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Fractal dimension results for continuous time random walks

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ABSTRACT

Continuous time random walks impose random waiting times between particle jumps. This paper computes the fractal dimensions of their process limits, which represent particle traces in anomalous diffusion.

1. Introduction

Given a sequence of space–time random vectors $\{(J_n, W_n),\ n \geq 1\}$ on $\mathbb{R}^d \times [0, \infty)$, a particle arrives at location $S(n) = J_1 + \cdots + J_n$ at time $T(n) = W_1 + \cdots + W_n$. The renewal process $N(t) = \max\{n \geq 0 : T(n) \leq t\}$ counts the number of jumps, and the continuous time random walk (CTRW) $S(N_t)$ with $T(0) = 0$ gives the particle location at time $t \geq 0$. Under suitable conditions, the normalized sample paths of the space–time random walk $(S(n), T(n))$ converge to a limit $\{(Y(u), D(u)),\ u \geq 0\}$, and continuous mapping arguments yield a CTRW scaling limit $\{Y(E(t)),\ t \geq 0\}$, where

$$E(t) = \inf\{x \geq 0 : D(u) > t\}, \quad \forall t \geq 0 \tag{1.1}$$

is the inverse of the time process $D(u)$. CTRW limits provide a microscopic model for particle motions, leading to macroscopic differential equation models that can involve fractional derivatives in space and time. See Meerschaert and Sikorskii (2012) for details.

Sample paths of the CTRW limit process $Y(E(t))$ model particle traces in statistical physics. This paper investigates the fractal properties of those particle traces. Our results provide physical insight into the microscopic behavior of individual particles undergoing anomalous diffusion. The process $E(t)$ models particle resting times. The graph of the inverse process $u = E(t)$ flips the axes on a graph of the time process $t = D(u)$. Thus, when $D(u)$ jumps, $E(t)$ remains constant, and the particle rests. Unless the resting periods are exponentially distributed, the CTRW limit $Y(E(t))$ is non-Markovian, and novel methods are required to compute the Hausdorff and packing dimensions of the range and graph.

For example, assume i.i.d. particle jumps J_n with finite variance and zero mean, and i.i.d. waiting times $\mathbb{P}(W_n > t) \sim Ct^{-\beta}$ independent of the jumps for some $C > 0$ and $0 < \beta < 1$. Then Theorem 4.2 in Meerschaert and Scheffler (2004) implies that the CTRW limit $Y(E(t))$ is a Brownian motion with an inverse β-stable time change. A typical sample path, shown

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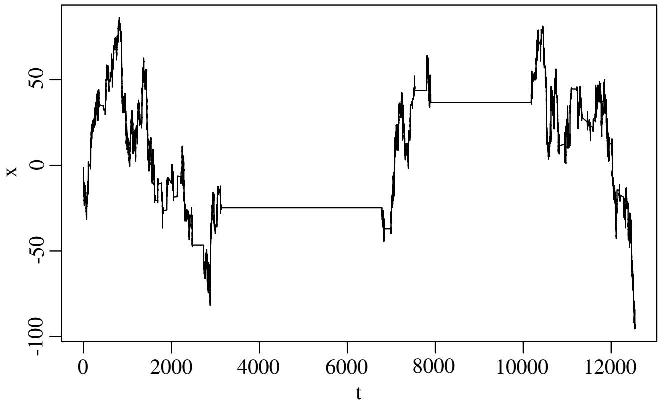

Fig. 1. Typical sample path of a CTRW limit process $Y(E(t))$. Here $Y(t)$ is a Brownian motion and $E(t)$ is the inverse of a 0.8-stable subordinator.

in Fig. 1, resembles a Brownian motion interrupted by long resting periods. This process appears in the theory of random conductance models; see Barlow and Černý (2011). Stochastic processes with "locally constant" paths are also discussed in Davydov (2012). Proposition 2.3 shows that the fractal (Hausdorff or packing) dimension is $1+\beta/2$, less than the dimension $3/2$ of a Brownian motion graph, showing that long resting times also affect the geometry of particle traces during motion.

2. CTRW dimension results

First we establish a general result concerning a time-changed stochastic process $X(t) = Y(E(t))$ for $t \geq 0$. We assume that $Y(u)$ is a stochastic process on $\mathbb{R}^d$ and $E(t)$ is a real-valued stochastic process with $E(0) = 0$ and nondecreasing continuous sample paths. We emphasize that these two processes are not necessarily independent. We use the standard definitions of Hausdorff and packing measures; e.g., see Falconer (1990), Kahane (1985), Taylor (1986) and Xiao (2004). Since $E(t)$ is continuous, naturally the range of $Y(E(t))$ is the same as the range of $Y(t)$ up to a random time. Hence the next result is intuitively obvious, but we include it here for completeness.

Proposition 2.1. *If $E(1) > 0$ a.s. and there exist constants c_1 and c_2 such that for all constants $0 < a < \infty$*

$$\dim_{\mathrm{H}} Y([0,a]) = c_1, \qquad \dim_{\mathrm{P}} Y([0,a]) = c_2 \quad a.s., \tag{2.1}$$

then almost surely

$$\dim_{\mathrm{H}} X([0,1]) = c_1 \quad and \quad \dim_{\mathrm{P}} X([0,1]) = c_2. \tag{2.2}$$

Proof. Since the process $t \mapsto E(t)$ is nondecreasing and continuous, the range $E([0,1])$ is the random interval $[0, E(1)]$. Hence $X([0,1]) = Y([0,E(1)])$.

