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Space-time duality for the fractional advection-dispersion equation

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Key Points:

- The space-fractional advection-dispersion equation models anomalous transport in rivers with a space fractional derivative
- Space-fractional models are mathematically equivalent to a corresponding time-fractional model
- This equivalence resolves a controversy in the hydrological literature

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Abstract The fractional advection-dispersion equation replaces the second spatial derivative in the usual advection-dispersion equation with a fractional derivative in the spatial variable. It was first applied to tracer tests in underground aquifers, and later to tracer tests in rivers. An alternative model replaces the first time derivative with a fractional derivative in time. Previous work has shown that both models provide a reasonable fit to breakthrough curves in rivers, which has led to a controversy regarding the physically appropriate fractional model. This paper shows that the relevant space-fractional model is mathematically equivalent to the corresponding time-fractional model, thus resolving the controversy.

1. Introduction

The fractional advection-dispersion equation (FADE) was introduced to model anomalous superdiffusion [*Benson et al.*, 2000a]. The FADE modifies the traditional advection-dispersion equation (ADE) by replacing the second derivative in space with a fractional derivative. The FADE has been successfully applied to tracer tests in underground aquifers [*Benson et al.*, 2000b, 2001; *Zhang et al.*, 2007] and in rivers [*Deng et al.*, 2004, 2006; *Zhang et al.*, 2005; *Kim and Kavvas*, 2006]. Point source solutions to the ADE are symmetric in space. Point source solutions to the FADE with a positive fractional derivative are positively skewed, with a long leading tail. Point source solutions to the FADE with a negative fractional derivative are negatively skewed, with a long trailing tail. A stochastic interpretation of the FADE model [*Meerschaert and Scheffler*, 2004; *Meerschaert*, 2012; *Benson et al.*, 2013] connects the positive fractional derivative in space with long particle movements in the flow direction, and the negative fractional derivative with upstream movements, against the flow. Hence, it is natural and physically meaningful to apply a positively skewed FADE, as is typical in applications to tracer tests in groundwater. Indeed, *Zhang et al.* [2009] caution that the negatively skewed FADE may not be physically realistic for applications to hydrology.

However, all known applications of the FADE to river flow hydrology employ a negative fractional derivative, which *Deng et al.* [2004] attribute to a "wide spectrum of dead zones" in the velocity field. Thus, it appears that, while the positive fractional derivative models early arrivals caused by preferential flow paths, the negative fractional derivative is modeling particle retention. Another modification of the traditional ADE replaces the first derivative in time by a fractional derivative in time. This time-FADE can be derived from a stochastic model with long resting periods between particle movements [*Meerschaert and Scheffler*, 2004; *Meerschaert*, 2012; *Benson et al.*, 2013]. This statistical physics model led *Zhang et al.* [2009] to recommend the time-FADE instead of the FADE with a negative fractional derivative to model tracer tests in rivers. *Baeumer et al.* [2009, Remark 4.5] revisit this controversy. They point out that, in the time-FADE, a particle remains at rest while the plume center of mass moves downstream. That particle is effectively displaced upstream relative to the center of mass, mimicking the effect of a negative fractional derivative in space.

In this paper, we resolve this controversy, by establishing a space-time duality between the negatively skewed FADE and the corresponding time-fractional model. First, we note that the point source solution to the FADE is given by a stable probability density function [*Benson et al.*, 2000a]. Then we apply a space-time duality result [*Baeumer et al.*, 2009, Corollary 4.1] to show that the solution to the negatively skewed FADE with no drift, restricted to the positive real axis, also solves a time-FADE with no drift. The proof of *Baeumer et al.* [2009, Corollary 4.1] relies on a deep duality result of *Zolotarev* [1961] for stable densities. Next we offer a very simple and revealing argument for space-time duality using the dispersion relation, followed by a new proof of space-time duality on the positive real axis using Fourier-Laplace transforms. Then we extend duality to the negative real axis, thereby proving space-time duality on the entire real axis. Finally, we apply space-time duality to the negatively skewed space-FADE with drift, by adopting a coordinate system that moves along with the plume center of mass. The resulting space-time duality resolves the controversy outlined in *Zhang et al.* [2009] and [*Baeumer et al.*, 2009, Remark 4.5] by showing that the negatively skewed space-FADE used to model tracer tests in rivers is just another way of writing a time-fractional model.

2. Simple Space-Time Duality

Here we outline a very simple argument for space-time duality. A more rigorous development will be presented later in section 4. The original FADE model for concentration $C=C(x,t)$ of a passive tracer is [*Benson et al.*, 2000a]

$$\frac{\partial C}{\partial t}=-v\frac{\partial C}{\partial x}+D\left(\frac{1+\beta}{2}\right)\frac{\partial^\alpha C}{\partial x^\alpha}+D\left(\frac{1-\beta}{2}\right)\frac{\partial^\alpha C}{\partial(-x)^\alpha}, \tag{1}$$

where the average plume velocity v [L/T] and fractional dispersivity D [L^α/T] are positive constants. The fractional index $\alpha\in(1,2]$, and skewness $\beta\in[-1,1]$ are dimensionless. When $\alpha=2$, the parameter β is superfluous, and equation (1) reduces to the traditional advection-dispersion equation (ADE) [*Bear*, 1972]. The positive Riemann-Liouville fractional derivative $\partial^\alpha C/\partial x^\alpha$ is the function with Fourier transform (FT) $(ik)^\alpha\hat{C}$, where $\hat{C}(k,t)=\int_{-\infty}^{\infty}e^{-ikx}C(x,t)\,dx$ is the usual FT. This definition extends the familiar FT formula for integer order derivatives. The negative Riemann-Liouville fractional derivative $\partial^\alpha C/\partial(-x)^\alpha$ has FT $(-ik)^\alpha\hat{C}$ [*Meerschaert and Sikorskii*, 2012]. Applications to groundwater hydrology typically use $\beta=1$, resulting in a positively skewed snapshot $x\mapsto C(x,t)$ with a long leading tail. Applications to river flows typically use $\beta=-1$, so that the snapshot is negatively skewed, and the breakthrough curve $t\mapsto C(x,t)$ has a long trailing tail. In the following derivations, we assume the negatively skewed FADE with $\beta=-1$.

