x), \quad t>0,\, x\in\mathbb{R}^n \\ u^+(0,x)=\phi(x) \end{cases} \tag{3.27}$$

and

$$\begin{cases} D_t^{\alpha/2} u^-(t,x) = -Au^-(t,x) = -i[-(-\Delta)^{1/2}]u^-(t,x), & t>0,\, x\in\mathbb{R}^n \\ u^-(0,x) = \phi(x) \end{cases} \tag{3.28}$$

respectively.

In the following we shall consider the special case $\alpha = 2$. When $\alpha = 2$, equation (3.28) is a special case of the free space-fractional Schrödinger equation [5, 12, 17], *and (3.27) is the time reverse of (3.28). By taking Fourier transforms, we get that the fundamental solutions to (3.27) and (3.28) can be represented as*

$$u_F^+(t,x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} e^{-i|\xi|t} d\xi$$

and

$$u_F^-(t,x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} e^{i|\xi|t} d\xi$$

respectively.

We have not been able to obtain an explicit closed expression for $u_F^+(t,x)$ or $u_F^-(t,x)$ in the general n-dimensional case even in the sense of distributions. However, we can compute their sum in one dimension and reduce it to the classical d'Alembert formula for the wave equation. More precisely, when $n = 1$, we have

$$\begin{aligned} & u_F^+(t,x) + u_F^-(t,x) \\ = \ & \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi x} e^{-i|\xi|t} d\xi + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi x} e^{i|\xi|t} d\xi \\ = \ & \frac{1}{2\pi}\int_{0}^{+\infty} e^{i\xi x} e^{-i\xi t} d\xi + \frac{1}{2\pi}\int_{-\infty}^{0} e^{i\xi x} e^{i\xi t} d\xi \\ & + \frac{1}{2\pi}\int_{0}^{+\infty} e^{i\xi x} e^{i\xi t} d\xi + \frac{1}{2\pi}\int_{-\infty}^{0} e^{i\xi x} e^{-i\xi t} d\xi \\ = \ & \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi x} e^{-i\xi t} d\xi + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi x} e^{i\xi t} d\xi \\ = \ & \delta(x-t) + \delta(x+t). \end{aligned}$$

Therefore the solution (3.26) to (3.25) is

$$u(t,x) = \frac{1}{2}(U(t)\phi(x) + U(-t)\phi(x)) = \frac{1}{2}(\phi(x-t) + \phi(x+t)),$$

which is the same as the classical d'Alembert solution.

4. D'Alembert formula for fractional telegraph equations

It was obtained by Kac [13, 14] that for the telegraph equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + 2h\frac{\partial u(t,x)}{\partial t} = c^2\frac{\partial^2 u(t,x)}{\partial x^2}, & t>0; \\ u(0,x) = \phi(x), \qquad u_t'(0,x) = 0. \end{cases} \tag{4.1}$$

the solution is given by

$$u(t,x) = \frac{1}{2}\mathbf{E}[\phi(x + c\xi_h(t)) + \phi(x - c\xi_h(t))], \tag{4.2}$$

where

$$\xi_h(t) = \int_0^t (-1)^{N_h(s)} ds, \quad t \geq 0$$

and $N_h(t)$ is a Poisson process with parameter h, that is, $\mathbb{P}\{N_h(t) = k\} = \frac{h^k t^k}{k!} e^{-ht}$. When $h = 0$, we have $N_h(t) \equiv 0$ and $\xi_h(t) = t$, then (4.1) and (4.2) are reduced to the classical wave equation and d'Alembert solution respectively. Therefore, (4.2) can be regarded as the d'Alembert formula solution for the telegraph equation.