It follows from the σ-stability of $\dim_{\mathrm{H}}$ and (2.1) that $\dim_{\mathrm{H}} Y([0,\infty)) = c_1$ a.s. Hence $\dim_{\mathrm{H}} X([0,1]) \leq c_1$ almost surely. On the other hand, (2.1) implies

$$\mathbb{P}\Big(\dim_{\mathrm{H}} Y([0,q]) = c_1,\ \forall q \in \mathbb{Q}_+\Big) = 1, \tag{2.3}$$

where $\mathbb{Q}_+$ denotes the set of positive rational numbers. Since $E(1) > 0$ almost surely, we see that there is an event $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$ we have $E(1,\omega) > 0$ and $\dim_{\mathrm{H}} Y([0,q],\omega) = c_1$ for all $q \in \mathbb{Q}_+$. Since for every $\omega \in \Omega'$ there is a $q \in \mathbb{Q}_+$ such that $0 < q < E(1,\omega)$, we derive that

$$\dim_{\mathrm{H}} X([0,1],\omega) = \dim_{\mathrm{H}} Y([0,E(1,\omega)],\omega) \geq \dim_{\mathrm{H}} Y([0,q],\omega) = c_1.$$

Combining the upper and lower bounds for $\dim_{\mathrm{H}} X([0,1])$ yields the first equation in (2.2). The proof of the second equation in (2.2) is similar and is omitted. □

The graph of the CTRW limit X is closely related to the range of the space–time limit $Z = \{(Y(u), D(u)),\ u \geq 0\}$. In fact, the graph of X is obtained by connecting the points in the range of Y by horizontal line segments, representing the particle resting periods. This motivates the following result.

Theorem 2.2. *If $E(1) > 0$ a.s. and there exist constants c_5 and c_6 such that for all constants $0 < a < \infty$*

$$\dim_{\mathrm{H}} Z([0,a]) = c_5 \quad and \quad \dim_{\mathrm{P}} Z([0,a]) = c_6 \quad a.s., \tag{2.4}$$

then

$$\dim_H \mathrm{Gr}X([0,1]) = \max\big\{1, \dim_H Z([0,1])\big\}, \quad a.s. \tag{2.5}$$

and

$$\dim_P \mathrm{Gr}X([0,1]) = \max\big\{1, \dim_P Z([0,1])\big\}, \quad a.s. \tag{2.6}$$

Proof. We only prove (2.5), and the proof of (2.6) is similar. The sample function $x \mapsto D(x)$ is a.s. strictly increasing and we can write the unit interval $[0, 1]$ in the state space of D as

$$[0,1] = D([0, E(1))) \cup \bigcup_{i=1}^{\infty} I_i, \tag{2.7}$$

where for each $i \geq 1$, I_i is a subinterval on which $E(t)$ is a constant. Using D we can express $I_i = [D(x_i-), D(x_i))$, which is the gap corresponding to the jumping site x_i of D, except in the case when $x_i = E(1)$. In the latter case, $I_i = [D(x_i-), 1]$.

Notice that I_i $(i \geq 1)$ are disjoint intervals and

$$E(t) = E(s) \quad \text{if and only if} \quad s, t \in I_i \text{ for some } i \geq 1.$$

Thus, over each interval I_i, the graph of X is a horizontal line segment. More precisely, we can decompose the graph set of X as

$$\begin{aligned} \mathrm{Gr}X([0,1]) &= \big\{(t, Y(E(t))) : t \in [0,1]\big\} \\ &= \big\{(t, Y(E(t))) : t \in D([0, E(1)))\big\} \cup \bigcup_{i=1}^{\infty} \big\{(t, Y(E(t))) : t \in I_i\big\}. \end{aligned} \tag{2.8}$$

Hence, by the σ-stability of $\dim_H$, we have

$$\dim_H \mathrm{Gr}X([0,1]) = \max\Big\{1, \dim_H\big\{(t, Y(E(t))) : t \in D([0, E(1)))\big\}\Big\}. \tag{2.9}$$

On the other hand, every $t \in D([0, E(1)))$ can be written as $t = D(x)$ for some $0 \leq x < E(1)$ and $E(t) = E(D(x)) = x$, we see that

$$\big\{(t, Y(E(t))) : t \in D([0, E(1)))\big\} = \big\{(D(x), Y(x)) : x \in [0, E(1))\big\}, \quad \text{a.s.} \tag{2.10}$$

It follows from (2.4) that

$$\mathbb{P}\Big(\omega : \dim_H\big\{(D(x,\omega), Y(x,\omega)) : x \in [0, q]\big\} = c_5,\ \forall q \in \mathbb{Q}_+\Big) = 1. \tag{2.11}$$

Combining this with the assumption that $E(1, \omega) > 0$ almost surely, we can find an event Ω_2'' such that $\mathbb{P}(\Omega_2'') = 1$ and for every $\omega \in \Omega_2''$ we derive from (2.11) that

$$\dim_H\big\{(D(x,\omega), Y(x,\omega)) : x \in [0, E(1,\omega))\big\} = c_5, \tag{2.12}$$

since $q_1 < E(1, \omega_2) < q_2$ for some $q_1, q_2 \in \mathbb{Q}_+$, and $U \subseteq V$ implies $\dim_H(U) \leq \dim_H(V)$. Combining (2.10) and (2.12) yields

$$\dim_H\big\{(t, Y(E(t))) : t \in D([0, E(1)))\big\} = c_5, \quad \text{a.s.} \tag{2.13}$$

Therefore, (2.5) follows from (2.9) and (2.13). □

Now we apply these general results to CTRW limits. We assume that the space–time random walk $(S(n), T(n))$ has a process limit $\{(Y(u), D(u)),\ u \geq 0\}$ such that $D(u)$ is a subordinator (a nondecreasing Lévy process) with $D(0) = 0$ and

$$\mathbb{E}[e^{-sD(u)}] = e^{-u\sigma(s)},$$

where the Laplace exponent

$$\sigma(s) = \int_0^{\infty} (1 - e^{-sy})\nu(dy), \tag{2.14}$$

and the Lévy measure ν of D_σ satisfies $\nu(0, \infty) = \infty$, so that the sample paths $x \mapsto D_\sigma(x)$ are a.s. strictly increasing. Then the sample paths $t \mapsto E(t)$ are a.s. continuous and nondecreasing, with $\mathbb{P}\big(E(1) > 0\big) = 1$; e.g., see Proposition A.1 in Veillette and Taqqu (2010).