Denote by C_0 the point source solution to the simplest negatively skewed FADE, taking $v=0$ and $D=1$, so that

$$\frac{\partial C_0}{\partial t}=\frac{\partial^\alpha C_0}{\partial(-x)^\alpha}. \tag{2}$$

Apply the Fourier transform in both variables,

$$\hat{C}_0(k,\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-ikx-i\omega t}C_0(x,t)\,dx\,dt, \tag{3}$$

to see that

$$[(i\omega)-(-ik)^\alpha]\hat{C}_0=0, \tag{4}$$

where k is the wave number and ω is the angular frequency. Viewing the FT as a weighted average of nonvanishing plane waves, the *dispersion relation* $(i\omega)=(-ik)^\alpha$ follows, which is equivalent to $(i\omega)^\gamma=(-ik)$ where $\gamma=1/\alpha$. Substituting back into (4) and inverting the FT leads to the dual equation:

$$\frac{\partial^\gamma C_0}{\partial t^\gamma}=-\frac{\partial C_0}{\partial x}, \tag{5}$$

since $\partial/\partial(-x)=-\partial/\partial x$. In the case $\alpha=2$, so that $\gamma=1/2$, this duality was observed by Heaviside in 1871, see *Das* [2011, section 3.7] for a modern presentation. The Heaviside solution to the diffusion equation was the first practical application of the fractional calculus, and the origin of what is now called *operational calculus*. In short, Heaviside took the square root of the operator on both sides of the diffusion equation, yielding a time-fractional equation of order $\gamma=1/2$. Similarly, we take the α root where $1<\alpha\le 2$, yielding a time-fractional equation of order $\gamma=1/\alpha$.

3. The FADE Model

The point source solution to the ADE, (1) with $\alpha=2$ and Dirac delta function initial condition $C(x,0)=\delta(x)$, is given by a normal probability density function (PDF):

$$C(x,t)=\frac{1}{\sqrt{4\pi Dt}}e^{-(x-vt)^2/(4Dt)},\tag{6}$$

for all $t>0$. To check this, take FT $x\mapsto k$ in (1) to get

$$\frac{d\hat{C}}{dt}=-v(ik)\hat{C}+D(ik)^2\hat{C};\quad \hat{C}(k,0)=1,\tag{7}$$

observe that the unique solution to (7) is $\hat{C}(k,t)=\exp[-v(ik)t+D(ik)^2t]$, and recognize that $\hat{C}(k,t)$ is the FT of (6). The normal PDF (6) has mean $x=vt$ and variance $2Dt$, so its standard deviation $\sqrt{2Dt}$ indicates a plume spreading proportional to the square root of the time variable.

The solution to the FADE (1) for any $1<\alpha\le 2$ can be written in terms of a stable PDF [*Benson et al.*, 2000a]. The stable PDF family includes the normal PDF as a special case. Taking FT in the FADE, (1) with $C(x,0)=\delta(x)$, yields

$$\frac{d\hat{C}}{dt}=-v(ik)\hat{C}+Dp(ik)^\alpha\hat{C}+Dq(-ik)^\alpha\hat{C};\quad \hat{C}(k,0)=1,$$

where $p=(1+\beta)/2$ and $q=(1-\beta)/2$. Then

$$\hat{C}(k,t)=\exp[-vt(ik)+Dpt(ik)^\alpha+Dqt(-ik)^\alpha].\tag{8}$$

For $\alpha\neq 1$, the FT of a stable PDF $f(x;\alpha,\beta,\sigma,\mu)$ can be written in the form [*Samorodnitsky and Taqqu*, 1994]:

$$\hat{f}(k;\alpha,\beta,\sigma,\mu)=\exp[-ik\mu-\sigma^\alpha|k|^\alpha(1+i\beta\,\mathrm{sgn}(k)\tan(\pi\alpha/2))],\tag{9}$$

with stable index $\alpha\in(0,2]$, skewness $\beta\in[-1,1]$, scale $\sigma>0$, and center $\mu\in(-\infty,\infty)$. A slightly different form applies when $\alpha=1$ [*Meerschaert and Scheffler*, 2001, Theorem 7.3.5]. A little algebra [*Meerschaert and Sikorskii*, 2012, equation (5.6)] shows that

$$p(ik)^\alpha+q(-ik)^\alpha=\cos(\pi\alpha/2)|k|^\alpha(1+i\beta\,\mathrm{sgn}(k)\tan(\pi\alpha/2)),\tag{10}$$

where $\beta=p-q$. Substitute (10) into (8) and compare with (9) to see that the point source solution to the FADE (1) can be written in the form of a stable PDF:

$$C(x,t)=f\left(x;\alpha,\beta,(Dt|\cos(\pi\alpha/2)|)^{1/\alpha},vt\right).\tag{11}$$

The stable index $\alpha\in(1,2]$ controls the tail behavior of the FADE solution (11): $C(x,t)\approx Aptx^{-\alpha-1}$ as $x\to\infty$ and $C(x,t)\approx Aqt|x|^{-\alpha-1}$ as $x\to-\infty$ [*Samorodnitsky and Taqqu*, 1994, Property 1.2.15]. When $\beta=0$, the snapshot $x\mapsto C(x,t)$ is symmetric about its mean $x=vt$. If $\beta=-1$ as in applications to river flow, the snapshot has a heavy trailing tail that models particle retention. If $\beta=1$ as in applications to groundwater flow, the snapshot has a heavy leading tail that models early arrivals. When $\alpha=2$, the skewness β is superfluous, and the stable PDF is normal with mean μ and variance $2\sigma^2$, so the stable PDF solution (11) reduces to the normal PDF solution (6). Although the FT of the stable PDF with $1<\alpha<2$ cannot be inverted in closed form, convenient computer codes are available to plot the stable PDF [*Nolan*, 1997] and these have been applied to the FADE [*Meerschaert and Sikorskii*, 2012, Chapter 5].