We may also give the d'Alembert's formula solution for the (sequential Caputo) fractional telegraph equation with proper initial conditions. Let $1 < \alpha \leq 2$, consider the following fractional telegraph equation:

$$\begin{cases} D_t^{\alpha/2} D_t^{\alpha/2} u(t) + 2h D_t^{\alpha/2} u(t) = Au(t) & t > 0; \\ u(0) = \phi, \qquad D_t^{\alpha/2} u(0) = \psi. \end{cases} \tag{4.3}$$

Formally, the equation (4.3) can be rewritten as

$$\begin{cases} (D_t^{\alpha/2} + h)^2 u(t) = (A + h^2) u(t) = B^2 u(t) \\ u(0) = \phi, \qquad D_t^{\alpha/2} u(0) = \psi, \end{cases}$$

if there is an operator B satisfying $A + h^2 = B^2$. Therefore, the operator $D_t^{\alpha/2} D_t^{\alpha/2} + 2h D_t^{\alpha/2} - A$ can be decomposed into the product of $D_t^{\alpha/2} + h + B$ and $D_t^{\alpha/2} + h - B$. We will then study the relations between the solution of (4.3) and the solutions of

$$\begin{cases} D_t^{\alpha/2} u^+_{\alpha/2}(t) = (B - h) u^+_{\alpha/2}(t), \\ u^+_{\alpha/2}(0) = \phi + B^{-1}(\psi + h\phi) \end{cases} \tag{4.4}$$

and

$$\begin{cases} D_t^{\alpha/2} u^-_{\alpha/2}(t) = (-B - h) u^-_{\alpha/2}(t), \\ u^-_{\alpha/2}(0) = \phi - B^{-1}(\psi + h\phi) \end{cases} \tag{4.5}$$

when $\psi + h\phi$ is in the range of B.

Theorem 4.1. *Suppose that B is an operator satisfying $B^2 = A + h^2$, and both $B - h$ and $-B - h$ generate $\alpha/2$ times resolvent families $S^+_{\alpha/2}(t)$ and $S^-_{\alpha/2}(t)$ on X, respectively. Then for $\phi \in D(A)$, and $\psi + h\phi = Bf$ for some $f \in D(A^2)$ the strong solutions of (4.4) and (4.5) can be expressed as*

$$u^+_{\alpha/2}(t) = S^+_{\alpha/2}(t)\phi + S^+_{\alpha/2}(t) B^{-1}(\psi + h\phi) \tag{4.6}$$

and

$$u^-_{\alpha/2}(t) = S^-_{\alpha/2}(t)\phi - S^-_{\alpha/2}(t) B^{-1}(\psi + h\phi) \tag{4.7}$$

respectively, and the d'Alembert's formula solution for (4.3) reads:

$$u_\alpha(t) = \frac{1}{2}[S^+_{\alpha/2}(t)\phi + S^-_{\alpha/2}(t)\phi] + \frac{1}{2}[S^+_{\alpha/2}(t) B^{-1}(\psi + h\phi) - S^-_{\alpha/2}(t) B^{-1}(\psi + h\phi)]. \tag{4.8}$$

Proof. First note that both ϕ and $B^{-1}(\psi + h\phi)$ are in the domain of B^2. Let $u_\alpha(t) = \frac{1}{2}(u^+_{\alpha/2}(t) + u^-_{\alpha/2}(t))$. Then we have

$$\begin{aligned}
& (D_t^{\alpha/2} + h)^2 u_\alpha(t) \\
= & \frac{1}{2}(D_t^{\alpha/2} + h)(D_t^{\alpha/2} + h)(u^+_{\alpha/2}(t) + u^-_{\alpha/2}(t)) \\
= & \frac{1}{2}(D_t^{\alpha/2} + h)(D_t^{\alpha/2} + h)u^+_{\alpha/2}(t) + \frac{1}{2}(D_t^{\alpha/2} + h)(D_t^{\alpha/2} + h)u^-_{\alpha/2}(t) \\
= & \frac{1}{2}(D_t^{\alpha/2} + h)Bu^+_{\alpha/2}(t) - \frac{1}{2}(D_t^{\alpha/2} + h)Bu^-_{\alpha/2}(t) \\
= & \frac{1}{2}B(D_t^{\alpha/2} + h)u^+_{\alpha/2}(t) - \frac{1}{2}B(D_t^{\alpha/2} + h)u^-_{\alpha/2}(t) \\
= & \frac{1}{2}B^2(u^+_{\alpha/2}(t) + u^-_{\alpha/2}(t)) \\
= & B^2 u_\alpha(t).
\end{aligned}$$

Moreover,

$$u_\alpha(0) = \frac{1}{2}[u^+_{\alpha/2}(0) + u^-_{\alpha/2}(0)] = \frac{1}{2}[\phi + B^{-1}(\psi + h\phi) + \phi - B^{-1}(\psi + h\phi)] = \phi.$$