2.1. Uncoupled CTRW

Consider a CTRW whose i.i.d. waiting times $\{W_n, n \geq 1\}$ belong to the domain of attraction of a positive β-stable random variable $D(1)$, and whose i.i.d. jumps $\{J_n, n \geq 1\}$ belong to the strict domain of attraction of a d-dimensional stable random vector $Y(1)$. We assume that $\{W_n\}$ and $\{J_n\}$ are independent; that is, the CTRW is uncoupled. It follows from Theorem 4.2 in Meerschaert and Scheffler (2004) that the scaling limit of this CTRW is a time-changed process $X(t) = Y(E(t))$, where $E(t)$ is the inverse (1.1) of a β-stable subordinator D. Since D is self-similar with index $1/\beta$, its inverse E is self-similar with index β. Since Y is independent of E, the CTRW scaling limit X is self-similar with index β/α.

Proposition 2.3. *The uncoupled CTRW limit has a.s.*

$$\dim_{\mathrm{H}} X([0,1]) = \dim_{\mathrm{P}} X([0,1]) = \min\{d, \alpha\} \tag{2.15}$$

and

$$\dim_{\mathrm{H}} \mathrm{Gr}X([0,1]) = \dim_{\mathrm{P}} \mathrm{Gr}X([0,1]) = \begin{cases} \max\{1, \alpha\} & \text{if } \alpha \leq d, \\ 1 + \beta\left(1 - \dfrac{1}{\alpha}\right) & \text{if } \alpha > d = 1. \end{cases} \tag{2.16}$$

Proof. The result (2.15) follows from Proposition 2.1 and the results of Blumenthal and Getoor (1960a,b) on the Hausdorff dimension and Pruitt and Taylor (1996) on the packing dimension of the range of the stable Lévy process Y.

To prove (2.16), recall from Pruitt and Taylor (1969) that for any constant $a > 0$,

$$\dim_{\mathrm{H}} Z([0,a]) = \begin{cases} \beta & \text{if } \alpha \leq \beta, \\ \alpha & \text{if } \beta < \alpha \leq d, \\ 1 + \beta\left(1 - \dfrac{1}{\alpha}\right) & \text{if } \alpha > d = 1, \end{cases} \quad \text{a.s.} \tag{2.17}$$

Theorem 3.2 in Meerschaert and Xiao (2005) shows that $\dim_{\mathrm{P}} Z([0,a])$ also equals the right hand side of (2.17). Then (2.16) follows using Theorem 2.2. □

Remark 2.4. In increasing generality, Blumenthal and Getoor (1962), Jain and Pruitt (1968), Pruitt and Taylor (1969) and Rezakhanlou and Taylor (1988) showed that

$$\dim_{\mathrm{H}} \mathrm{Gr}Y([0,1]) = \dim_{\mathrm{P}} \mathrm{Gr}Y([0,1]) = \begin{cases} \max\{1, \alpha\} & \text{if } \alpha \leq d \\ 2 - \dfrac{1}{\alpha} & \text{if } \alpha > d = 1 \end{cases} \quad \text{a.s.}$$

Compare with (2.16) to see that the inverse stable time change modifies the fractal dimension of the CTRW limit graph in dimension $d = 1$ when $\alpha > 1$.

2.2. Coupled CTRW

In a coupled CTRW, the space–time jumps $\{(J_n, W_n), n \geq 1\}$ are i.i.d., but J_n can depend on the waiting time W_n. Now the CTRW $S(N(t))$ has scaling limit $Y(E(t-))$ and the so-called oracle CTRW $S(N(t)+1)$ has scaling limit $Y(E(t))$; see Henry and Straka (2011) or Jurlewicz et al. (2012). In the uncoupled case, the two limit processes are the same. The proof of Theorem 2.2 extends immediately to the process $Y(E(t-))$, with the same dimension results, because the graphs of $Z(u) = (D(u), Y(u))$ and $Z'(u) = (D(u), Y(u-))$ have the same Hausdorff and packing dimensions, as they differ by at most a countable number of discrete points. In the following, we discuss examples for $Y(E(t))$, with the understanding that the same dimension results hold for $Y(E(t-))$.

The simplest case is $W_n = J_n$, so that $X(t) = D(E(t))$. This process is self-similar with index 1; see for example Becker-Kern et al. (2004). It follows from Proposition 2.1, Theorem 2.2 and the fact that for any constant $a > 0$,

$$\dim_{\mathrm{H}} D([0,a]) = \dim_{\mathrm{H}}\{(D(x), D(x)) : x \in [0,a]\} = \beta, \quad \text{a.s.}$$

that

$$\dim_{\mathrm{H}} X([0,1]) = \dim_{\mathrm{P}} X([0,1]) = \beta, \quad \text{a.s.} \tag{2.18}$$

and

$$\dim_{\mathrm{H}} \mathrm{Gr}X([0,1]) = \dim_{\mathrm{P}} \mathrm{Gr}X([0,1]) = 1, \quad \text{a.s.} \tag{2.19}$$

These results can also be obtained using "uniform" Hausdorff and packing dimension results for the β-stable subordinator; see Perkins and Taylor (1987).