For our purposes, it will also be useful to nondimensionalize the FADE (1). Recall that the FT of the point source solution $C(x,t)$ to the FADE (1) is given by (8). Let $C_0(x,t)$ be the solution to the FADE (1) with $v=0$ and $D=1$, so that $\hat{C}_0(k,t)=\exp[pt(ik)^\alpha+qt(-ik)^\alpha]$, and consider the function $C_0(x-vt,Dt)$. Substitute $y=x-vt$ and simplify to get

$$\int_{-\infty}^{\infty}e^{-ikx}C_0(x-vt,Dt)\,dx=\int_{-\infty}^{\infty}e^{-ik(y+vt)}C_0(y,Dt)\,dy$$
$$=\exp[-ikvt+pDt(ik)^\alpha+qDt(-ik)^\alpha].$$

Compare (8) and use the uniqueness of the FT to see that $C(x,t)=C_0(x-vt,Dt)$, where $C_0(x,t)$ solves the fractional dispersion equation (FDE):

$$\frac{\partial C_0}{\partial t}=\left(\frac{1+\beta}{2}\right)\frac{\partial^\alpha C_0}{\partial x^\alpha}+\left(\frac{1-\beta}{2}\right)\frac{\partial^\alpha C_0}{\partial(-x)^\alpha},\tag{12}$$

for all x and all $t>0$. The change of variables yields a new coordinate system that moves along with the plume center of mass $x=vt$. Hence, $C(x,t)=C_0(x-vt,Dt)$ solves the FADE (1) if and only if $C_0(x,t)$ solves the FDE (12). A very similar FT argument shows that the FDE solution has a useful scaling property:

$$C_0(x,t)=t^{-1/\alpha}C_0(t^{-1/\alpha}x,1), \tag{13}$$

which means that the plume spreads like $t^{1/\alpha}$ and the peak concentration declines at the same rate.

4. Space-Time Duality

In this section, we develop the space-time duality relation for the FADE, extending the basic idea presented in section 2. Consider the solution $C_0(x,t)$ to the negatively skewed dimensionless FDE, (2) with point source initial condition $C_0(x,0)=\delta(x)$, and recall that $1<\alpha\le 2$. Substitute $\beta=-1$ and $v=0$ into (11) to see that

$$C_0(x,t)=f\left(x;\alpha,-1,t^{1/\alpha}|\cos(\pi\alpha/2)|^{1/\alpha},0\right). \tag{14}$$

Now we derive an alternative, equivalent solution for $x>0$. Apply the Fourier-Laplace transform (FLT),

$$\overline{C_0}(k,s)=\int_0^\infty\int_{-\infty}^\infty e^{-st}e^{-ikx}C_0(x,t)\,dx\,dt, \tag{15}$$

to both sides of (2), noting that the FT of the Dirac delta function is $\hat{C}_0(k,0)=1$ for all k, to get $s\overline{C_0}(k,s)-1=(-ik)^\alpha\overline{C_0}(k,s)$. Solve for $\overline{C_0}(k,s)$ to obtain

$$\overline{C_0}(k,s)=\frac{1}{s-(-ik)^\alpha}. \tag{16}$$

The inverse FT of (16) can be expressed as [*Morse and Feshbach*, 1953, (4.8.18)]

$$\tilde{C}_0(x,s)=\frac{1}{2\pi}\lim_{T\to\infty}\int_{-T+i\tau}^{T+i\tau}\frac{e^{ikx}}{s-(-ik)^\alpha}dk, \tag{17}$$

where $\tau>0$ is chosen to avoid the branch cut along the negative real axis. Note that the integrand has a single pole at $k^*=is^{1/\alpha}$ and remains analytic for all other points in the upper half plane. Convert the integral in (17) to a contour integral by attaching a semicircle of radius T in the upper half plane, and apply the Cauchy residue theorem; details are shown in Appendix A. For $x>0$, the integral over the semicircle tends to zero as $T\to\infty$, and we obtain

$$\tilde{C}_0(x,s)=\gamma s^{\gamma-1}\exp(-xs^\gamma)\quad\text{for } x>0, \tag{18}$$

where $\gamma=1/\alpha\in[1/2,1)$. Equation (18) can be inverted using the formula [*Meerschaert and Sikorskii*, 2012, (4.42)]:

$$\tilde{h}_+(x,s)=s^{\gamma-1}\exp(-xs^\gamma), \tag{19}$$

for the Laplace transform (LT) of the inverse stable subordinator PDF [*Meerschaert and Sikorskii*, 2012, equation (4.47) and Remark 5.6]:

$$h_+(x,t)=\frac{t}{\gamma x^{1+1/\gamma}}f(tx^{-1/\gamma};\gamma,1,|\cos(\pi\gamma/2)|^{1/\gamma},0)\quad\text{for } x>0. \tag{20}$$

Comparing (18) with (19) and using the uniqueness of the LT leads to

$$C_0(x,t)=\gamma h_+(x,t)=tx^{-1-1/\gamma}f(tx^{-1/\gamma};\gamma,1,|\cos(\pi\gamma/2)|^{1/\gamma},0)\quad\text{for all } x>0. \tag{21}$$

Thus, for $x>0$, we have an alternative solution to the negatively skewed FDE (2) in terms of a positively skewed stable PDF with index $\gamma=1/\alpha$.