Finally, we have

$$\begin{aligned}
2\lim_{t\to 0}(D_t^{\alpha/2}u_\alpha)(t) &= \lim_{t\to 0}(D_t^{\alpha/2}u^+_{\alpha/2})(t) + \lim_{t\to 0}(D_t^{\alpha/2}u^-_{\alpha/2})(t) \\
&= Bu^+_{\alpha/2}(0) - hu^+_{\alpha/2}(0) - Bu^-_{\alpha/2}(0) - hu^-_{\alpha/2}(0) \\
&= B[B^{-1}(\psi + h\phi) + B^{-1}(\psi + h\phi)] - h(u^+_{\alpha/2}(0) + u^-_{\alpha/2}(0)) \\
&= 2\psi + 2h\phi - 2h\phi = 2\psi.
\end{aligned}$$

Thus (4.8) is the strong solution of (4.3). □

Remark 4.2. (1) When $h = 0$, Theorem 4.1 reduces to Theorem 2.8.

(2) Suppose that $-(\omega_0 + A)$ is a sectorial operator with angle $0 < \varphi < \pi - \pi\alpha$ for some constant $\omega_0 > 0$. Then $h^2 + A$ generates a bounded analytic α-times resolvent family for each constant h satisfying $|h| \le \sqrt{\omega_0}$. Let $B = i[-(h^2 + A)]^{1/2}$, then $(\pm B)^2 = A + h^2$, B is invertible, and $\pm B$ generate bounded analytic $\alpha/2$-times resolvent families. Therefore, both $B - h$ and $-B - h$ generate exponentially bounded $\alpha/2$-times resolvent family. See [20, Section 7] for more details.

Example 4.3. *Let $1 < \alpha \le 2$. Consider*

$$\text{(4.9)} \qquad \begin{cases} D_t^{\alpha/2}D_t^{\alpha/2}u(t,x) + 2hD_t^{\alpha/2}u(t,x) = c^2\frac{\partial^2 u(t,x)}{\partial x^2} & t > 0, \quad x \in \mathbb{R}; \\ u(0,x) = \phi(x), \qquad D_t^{\alpha/2}u(0,x) = 0. \end{cases}$$

which is equivalent to

$$\text{(4.10)} \qquad \begin{cases} D_t^{\alpha}u(t,x) + 2hD_t^{\alpha/2}u(t,x) = c^2\frac{\partial^2 u(t,x)}{\partial x^2} & t > 0, \quad x \in \mathbb{R}; \\ u(0,x) = \phi(x), \qquad u_t(0,x) = 0. \end{cases}$$

Denote by $A = c^2\frac{d^2}{dx^2}$. Then the conditions in Theorem 4.1 and Remark 4.2 are fulfilled. Therefore the solution to (4.9) can be represented as

$$u(t,x) = \frac{1}{2}[S^+_{\alpha/2}(t)(1 + hB^{-1})\phi(x) + S^-_{\alpha/2}(t)(1 - hB^{-1})\phi(x)]$$

Orsingher and Beghin ([28]) have obtained the following Fourier transform of the solution to (4.10):

$$\begin{aligned}\tilde{u}(t,\xi) &= E_{\alpha,1}(\eta_1 t^\alpha) + \frac{(2h+\eta_2)t^\alpha}{\eta_1-\eta_2}[\eta_1 E_{\alpha,\alpha+1}(\eta_1 t^\alpha) - \eta_2 E_{\alpha,\alpha+1}(\eta_2 t^\alpha)]\phi(\xi) \\ &= \frac{1}{2}\big[(1+\frac{h}{\sqrt{h^2-c^2\xi^2}})E_{\alpha,1}(\eta_1 t^\alpha) + (1-\frac{h}{\sqrt{h^2-c^2\xi^2}})E_{\alpha,1}(\eta_2 t^\alpha)\big]\phi(\xi)\end{aligned}$$

where

$$\eta_1 = -h + \sqrt{h^2-c^2\xi^2}, \qquad \eta_2 = -h - \sqrt{h^2-c^2\xi^2}.$$

They also remarked that, for $\alpha = 2$, $\tilde{u}(t,\xi)$ can be reduced to the characteristic function of the telegraph process

$$T(t) = V(0)\xi_h(t) = V(0)\int_0^t (-1)^{N_h(s)} ds,$$

where $V(0)$ is a random variable with the probability distribution $P(V(0) = \pm c) = 1/2$ independent of $N_h(t)$.