Shlesinger et al. (1982) consider a CTRW where the waiting times W_n are i.i.d. with a β-stable random variable D such that $\mathbb{E}(e^{-sD}) = e^{-s^\beta}$ and, conditional on $W_n = t$, the jump J_n is normal with mean zero and variance $2t$. Then J_n is symmetric stable with index $\alpha = 2\beta$. This model was applied to stock market prices by Meerschaert and Scalas (2006). Becker-Kern et al. (2004) show that the CTRW limit is $X(t) = Y(E(t))$, where Y is a real-valued stable Lévy process with index $\alpha = 2\beta$ and $E(t)$ is the inverse of a β-stable subordinator, which is not independent of Y. Here $X(t)$ is self-similar with index $1/2$, the same as Brownian motion. However, the Hausdorff dimensions of the range and graph of X are completely different than those for Brownian motion.

Proposition 2.1 gives that $\dim_{\mathrm{H}} X([0,1]) = \min\{1, 2\beta\}$ a.s. To determine the Hausdorff dimension of the graph of $X(t)$, we first verify that the Fourier–Laplace transform of $(D(1), Y(1))$ is

$$\mathbb{E}\left(e^{i\xi Y(1)-\eta D(1)}\right) = \mathbb{E}\left[e^{-\eta D(1)}\mathbb{E}\left(e^{i\xi Y(1)}|D(1)\right)\right] = \mathbb{E}\left(e^{-(\eta+\xi^2)D(1)}\right) = e^{-(\eta+\xi^2)^\beta}.$$

It follows that the Lévy process $Z(u) = (D(u), Y(u))$ is operator stable (cf. Meerschaert and Scheffler, 2001) with the unique exponent

$$C = \begin{pmatrix} \beta^{-1} & 0 \\ 0 & (2\beta)^{-1} \end{pmatrix}. \tag{2.20}$$

Now Theorem 3.2 in Meerschaert and Xiao (2005) implies that for any $a > 0$,

$$\dim_{\mathrm{H}} Z([0,a]) = \dim_{\mathrm{P}} Z([0,a]) = \begin{cases} 2\beta & \text{if } 2\beta \le 1, \\ \dfrac{1}{2} + \beta & \text{if } 2\beta > 1, \end{cases} \quad \text{a.s.} \tag{2.21}$$

Consequently, we use Theorem 2.2 to derive

$$\dim_{\mathrm{H}} \mathrm{Gr}X([0,1]) = \dim_{\mathrm{P}} \mathrm{Gr}X([0,1]) = \max\left\{1, \beta + \frac{1}{2}\right\}, \quad \text{a.s.} \tag{2.22}$$

2.3. CTRW triangular arrays

Meerschaert and Scheffler (2008) prove limit theorems for CTRW triangular arrays. Here $Z(u) = (Y(u), D(u))$ is a Lévy process with values in $\mathbb{R}^p$ and characteristic exponent Φ (i.e., $\mathbb{E}(e^{i\langle\xi, Z(u)\rangle}) = e^{-u\Phi(\xi)}$).

Proposition 2.5. *Let $X = \{Y(E(t)), t \ge 0\}$, where Y is a Lévy process with values in $\mathbb{R}^d$ and characteristic exponent ψ, and $E(t)$ is the inverse (1.1) of a subordinator $\{D(u), u \ge 0\}$ with characteristic exponent σ. Suppose $Z = \{(D(u), Y(u)), u \ge 0\}$ is a Lévy process on $\mathbb{R}^{1+d}$ whose characteristic exponent Φ satisfies*

$$K^{-1}\,\mathrm{Re}\left(\frac{1}{1+\sigma(\eta)+\psi(\xi)}\right) \le \mathrm{Re}\left(\frac{1}{1+\Phi(\eta,\xi)}\right) \le K\,\mathrm{Re}\left(\frac{1}{1+\sigma(\eta)+\psi(\xi)}\right) \tag{2.23}$$

for all $(\eta, \xi) \in \mathbb{R}^{1+d}$ with $|\eta| + \|\xi\|$ large, where $K \ge 1$ is a constant. Then a.s.

$$\dim_{\mathrm{H}} X([0,1]) = \sup\left\{\gamma < d : \int_{\{\xi\in\mathbb{R}^d:\ \|\xi\|\ge 1\}} \mathrm{Re}\left(\frac{1}{1+\psi(\xi)}\right)\frac{d\xi}{\|\xi\|^\gamma} < \infty\right\}$$

and $\dim_{\mathrm{H}} \mathrm{Gr}X([0,1]) = \max\{1, \chi\}$ *a.s., where*

$$\chi = \sup\left\{\gamma < 1 + d : \int_{\{|\eta|+\|\xi\|\ge 1\}} \mathrm{Re}\left(\frac{1}{1+\sigma(\eta)+\psi(\xi)}\right)\frac{d\eta d\xi}{(|\eta|+\|\xi\|)^\gamma} < \infty\right\}.$$

Proof. Corollary 1.8 in Khoshnevisan et al. (2003) shows that for any $a > 0$,

$$\dim_{\mathrm{H}} Z([0,a]) = \sup\left\{\gamma < p : \int_{\{\xi\in\mathbb{R}^p:\ \|\xi\|\ge 1\}} \mathrm{Re}\left(\frac{1}{1+\Phi(\xi)}\right)\frac{d\xi}{\|\xi\|^\gamma} < \infty\right\}, \quad \text{a.s.} \tag{2.24}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^p$. Apply Proposition 2.1 and Theorem 2.2. □