The two solutions given by equations (14) and (21) are equal for any $t>0$ and $x>0$, and this leads to a remarkable connection between space-fractional and time-fractional dispersion models. Denote by $(\partial/\partial t)^\gamma g(t)$ the *Caputo* fractional derivative of order $0<\gamma<1$, with LT $s^\gamma\tilde{g}(s)-s^{\gamma-1}g(0)$ [*Meerschaert and Sikorskii*, 2012, pp. 38–39]. Recall that $1/(a+ik)$ is the FT of $e^{-ax}H(x)$, where $H(x)$ is the Heaviside function, and take FT in (19) to see that $h_+(x,t)$ has FLT:

$$\bar{h}_+(k,s)=\frac{s^{\gamma-1}}{ik+s^\gamma}, \tag{22}$$

for all k and all $s>0$. Rewrite in the form $s^\gamma\bar{h}_+(k,s)-s^{\gamma-1}=-(ik)\bar{h}_+(k,s)$ and then invert to see that $h_+(x,t)$ solves the time-fractional dispersion equation:

$$\left(\frac{\partial}{\partial t}\right)^\gamma h_+(x,t)=-\frac{\partial}{\partial x}h_+(x,t);\quad h_+(x,0)=\delta(x). \tag{23}$$

Since $C_0(x,t)$ is proportional to $h_+(x,t)$ for all $x>0$ and $t>0$, this implies that the point source solution to the negatively skewed FDE (2) also solves the time-fractional dispersion equation:

$$\left(\frac{\partial}{\partial t}\right)^\gamma C_0(x,t)=-\frac{\partial}{\partial x}C_0(x,t)\quad \text{for } x>0 \text{ and } t>0. \tag{24}$$

This space-time duality was first established by *Baeumer et al*. [2009, Corollary 4.1] using a completely different argument.

4.1. Random Walk Interpretation

Next we briefly discuss the random walk models behind the two equivalent model equations (2) and (24), see *Benson et al*. [2013] for more detail. The time-fractional dispersion equation (24) governs the long-time limiting particle density of a delayed random walk, where particles move a small distance downstream after each random waiting time, and the waiting times satisfy $P[T>t]\approx t^{-\gamma}$ for $t>0$ large. The heavy tailed waiting times T model occasional long particle retention times in the immobile zone. The negatively skewed FDE (2) governs the long-time limiting particle density of a random walk, where the jump variable X satisfies $P[X<-x]\approx x^{-\alpha}$ for $x>0$ large. This models occasional particle movements upstream *relative to the center of mass*. *Zhang et al*. [2009] argue that time-fractional models like (24) provide a more suitable physical description of particle movements in a river flow, since the negatively skewed FADE (2) models particles that jump upstream. On the other hand, *Deng et al*. [2004] and *Hunt* [2006] interpret the negative fractional derivative as a model for retention. The equivalence demonstrated here between the time-fractional model (24) and the space-fractional model (2) resolves that controversy, by showing that in fact the two models are equivalent.

4.2. Incorporating the Advection Term

Next we show how the general negatively skewed FADE (1) with $\beta=-1$ and a nonzero drift term $v\neq 0$ relates to a time-fractional model. Recall that the solution $C(x,t)$ to (1) can be written as: $C(x,t)=C_0(x-vt,Dt)$, where C_0 solves (2). Then it follows from (21) that

$$C(x,t)=C_0(x-vt,Dt)=\gamma h_+(x-vt,Dt)\quad \text{for } x>vt. \tag{25}$$

Note that $h_+(x,t)=0$ for $x<0$, and take FT $x\mapsto k$ to see that [*Bingham*, 1971]

$$\hat{h}_+(k,t)=\int_0^\infty e^{-ikx}h_+(x,t)\,dx=E_\gamma(-ikt^\gamma), \tag{26}$$

using the Mittag-Leffler function:

$$E_\gamma(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(1+\gamma j)}, \tag{27}$$

defined for all complex numbers z [*Mainardi and Gorenflo*, 2000]. Take LT in the remaining variable $t\mapsto s$ in (26), noting that $E_\gamma(-zt^\gamma)$ has LT $s^{\gamma-1}/(z+s^\gamma)$ [*Meerschaert and Sikorskii*, 2012, equation (2.29)], to arrive back at (22).

A change of variable $y=x-vt$ shows that

$$\begin{aligned}\hat{C}(k,t)&=\int_{vt}^{\infty}e^{-ikx}\gamma h_{+}(x-vt,Dt)\,dx\\&=\gamma\int_{0}^{\infty}e^{-ik(y+vt)}h_{+}(y,Dt)\,dy\\&=\gamma e^{-ikvt}E_{\gamma}(-ik(Dt)^{\gamma})\end{aligned}$$

and then evaluating the LT $t\mapsto s$ and using a substitution $u=Dt$ we arrive at

$$\begin{aligned}\bar{C}(k,s)&=\int_{0}^{\infty}e^{-st}\hat{C}(k,t)\,dt\\&=\int_{0}^{\infty}e^{-st}\gamma e^{-ikvt}E_{\gamma}(-ik(Dt)^{\gamma})\,dt\\&=\gamma\int_{0}^{\infty}e^{-(s+ikv)D^{-1}u}E_{\gamma}(-iku^{\gamma})\,D^{-1}du\\&=\frac{\gamma(s+ikv)^{\gamma-1}}{D^{\gamma}ik+(s+ikv)^{\gamma}}.\end{aligned}$$

Rearrange to get $(s+ikv)^{\gamma}\bar{C}(k,s)=-ikD^{\gamma}\bar{C}(k,s)+\gamma(s+ikv)^{\gamma-1}$ and invert to see that the same $C(x,t)$ that solves the FADE (1) with $\beta=-1$ also solves the coupled space-time-fractional governing equation:

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)^{\gamma}C(x,t)=-D^{\gamma}\frac{\partial}{\partial x}C(x,t)+\gamma\delta(x-vt)\frac{t^{-\gamma}H(t)}{\Gamma(1-\gamma)}\quad\text{for }x>vt, \tag{28}$$

using the LFT formula [*Meerschaert et al.*, 2002]:

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)^{\gamma}f(x,t)\mapsto(s+ikv)^{\gamma}\bar{f}(k,s) \tag{29}$$

and using the LT formula $t^{-\gamma}/\Gamma(1-\gamma)\mapsto s^{\gamma-1}$ again to see that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-st}e^{-ikx}\delta(x-vt)\frac{t^{-\gamma}H(t)}{\Gamma(1-\gamma)}\,dx\,dt=\int_{0}^{\infty}e^{-st}e^{-ikvt}\frac{t^{-\gamma}}{\Gamma(1-\gamma)}\,dt=(s+ikv)^{\gamma-1},$$

for all $k>0$ and $s>0$.