On the other hand, we can obtain the d'Alembert formula solution for (4.9) by Bernstein and subordinator theory [1, 33]. Indeed, since $z(\lambda) = (\lambda^\alpha + 2h\lambda^{\alpha/2})^{1/2}$ is a Bernstein function for $0 < \alpha \le 2$, $h \ge 0$, there is a subordinator $D_z(t)$ with Laplace exponent $z(\lambda)$, that is to say,

$$\mathbb{E}\big(e^{-\lambda D_z(t)}\big) = e^{-tz(\lambda)}.$$

Denote the inverse of $D_z(t)$ by $Y_z(t)$,i.e.,$Y_z(t) = inf\{y > 0 : D_z(y) > t\}$. Then we have

$$\int_0^\infty e^{-\lambda t} dF(t,x) = \frac{z(\lambda)}{\lambda} e^{-xz(\lambda)}, \quad x \ge 0 \tag{4.11}$$

where $F(t,x)$ is the distribution function of $Y_z(t)$.

Thanks to the subordination principle and the d'Alembert's solution (1.2) to (1.1), the solution to (4.9) can be represented as

$$u(t,x) = \frac{1}{2}\mathbb{E}[\phi(x+cY_z(t)) + \phi(x-cY_z(t))] \tag{4.12}$$

Next, we will clarify the the relation between (4.2) and (4.12) in the special case $\alpha = 2$. Let $z(\lambda) = \sqrt{\lambda^2+2h\lambda}$. Denote the density of $\xi_h(t)$ by $g(t,x)$, $x \in (-\infty,\infty)$. Then $|\xi_h(t)|$ has density

$$w(t,r) = \begin{cases} g(t,x) + g(t,-x), & x \ge 0, \\ 0, & x < 0. \end{cases}$$

Moreover, we have [7]

$$\int_0^\infty e^{-\lambda t} g(t,x)dt = \begin{cases} \frac{1}{2}[\frac{z(\lambda)}{\lambda}+1]e^{-xz(\lambda)}, & x > 0, \\ \frac{1}{2}[\frac{z(\lambda)}{\lambda}-1]e^{xz(\lambda)}, & x < 0, \\ \frac{z(\lambda)}{2\lambda}, & x = 0, \end{cases}$$

and

$$\int_0^\infty e^{-\lambda t} w(t,x)dt = \frac{z(\lambda)}{\lambda} e^{-xz(\lambda)}, \qquad x \ge 0. \tag{4.13}$$

Since the solution of (4.1) can be represented as (4.2):

$$\begin{aligned} u(t,x) &= \frac{1}{2}\mathbb{E}[\phi(x+c\xi_h(t))+\phi(x-c\xi_h(t))] \\ &= \int_{-\infty}^{\infty}\frac{1}{2}[\phi(x+r)+\phi(x-r)]g(t,r)dr, \end{aligned}$$

the solution of (4.1) can be represented in another way:

$$\begin{aligned} u(t,x) &= \frac{1}{2}\mathbb{E}[\phi(x+c|\xi_h(t)|)+\phi(x-c|\xi_h(t)|)] \\ &= \int_{0}^{\infty}\frac{1}{2}[\phi(x+r)+\phi(x-r)]w(t,r)dr. \end{aligned}$$

It is easy to see $|\xi_h(t)|$ and the inverse subordinator $Y_z(t)$ are identically distributed from (4.11) and (4.13).

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Department of Mathematics, Southwest Jiaotong University, Chengdu 611756, P.R.China.
E-mail address: `lichenggang@swjtu.edu.cn`

Department of Mathematics, Sichuan University
E-mail address: `mli@scu.edu.cn`

Science Research Computer Center, Lomonosov Moscow State University
E-mail address: `piskarev@gmail.com`

Department of Statistics and probability, Michigan State University
E-mail address: `mcubed@stt.msu.edu`