Proposition 2.6. *Let X, Z be as in Proposition 2.5. If the characteristic exponent Φ of Z satisfies (2.23), then*

$$\dim_P X([0,1]) = \sup\left\{\gamma \ge 0 : \liminf_{r\to 0^+} \frac{W(r)}{r^\gamma} = 0\right\}, \quad \text{a.s.,}$$

where $W(r)$ is defined by

$$W(r) = \int_{\mathbb{R}^d} \mathrm{Re}\left(\frac{1}{1+\psi(\xi/r)}\right)\prod_{j=1}^d \frac{1}{1+\xi_j^2}\,d\xi,$$

and $\dim_P \mathrm{Gr}X([0,1]) = \max\{1, \chi'\}$ almost surely, where

$$\chi' = \sup\left\{\gamma \ge 0 : \liminf_{r\to 0^+} \frac{\widetilde{W}(r)}{r^\gamma} = 0\right\}$$

and where

$$\widetilde{W}(r) = \int_{\mathbb{R}^{1+d}} \mathrm{Re}\left(\frac{1}{1+\sigma(\eta/r)+\psi(\xi/r)}\right)\frac{1}{1+\eta^2}\prod_{j=1}^d \frac{1}{1+\xi_j^2}\,d\eta\,d\xi. \tag{2.25}$$

Proof. Theorem 1.1 in Khoshnevisan and Xiao (2008) shows that for any $a > 0$,

$$\dim_P Z([0,a]) = \sup\left\{\gamma \ge 0 : \liminf_{r\to 0^+} \frac{W(r)}{r^\gamma} = 0\right\}, \quad \text{a.s.,}$$

with $W(r)$ given by (2.25). Apply Proposition 2.1 and Theorem 2.2. □

Our first example is a stochastic model for ultraslow diffusion from Meerschaert and Scheffler (2006); see also Chechkin et al. (2002). At each scale $c > 0$ we are given i.i.d. waiting times (W_n^c) and i.i.d. jumps (J_n^c). Let $S^c(n) = J_1^c + \cdots + J_n^c$ and $T^c(n) = W_1^c + \cdots + W_n^c$, and suppose $S^c(ct) \Rightarrow Y(t)$ and $T^c(ct) \Rightarrow D(t)$ as $c \to \infty$, where $Y(t)$ and $D(t)$ are independent Lévy processes. Letting $N_t^c = \max\{n \ge 0 : T^c(n) \le t\}$, the CTRW scaling limit $S^c(N_t^c) \Rightarrow Y(E_t)$ by Theorem 2.1 in Meerschaert and Scheffler (2008). To be specific, take $\{B_i\}$ i.i.d. with $0 < B_i < 1$ and assume $\mathbb{P}\{W_i^c > u | B_i = \beta\} = c^{-1}u^{-\beta}$ for $u \ge c^{-1/\beta}$, so that the waiting times are Pareto distributed, conditional on the mixing variables. Then $\mathbb{E}[e^{-sD(t)}] = e^{-t\sigma(s)}$, where (2.14) holds with

$$\nu(t,\infty) = \int_0^1 t^{-\beta}\mu(d\beta), \tag{2.26}$$

and μ is the distribution of the mixing variable. Suppose $0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1$ and take $\mu(d\beta) = \sum_{k=1}^n d_k^{\beta_k}(\Gamma(1-\beta_k))^{-1}\delta_{\beta_k}(d\beta)$, where δ_a is the unit mass at a. Then $D(u) = \sum_{k=1}^n d_k D_k(u)$, a mixture of independent β_k-stable subordinators.

Lemma 2.7. *Let $Y = \{Y(x), x \ge 0\}$ be a strictly stable Lévy motion of index $\alpha \in (0,2]$ with values in $\mathbb{R}^d$, independent of D, and let Φ be the characteristic exponent of the Lévy process $Z(u) = (D(u), Y(u))$. Then for all $(\eta,\xi) \in \mathbb{R}^{1+d}$ that satisfies $|\eta| + \|\xi\| > 1$, we have*

$$\frac{K^{-1}}{|\eta|^{\beta_n} + \|\xi\|^\alpha} \le \mathrm{Re}\left(\frac{1}{1+\Phi(\eta,\xi)}\right) \le \frac{K}{|\eta|^{\beta_n} + \|\xi\|^\alpha}, \tag{2.27}$$

where $K \ge 1$ is a constant that may depend on n, α, β_k, d_k.

Proof. For simplicity, assume that Y has characteristic exponent $\psi(\xi) = \|\xi\|^\alpha$. Then

$$\begin{aligned}
\Phi(\eta,\xi) &= \sum_{k=1}^n \left(-id_k\eta\right)^{\beta_k} + \|\xi\|^\alpha \\
&= \sum_{k=1}^n |d_k\eta|^{\beta_k}[\cos(\pi\beta_k/2) - i\sin(\pi\beta_k/2)] + \|\xi\|^\alpha \\
&=: f(\eta,\xi) - ig(\eta).
\end{aligned} \tag{2.28}$$

Since $\beta_k \in (0, 1)$, we have $f(\eta, \xi) \geq 0$ for all $\eta, \xi \in \mathbb{R}^{1+d}$. Moreover, $0 \leq g(\eta) \leq Kf(\eta, \xi)$ for some constant $K > 0$. Hence

$$\frac{1}{(1+K^2)(1+f(\eta,\xi))} \leq \mathrm{Re}\left(\frac{1}{1+\Phi(\eta,\xi)}\right) \leq \frac{1}{1+f(\eta,\xi)}.$$