The operator defined in (29) is the *fractional material derivative*, first studied in *Sokolov and Metzler* [2003]. A multidimensional definition of the fractional substantial derivative was independently proposed in *Friedrich et al.* [2006] to study the fractional Kramers-Fokker-Planck equation. The standard material (or substantial) derivative gives the temporal rate of change of a quantity (e.g., concentration) in a moving reference frame as viewed from a fixed ground frame. Hence, (29) is a nonlocal generalization of the standard material, or substantial, derivative [*Bear*, 1972, 4.1.4] that models retention in a moving reference frame.

In conclusion, the point source solution $C(x, t)$ to the negatively skewed space-FADE (1) with $\beta=-1$, the model used in river flows, also solves the space-time-FADE (28) for all $x>vt$. Hence, the space-fractional model (1) with $\beta=-1$ is completely equivalent to the space-time-fractional model (28) of plume motion in river flows.

4.3. The Upstream Tail

Next we extend the space-time duality to both sides of the plume, including the portion upstream of the plume center of mass. Mathematical details are included in Appendix B, to show that for $x<0$, the solution to the negatively skewed dimensionless FADE (2) is given by $C_0(x,t)=\gamma h_{-}(-x,t)$ for $x<0$, where the PDF:

$$h_{-}(x,t)=\frac{t}{\gamma x^{1+1/\gamma}}f\left(tx^{-1/\gamma};\gamma,\beta',\sigma',0\right)H(x), \tag{30}$$

where β' is given by (B7), and σ' by (B8). Hence, the solution to (2) on the entire real line is

$$C_0(x,t)=\gamma h_{+}(x,t)H(x)+\gamma h_{-}(-x,t)H(-x), \tag{31}$$

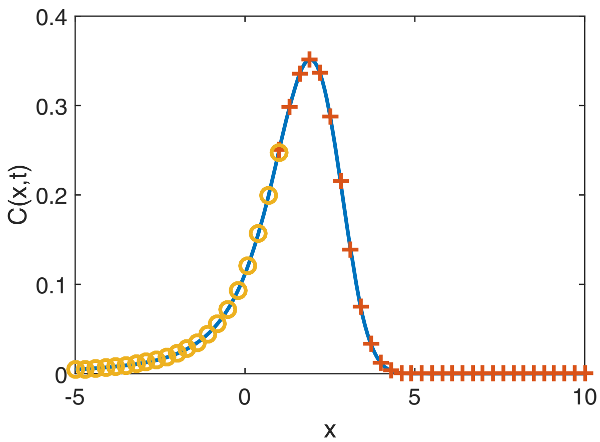

Figure 1. Snapshot of (32) with parameters are $\alpha=3/2$, $v=1$, $t=1$, and $D=1$. Solid line is the stable PDF on the right-hand side. Pluses mark the first term on the left-hand side, and circles mark the second term.

where we include the Heaviside terms to emphasize that only the h_+ term contributes when $x>0$, and only the h_- term operates on $x<0$. Then the general solution $C(x,t)=C_0(x-vt,Dt)$ to the negatively skewed FADE, (1) with $\beta=-1$, can be written as

$$C(x,t)=\gamma h_+(x-vt,Dt)H(x-vt) \\ +\gamma h_-(vt-x,Dt)H(vt-x). \tag{32}$$

To illustrate the two components in (32), Figure 1 plots an example snapshot $x\mapsto C(x,t)$ of the left-hand side against the two components on the right-hand side. Figure 2 plots a breakthrough curve (BTC) $t\mapsto C(x,t)$. Note that the power law tail of the BTC is modeled by the second term in (32).

Further calculations in Appendix C show that $h_-(x,t)$ is the point source solution to

$$p'\left(\frac{\partial}{\partial t}\right)^{\gamma}h_-(x,t)+q'\frac{\partial^{\gamma}h_-(x,t)}{\partial(-t)^{\gamma}}=Q'\frac{\partial h_-(x,t)}{\partial x};\quad h_-(x,0)=\delta(x), \tag{33}$$

for $t>0$ and $x>0$, where $p'=(1+\beta')/2$, $q'=(1-\beta')/2$, β' is given by (B7), and $Q'=\cos(\pi\gamma/2)/\cos(3\pi\gamma/2)$. The first term in (33) uses the Caputo derivative. This equation is more complicated than the corresponding dual (24) for $x>0$, because for $x<0$ the skewness varies from $-1/3$ to $+1$, and Q' varies from $-1/3$ to -1 as α increases from 1 to 2. Using (C3), a calculation similar to (28) shows that

$$p'\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)^{\gamma}C(x,t)+q'\left(-\frac{\partial}{\partial t}-v\frac{\partial}{\partial x}\right)^{\gamma}C(x,t)=\tilde{D}\frac{\partial}{\partial x}C(x,t)+r(x,t), \tag{34}$$

for $x<vt$, where $\tilde{D}=D^{\gamma}Q'$ and the boundary term:

$$r(x,t)=\gamma p'\delta(x-vt)\frac{t^{-\gamma}H(t)}{\Gamma(1-\gamma)}+\gamma q'\delta(x-vt)\frac{(-t)^{-\gamma}H(-t)}{\Gamma(1-\gamma)}, \tag{35}$$

see Appendix D for details. Note that for $t>0$, the boundary term simplifies to

$$r(x,t)=\gamma p'\delta(x-vt)\frac{t^{-\gamma}H(t)}{\Gamma(1-\gamma)}. \tag{36}$$