From here it is elementary to verify (2.27). □

Proposition 2.8. *For the triangular array CTRW limit described above, we have a.s.*

$$\dim_{\mathrm{H}} X([0,1]) = \dim_{\mathrm{P}} X([0,1]) = \min\{d, \alpha\}, \tag{2.29}$$

and

$$\dim_{\mathrm{H}} \mathrm{Gr}X([0,1]) = \dim_{\mathrm{P}} \mathrm{Gr}X([0,1]) = \begin{cases} \max\{1,\alpha\} & \text{if } \alpha \leq d, \\ 1+\beta_n\left(1-\dfrac{1}{\alpha}\right) & \text{if } \alpha > d = 1. \end{cases} \tag{2.30}$$

Proof. The proof is similar to Proposition 4.1 in Meerschaert and Xiao (2005), using Lemma 2.7, and Propositions 2.5 and 2.6, hence we omit the details. □

2.4. CTRW with correlated jumps

Now we consider an uncoupled CTRW whose jumps $\{J_n\}$ form a correlated sequence of random variables, and whose waiting times $\{W_n\}$ are i.i.d. and belong to the domain of attraction of a positive β-stable random variable $D(1)$. In this case, Meerschaert et al. (2009) show that, under certain conditions on the correlation structure of jumps, the CTRW scaling limit is the $(H\beta)$-self-similar process $X = \{Y(E(t)) : t \geq 0\}$, where Y is a fractional Brownian motion with index $H \in (0, 1)$, and $E(t)$ is the inverse of a β-stable subordinator D, independent of Y.

Proposition 2.9. *The correlated CTRW limit X described above satisfies a.s.*

$$\dim_{\mathrm{H}} X([0,1]) = \dim_{\mathrm{P}} X([0,1]) = \min\left\{1, \frac{1}{H}\right\}, \tag{2.31}$$

and

$$\dim_{\mathrm{H}} \mathrm{Gr}X([0,1]) = \dim_{\mathrm{P}} \mathrm{Gr}X([0,1]) = \beta + (1 - H\beta). \tag{2.32}$$

The proof of Proposition 2.9 requires a few preliminary results. Let $c_7 > 0$ be a fixed constant. A collection $\Lambda(b)$ of intervals of length b in $\mathbb{R}$ is called c_7-nested if no interval of length b in $\mathbb{R}$ can intersect more than c_7 intervals of $\Lambda(b)$. Note that for each integer $n \geq 1$, the collection of dyadic intervals $I_{n,j} = [j/2^n, (j+1)/2^n]$ is c_7-nested with $c_7 = 3$.

Lemma 2.10. *Let $\{D(t), t \geq 0\}$ be a β-stable subordinator and let $\Lambda(b)$ be a c_7-nested family. Denote by $M_u(b, s)$ the number of intervals in $\Lambda(b)$ which intersect $D([u, u+s])$. Then there exists a positive constant c_8 such that for all $u \geq 0$ and all $0 < b^\beta \leq s$,*

$$\mathbb{E}\big(M_u(b,s)\big) \leq c_8\, s b^{-\beta}. \tag{2.33}$$

If one takes $b = s \leq 1$, then we have

$$\mathbb{E}\big(M_u(b,s)\big) \leq c_8. \tag{2.34}$$

Proof. This is an immediate consequence of Lemma 6.1 in Pruitt and Taylor (1969). It can also be derived from Lemma 3.2 in Liu and Xiao (1998) where general self-similar Markov processes are considered. □

Lemma 2.11. *Let the assumptions of Proposition 2.9 hold and let $a > 0$ be a constant. Then*

$$\dim_{\mathrm{P}} Z([0,a]) \leq \begin{cases} 1/H & \text{if } 1 \leq Hd, \\ \beta + (1-H\beta)d & \text{if } 1 > Hd, \end{cases} \quad \text{a.s.} \tag{2.35}$$

Proof. The proof is based on a moment argument. Note that for every $\varepsilon > 0$ the function $Y(u)$ $(0 \leq u \leq a)$ satisfies a uniform Hölder condition of order $H - \varepsilon$. We divide the interval $[0, a]$ into $(\lfloor a \rfloor + 1)2^n$ dyadic intervals $I_{n,j}$ of length 2^{-n}.

First we construct a covering of the range $Z([0, a])$ by using balls in $\mathbb{R}^{d+1}$ of radius 2^{-Hn} as follows. Define $t_{n,j} = j/2^n$ so that for each $I_{n,j} = [t_{n,j}, t_{n,j} + 2^{-n}]$, the image $Y(I_{n,j})$ is contained in a ball in $\mathbb{R}^d$ of radius $\sup_{s\in I_{n,j}} \|Y(s) - Y(t_{n,j})\|$ and can be covered by at most

$$N_{n,j} = c_9 \left(\frac{\sup_{s\in I_{n,j}} \|Y(s) - Y(t_{n,j})\|}{2^{-Hn}} \right)^d \tag{2.36}$$

balls of radius 2^{-Hn}. By the self-similarity and stationarity of increments of Y, we have

$$\begin{aligned} \mathbb{E}\big(N_{n,j}\big) &= c_9\, 2^{Hdn} \mathbb{E}\left[\left(\sup_{s\in I_{n,j}} \|Y(s) - Y(t_{n,j})\| \right)^d \right] \\ &= c_9 \mathbb{E}\left[\left(\sup_{s\in[0,1]} \|Y(s)\| \right)^d \right] := c_{10} < \infty, \end{aligned} \tag{2.37}$$

where the last inequality follows from the well known tail probability for the supremum of Gaussian processes (e.g., Fernique's inequality).