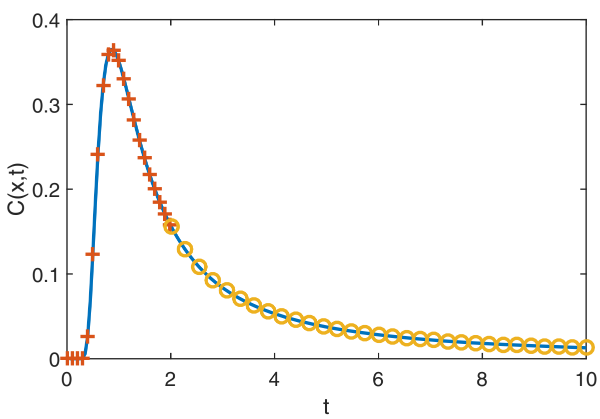

Figure 2. Breakthrough curve for (32) with parameters $\alpha=3/2$, $v=1$, $x=2$, and $D=1$. Solid line is the stable PDF on the right-hand side. Pluses mark the first term on the left-hand side, and circles mark the second term.

4.4. Duality for $\alpha=2$

For the dimensionless ADE, (12) with $\alpha=2$, the duality relationship given by (32) becomes

$$C(x,t)=\frac{1}{2}[h_+(x,t)H(x)+h_-(-x,t)H(-x)], \tag{37}$$

where $h_+(x,t)$ solves (38) with $\gamma=1/2$, and $h_-(-x,t)$ solves a very similar equation:

$$\left(\frac{\partial}{\partial t}\right)^{\gamma}h_-(x,t)=\frac{\partial}{\partial x}h_-(x,t); \\ h_+(x,0)=\delta(x), \tag{38}$$

since $Q'=-1$ and $\beta'=1$ in (33), so that $p'=1$ and $q'=0$. This classical factorization of the diffusion equation was established by Heaviside in 1871, see [*Das*, 2011, section 3.7] or [*Heaviside*, 2008].

5. Discussion

Deng et al. [2004] have attributed the skewness and long trailing tails of observed breakthrough curves in river flows to particle retention. Retention may be modeled via a continuous time random walk (CTRW) [*Metzler and Klafter*, 2004; *Montroll and Weiss*, 1965] where particles can undergo long waiting times between jumps. The scaling limit of such a CTRW is governed by a time-fractional diffusion equation [*Meerschaert et al*., 2002], providing a sound physical argument in favor of the time-fractional model. However, space-fractional models for solute transport in rivers have also been derived *ab initio* using a fractional conservation of mass argument [*Kim and Kavvas*, 2006]. At present, neither time-fractional nor space-fractional models can be invalidated using either data or physics.

In this paper, we presented a space-time duality result for the negatively skewed FADE, which implies that the space-fractional models and time-fractional models commonly applied to tracer tests in rivers may be viewed as two faces of the same coin. That is, under certain conditions, the space-fractional model and time-fractional model describe the *same* underlying physics and possess the same class of solutions; the two models appear to differ since they are clothed with different mathematical operators.

Baeumer et al. [2009] proved a space-time duality result for the negatively skewed FADE with no drift, $v=0$ and $\beta=-1$ in (1). They showed that a point source solution to that FADE, restricted to the positive real line $x>0$, also solves a time-FADE (24) with no drift. This result is not immediately applicable to tracer tests in river flows, where a nonzero drift is required. Hence, in this paper we extended that result, to establish a space-time duality for the negatively skewed FADE with drift. First, we offered a simple, intuitive argument for the original result of *Baeumer et al*. [2009] based on the dispersion relation (4). Then we provided a new proof of the result in *Baeumer et al*. [2009] based on Fourier and Laplace transforms: We considered the point source solution to the simplest negatively skewed FADE (2) with zero drift. For a point source solution starting at $x=0$, we took a Fourier transform in the space variable x and a Laplace transform in the time variable t. Rearranging, we inverted the Laplace transform, and then the Fourier transform, to recover the negatively skewed FADE with no drift. Inverting in the opposite order, first the Laplace transform and then the Fourier transform, led to the time-FADE with no drift. Hence, there is an equivalence between the negatively skewed space-FADE with no drift, and the time-FADE. As in *Baeumer et al*. [2009], that argument was only valid for $x>0$. Next we extended that argument to the negative real line $x<0$, and to the case of a nonzero drift, both of which are required for applications to tracer tests in rivers. The resulting duality relation shows that the negatively skewed FADE used to model tracer tests in rivers is mathematically equivalent to a corresponding time-fractional model. This equivalence resolves a controversy in the hydrological literature [*Berkowitz et al*., 2006; *Zhang et al*., 2009]. Rather than viewing the space-FADE and time-FADE (or CTRW) as competing models, one should see them as complementary versions of the same underlying physical process.

6. Conclusion

This paper establishes the mathematical equivalence of certain space-fractional and time-fractional models in hydrology. Previous research established a link between space-fractional models and long particle movements, as well as a relation between time-fractional models and long resting times between particle movements. Although seemingly incompatible, the results in this paper show that these phenomena are two sides of the same coin. In essence, a particle (or bolus of solute) experiencing a long resting time in the immobile zone, while the bulk of the plume flows downstream, is displaced in the upstream direction *relative to the plume center of mass*, since the center of mass moves downstream while the particle remains at rest. Hence, the FADE with negative skewness and the CTRW are completely compatible models, and both are equally valid representations of the underlying physics.

Appendix A: Complex Contour Integration Details

We transform the integral in (17) into a complex contour integral, by attaching a semicircle C_T of radius T in the upper half plane, see Figure A1. Then we show that the integral over the semicircle C_T approaches zero

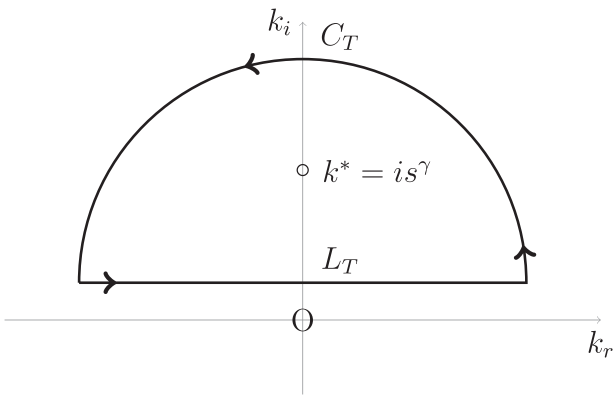

Figure A1. Contour $\mathcal{C}=L_T+C_T$ for $x>0$.

as $T\to\infty$ when $x>0$: Along C_T, let $k=T\exp(i\theta)$, where $0\le\theta\le\pi$, define the dispersion symbol $\psi(k,s)=s-(-ik)^\alpha$ and write

$$\begin{aligned}\left|\int_{C_T}\frac{e^{ikx}}{\psi(k,s)}dk\right| &= T\left|\int_0^\pi\frac{\exp(ixT\exp(i\theta))}{\psi(T\exp(i\theta),s)}d\theta\right|\\ &= T\int_0^\pi\frac{|\exp(ixT(\cos\theta+i\sin\theta))|}{|\psi(T\exp(i\theta),s)|}d\theta\\ &\le T\int_0^\pi\frac{\exp(-Tx\sin\theta)}{|s-T^\alpha e^{i\alpha(\theta-\pi/2)}|}d\theta.\end{aligned} \tag{A1}$$

Given $s>0$, for any $T>s+1$ we have $|s-T^\alpha e^{i\alpha(\theta-\pi/2)}|\ge|s-T^\alpha|>1$ so that the integrand is bounded above by its denominator. Since $\sin\theta\ge 2\theta/\pi$ for $0<\theta\le\pi/2$, we have $\exp(-Tx\sin\theta)\le\exp(-Tx2\theta/\pi)$, and then since $\sin\theta=\sin(\pi-\theta)$ it follows that the final term in (A1) is bounded above by

$$2T\int_0^{\pi/2}\exp(-2Tx\theta/\pi)\,d\theta\,, \tag{A2}$$

which tends to zero as $T\to\infty$. Hence, (A1) can be evaluated by applying the residue theorem to the interior of $\mathcal{C}$. Set $\psi(k,s)=0$ and solve for k to see that the integrand has a (simple) pole $k^*=is^{1/\alpha}$ for any $s>0$. Differentiate $\psi(k,s)$ with respect to k and evaluate at the pole to get $\psi'(k^*,s)=i\alpha s^{1-1/\alpha}$, then calculate the residue [*Brown et al.*, 1996, p. 195] to get

$$\tilde{C}(x,s)\quad=\frac{1}{2\pi}(2\pi i)\frac{e^{ik^*x}}{\psi'(k^*,s)}=\frac{1}{\alpha}s^{1/\alpha-1}e^{-xs^{1/\alpha}}\,, \tag{A3}$$

for any $s>0$. Substitute $\gamma=1/\alpha$ to arrive at (18).

Appendix B: Duality for $x<0$

Zolotarev [1986] defines the stable PDF $p_\alpha(x;\eta,b,\mu)$ as the function with FT:

$$\hat{p}_\alpha(k;\eta,b,\mu)=\exp\left[-ik\mu-b|k|^\alpha\exp\left(-i\eta\,\mathrm{sgn}(k)\frac{\pi}{2}\right)\right]. \tag{B1}$$

Here α and μ are as before in (9), $|\eta|\le\alpha$ for $0<\alpha\le1$, $|\eta|\le 2-\alpha$ for $1<\alpha\le2$ and $b>0$. *Zolotarev* [1961] proved a duality result for the stable PDF, which states that for any $\alpha\ge1$ and any $x>0$ we have

$$p_\alpha(x;\eta,1,0)=x^{-1-\alpha}p_{\alpha^*}(x^{-\alpha};\eta^*,1,0), \tag{B2}$$

where $\alpha^*=1/\alpha$ and $\eta^*=(\eta-1)/\alpha+1$. A little algebra [*Zolotarev*, 1986] shows that (B1) is equivalent to (9) with

$$\beta=\cot\left(\frac{\pi\alpha}{2}\right)\tan\left(\frac{\pi\eta}{2}\right)\quad\text{and}\quad\sigma^\alpha=b\cos\left(\frac{\pi\eta}{2}\right). \tag{B3}$$

Next we convert the duality relation (B2) to the parameterization of (9). Specializing to the case of a negatively skewed FADE with $1<\alpha\le2$ and $\beta=-1$, it follows from $\beta=\cot(\pi\alpha/2)\tan(\pi\eta/2)$ with $|\eta|\le2-\alpha$, using $\tan(\pi-u)=-\tan u$, that $\eta=2-\alpha$. Then $\eta^*=(1-\alpha)/\alpha+1=1/\alpha=\gamma$, and this corresponds to a γ-stable PDF with skewness $\beta=\cot(\pi\gamma/2)\tan(\pi\gamma/2)=1$. Then we can rewrite the negatively skewed case of (B2) for $x>0$ in the equivalent form:

$$f\left(x;\alpha,-1,|\cos(\pi\alpha/2)|^{1/\alpha},0\right)=x^{-1-1/\gamma}f\left(x^{-1/\gamma},\gamma,+1,|\cos(\pi\gamma/2)|^{1/\gamma},0\right), \tag{B4}$$

an equality between an α-stable PDF and a γ-stable PDF with $\gamma=1/\alpha$. The duality relation (21) follows using the scaling relation (13) along with (11).