To get a covering for $D(I_{n,i})$, let $\Gamma(2^{-n})$ be the collection of dyadic intervals of order n in $\mathbb{R}_+$. Let $M_{n,j}$ be the number of dyadic intervals in $\Gamma(2^{-n})$ which intersect $D(I_{n,j})$. Applying (2.34) in Lemma 2.10 with $b_n = s_n = 2^{-n}$, we obtain that

$$\mathbb{E}(M_{n,j}) \le c_8, \quad \forall 1 \le j \le (\lfloor a \rfloor + 1)2^n. \tag{2.38}$$

Since $2^{-n} < 2^{-Hn}$, we see that $Z(I_{n,j}) = \{(D(x), Y(x)) : x \in I_{n,j}\}$ can be covered by at most $M_{n,j}N_{n,j}$ balls in $\mathbb{R}^{d+1}$ of radius 2^{-Hn}. Denote by $N\big(Z([0, a]), 2^{-Hn}\big)$ the smallest number of balls in $\mathbb{R}^{d+1}$ of radius 2^{-Hn} that cover $Z([0, a])$, then

$$N\big(Z([0, a]), 2^{-Hn}\big) \le \sum_{j=1}^{(\lfloor a \rfloor+1)2^n} M_{n,j}N_{n,j}.$$

It follows from (2.37), (2.38) and the independence of Y and D that

$$\mathbb{E}\Big[N\big(Z([0, a]), 2^{-Hn}\big)\Big] \le (\lfloor a \rfloor + 1)c_8 c_{10}\, 2^n.$$

Hence, for any $\varepsilon > 0$,

$$\mathbb{P}\Big\{N\big(Z([0, a]), 2^{-Hn}\big) \ge (\lfloor a \rfloor + 1)c_8 c_{10}\, 2^{n(1+\varepsilon)}\Big\} \le 2^{-n\varepsilon}.$$

It follows from the Borel–Cantelli lemma that almost surely

$$N\big(Z([0, a]), 2^{-Hn}\big) < (\lfloor a \rfloor + 1)c_8 c_{10}\, 2^{n(1+\varepsilon)}$$

for all n large enough. The upper box-counting dimension of F is defined as

$$\overline{\dim}_{\mathrm{M}} F = \limsup_{\varepsilon\to 0} \frac{\log N(F, \varepsilon)}{-\log \varepsilon}. \tag{2.39}$$

Hence $\overline{\dim}_{\mathrm{M}} Z([0, a]) \le (1 + \varepsilon)/H$ a.s. It is well known that for every (bounded) set $F \subseteq \mathbb{R}^d$,

$$0 \le \dim_{\mathrm{H}} F \le \dim_{\mathrm{P}} F \le \overline{\dim}_{\mathrm{M}} F \le d. \tag{2.40}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\dim_{\mathrm{P}} Z([0, a]) \le 1/H$ almost surely.

Next we construct a covering for the range $Z([0, a])$ by using balls in $\mathbb{R}^{d+1}$ of radius $2^{-n/\beta}$. Let $\Gamma(2^{-n/\beta})$ be the collection of intervals in $\mathbb{R}_+$ of the form $I'_{n,k} = [\frac{k}{2^{n/\beta}}, \frac{k+1}{2^{n/\beta}}]$, where k is an integer. Then the class $\Gamma(2^{-n/\beta})$ is 3-nested. Let $M'_{n,j}$ be the number of intervals in $\Gamma(2^{-n/\beta})$ that intersect $D(I_{n,j})$. By Lemma 2.10 with $b_n = 2^{-n/\beta}$ and $s_n = 2^{-n}$, we derive $\mathbb{E}(M'_{n,j}) \le c_8$. Thus $D(I_{n,j})$ can almost surely be covered by $M'_{n,j}$ intervals of length $2^{-n/\beta}$ from $\Gamma(2^{-n/\beta})$.

On the other hand, the image $Y(I_{n,j})$ can be covered by at most

$$N'_{n,j} = c_9 \left(\frac{\sup_{s\in I_{n,j}} \|Y(s) - Y(t_{n,j})\|}{2^{-n/\beta}} \right)^d$$

balls of radius $2^{-n/\beta}$, where $t_{n,j} = j/2^n$, and then similar to (2.37) we derive

$$\mathbb{E}\big(N'_{n,j}\big) = c_{10} 2^{n\left(\frac{1}{\beta}-H\right)d}. \tag{2.41}$$

Denote by $N\big(Z([0, a]), 2^{-n/\beta}\big)$ the smallest number of balls in $\mathbb{R}^{d+1}$ of radius $2^{-n/\beta}$ that cover $Z([0, a])$, then

$$N\big(Z([0, a]), 2^{-n/\beta}\big) \le \sum_{j=1}^{(\lfloor a\rfloor+1)2^n} M'_{n,j} N'_{n,j}.$$

By (2.41) and the independence of Y and D we have

$$\mathbb{E}\Big[N\big(Z([0, a]), 2^{-n/\beta}\big)\Big] \le (\lfloor a\rfloor + 1) c_8 c_{10}\, 2^{n\left(1+\left(\frac{1}{\beta}-H\right)d\right)}.$$

Hence, for any $\varepsilon > 0$, the Borel–Cantelli Lemma implies that a.s.

$$N\big(Z([0, a]), 2^{-n/\beta}\big) < (\lfloor a\rfloor + 1) c_8 c_{10}\, 2^{n\left(1+\left(\frac{1}{\beta}-H\right)d+\varepsilon\right)}$$

for all n large enough. This and (2.39) imply that $\overline{\dim}_{\mathrm{M}} Z([0, a]) \le \beta + (1 - \beta H)d + \beta\varepsilon$ almost surely which, in turn, implies $\dim_{\mathrm{P}} Z([0, a]) \le \beta + (1 - \beta H)d$ a.s.

Combining the above we have

$$\dim_{\mathrm{P}} Z([0, a]) \le \min\left\{\frac{1}{H}, \beta + (1 - \beta H)d\right\} \quad \text{a.s.}$$

This proves (2.35). □

Lemma 2.12. *Under the assumptions of Proposition 2.9, we have*

$$\dim_{\mathrm{H}} Z([0, 1]) \ge \begin{cases} 1/H & \text{if } 1 \le Hd, \\ \beta + (1 - H\beta)d & \text{if } 1 > Hd, \end{cases} \quad \text{a.s.} \tag{2.42}$$

Proof. The projection of $Z([0, 1])$ into $\mathbb{R}^d$ is $Y([0, 1])$, and $\dim_{\mathrm{H}} Y([0, 1]) = \frac{1}{H}$ a.s. when $1 \le Hd$. This implies the first inequality in (2.42).

To prove the inequality in (2.42) for the case $1 > Hd$, by Frostman's theorem (cf. Kahane, 1985, p. 133) along with the inequality

$$\|Z(x) - Z(y)\| \ge \frac{1}{2}\big[|D(x) - D(y)| + \|Y(x) - Y(y)\|\big],$$

it is sufficient to prove that for every constant $\gamma \in (0, \beta + (1 - H\beta)d)$, we have

$$\mathbb{E}\int_0^a\int_0^a \frac{dx\,dy}{\big[|D(x) - D(y)| + \|Y(x) - Y(y)\|\big]^{\gamma}} < \infty. \tag{2.43}$$

Since $1 > Hd$, we have $\beta + (1 - H\beta)d > d$. We only need to verify (2.43) for every $\gamma \in (d, \beta + (1 - H\beta)d)$.

For this purpose, we will make use of the following easily verifiable fact (see, e.g., Kahane, 1985, p. 279): if Ξ is a standard normal vector in $\mathbb{R}^d$, then there is a finite constant $c_{11} > 0$ such that for any constants $\gamma > d$ and $\rho \ge 0$,

$$\mathbb{E}\left[\frac{1}{\big(\rho + \|\Xi\|\big)^{\gamma}}\right] \le c_{11}\, \rho^{-(\gamma-d)}.$$

Fix $x, y \in [0, a]$ such that $x \ne y$. We use $\mathbb{E}_1$ to denote the conditional expectation given the subordinator D, apply the above fact with $\rho = |D(x) - D(y)|\,|x - y|^{-H}$ and use the self-similarity of D to derive

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\big[|D(x) - D(y)| + \|Y(x) - Y(y)\|\big]^{\gamma}}\right) &= |x - y|^{-H\gamma}\mathbb{E}\left[\mathbb{E}_1\left(\frac{1}{(\rho + \|\Xi\|)^{\gamma}}\right)\right] \\ &\le c_{11}|x - y|^{-H\gamma}\,\mathbb{E}\left[\frac{|x - y|^{H(\gamma-d)}}{|D(x) - D(y)|^{\gamma-d}}\right] \\ &= c_{12}\frac{1}{|x - y|^{Hd+(\gamma-d)/\beta}},\end{aligned} \tag{2.44}$$

where the last equality follows from the $1/\beta$-self-similarity of D and the constant $c_{12} = c_{11}\mathbb{E}\big(D(1)^{-(\gamma-d)}\big)$. Recall from Hawkes (1971, Lemma 1) that, as $r \to 0+$,

$$\mathbb{P}(D(1) \le r) \sim c_{13} r^{\beta/(2(1-\beta))} \exp\left(-(1-\beta)\beta^{\beta/(1-\beta)}\, r^{-\beta/(1-\beta)}\right),$$

where $c_{13} = \left[2\pi(1-\beta)\beta^{\beta/(2(1-\beta))}\right]^{-1/2}$. We verify easily $c_{12} < \infty$.

It follows from Fubini's theorem and (2.44) that

$$\mathbb{E}\int_0^a \int_0^a \frac{dx\,dy}{\left[|D(x)-D(y)| + \|Y(x)-Y(y)\|\right]^{\gamma}} \le c_{12}\int_0^a\int_0^a \frac{dx\,dy}{|x-y|^{Hd+(\gamma-d)/\beta}} < \infty, \tag{2.45}$$

the last integral is convergent because $Hd + (\gamma - d)/\beta < 1$. This proves (2.43) and thus the lemma. □

Proof of Proposition 2.9. Eq. (2.31) follows from Proposition 2.1 and the well-known results on the Hausdorff and packing dimensions for the range of a fractional Brownian motion (see, e.g., Chapter 18 of Kahane, 1985). In order to prove (2.32), apply Theorem 2.2 and (2.40) along with Lemmas 2.11 and 2.12. □

Remark 2.13. For a CTRW with dependent heavy tailed jumps, the outer process Y can be a linear fractional stable motion; see Meerschaert et al. (2009). Proposition 2.1 and Theorem 2.2 are applicable here, but the Hausdorff dimension of the range and graph sets of the processes Y and $Z(u) = (D(u), Y(u))$ are unknown in general. For some partial results in this direction, see Shieh and Xiao (2010) and Xiao and Lin (1994).

3. Discussion

CTRW limits are random fractals. The fractal dimension is altered by a random time change that represents particle resting times between diffusive movements. An important and useful example is the inverse stable subordinator with index $0 < \beta < 1$, that corresponds to power law waiting times with an infinite mean. The graph of a time-changed Brownian motion goes from dimension $3/2$ to dimension $1+\beta/2$. The fractal dimension of a fractional Brownian motion graph changes from $2 - H$ to $\beta + (1 - H\beta)$. The graph of a stable Lévy motion in one dimension with index $1 < \alpha < 2$ is a random fractal with index $2 - \alpha^{-1}$. After the time change, the graph has dimension $1 + \beta(1 - \alpha^{-1})$. In every case, the inverse β-stable subordinator alters the fractal dimension, such that substituting $\beta = 1$ recovers the dimension formula for the original outer process.

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