For $x<0$, use the definition of the FT and a simple change of variable in (B1) to see that

$$p_\alpha(-x;\eta,b,0)=p_\alpha(x;-\eta,b,0). \tag{B5}$$

Apply Zolotarev duality, (B2), to see that for any $\alpha\geq 1$ and $x<0$ we have

$$\begin{aligned} p_\alpha(x;\eta,1,0) &= p_\alpha(-|x|;\eta,1,0)\\ &= p_\alpha(|x|;-\eta,1,0)\\ &= |x|^{-1-\alpha}p_\gamma(|x|^{-\alpha};\eta',1,0) \end{aligned}, \tag{B6}$$

with $\gamma=1/\alpha$ and $\eta'=((-\eta)-1)/\alpha+1=2-3\gamma$. Specializing to the case of a negatively skewed FADE with $1<\alpha\leq 2$ and $\beta=-1$, we see that the right-hand side of (B6) corresponds to a γ-stable PDF with skewness:

$$\beta'=\cot\left(\frac{\pi\gamma}{2}\right)\tan\left(\frac{\pi\eta'}{2}\right)=-\cot\left(\frac{\pi\gamma}{2}\right)\tan\left(\frac{3\pi\gamma}{2}\right), \tag{B7}$$

using $\tan(\pi-u)=-\tan u$, and scale,

$$\sigma'=\cos\left(\frac{\pi\eta'}{2}\right)=-\cos\left(\frac{3\pi\gamma}{2}\right), \tag{B8}$$

using $\cos(\pi-u)=\cos u$. Then we can rewrite the negatively skewed case of (B2) for $x<0$ in the equivalent form:

$$f\left(x;\alpha,-1,|\cos(\pi\alpha/2)|^{1/\alpha},0\right)=|x|^{-1-1/\gamma}f\left(|x|^{-1/\gamma},\gamma,\beta',\sigma',0\right). \tag{B9}$$

The duality relation (31) for the entire real line follows easily using (21), the scaling relation (13), and the definition (30).

Appendix C: Governing Equation for $x<0$

Define $g_\gamma(t)=f(t,\gamma,\beta',\sigma',0)$ and $G_\gamma(t)=\int_{-\infty}^t g_\gamma(\tau)\,d\tau$. Rewrite (30) as

$$\begin{aligned} h_-(x,t) &= \alpha t x^{-1-\alpha}g_\gamma(tx^{-\alpha})H(x)\\ &= -\frac{d}{dx}G_\gamma(tx^{-\alpha})H(x)\\ &= -\frac{d}{dx}\int_{-\infty}^t x^{-\alpha}g_\gamma(\tau x^{-\alpha})\,H(x)d\tau. \end{aligned} \tag{C1}$$

Use equations (9) and (10) to see that

$$\hat{g}_\gamma(\omega)=\int_{-\infty}^{\infty}e^{-i\omega t}g_\gamma(t)\,dt=e^{-\psi(\omega)},$$

where $\psi(\omega)=D'[p'(i\omega)^\gamma+q'(-i\omega)^\gamma]$, $p'=(1+\beta')/2, '=(1-\beta')/2$, β' is given by (B7), and $D'=\cos(3\pi\gamma/2)/\cos(\pi\gamma/2)$. A change of variable in the definition of the FT shows that $g(x,t)=x^{-\alpha}g_\gamma(tx^{-\alpha})$ has FT $\hat{g}(x,\omega)=e^{-x\psi(\omega)}$, and then it follows from (C1) that

$$\tilde{h}_-(x,\omega)=-H(x)\frac{d}{dx}(i\omega)^{-1}e^{-x\psi(\omega)}, \tag{C2}$$

using FT property $\int_{-\infty}^t f(\tau)\,d\tau\mapsto(i\omega)^{-1}\hat{f}(\omega)$ [*Howell*, 2001]. Evaluate the derivative in (C2), and then take FT in the remaining variable using $H(x)e^{-ax}\mapsto 1/(ik+a)$ to get

$$\bar{h}_-(k,\omega)=\frac{(i\omega)^{-1}\psi(\omega)}{ik+\psi(\omega)}. \tag{C3}$$

Rearrange and substitute $\psi(\omega)$ to get

$$p'(i\omega)^\gamma\bar{h}_-(k,\omega)+q'(i\omega)^\gamma\bar{h}_-(k,\omega)=-ikQ'\bar{h}_-(k,\omega)+p'(i\omega)^{\gamma-1}+q'(-i\omega)^{\gamma-1}, \tag{C4}$$

where $Q'=1/D'=\cos(\pi\gamma/2)/\cos(3\pi\gamma/2)$. Recalling that $\partial^\gamma f/\partial(\pm t)^\gamma\mapsto(\pm i\omega)^\gamma\hat{f}(\omega)$ and inverting the FT in both variables yields

$$p'\frac{\partial^\gamma h_-}{\partial t^\gamma}+q'\frac{\partial^\gamma h_-}{\partial(-t)^\gamma}=Q'\frac{\partial h_-}{\partial x}+p'\delta(x)b(t)+q'\delta(x)b(-t), \tag{C5}$$

where the source term $b(t)=H(t)t^{-\gamma}/\Gamma(1-\gamma)$, using $H(t)t^{-\gamma}/\Gamma(1-\gamma)\mapsto(i\omega)^{\gamma-1}$. For $t>0$, the term $b(-t)$ vanishes because of the Heaviside term. Combine the $b(t)$ term with the positive Riemann-Liouville derivative using the relationship [*Meerschaert and Sikorskii*, 2012, equation (2.33)]:

$$\left(\frac{\partial}{\partial t}\right)^\gamma f(t)=\frac{\partial^\gamma f(t)}{\partial t^\gamma}-\frac{f(0)t^{-\gamma}}{\Gamma(1-\gamma)}, \tag{C6}$$

to arrive at (33).

Appendix D: Governing Equation for $x<vt$

Here we write the upstream governing equation in the original coordinates. The calculation is similar to (28). Recall that $C(x,t)=\gamma h_-(vt-x,Dt)$ for $x<vt$ and substitute $y=vt-x$ to get

$$\int_{-\infty}^{vt} e^{-ikx}\gamma h_-(vt-x,Dt)\,dx=\gamma e^{-ikvt}\hat{h}_-(-k,Dt),$$

use (C3) to see that

$$\int_0^\infty e^{-i\omega t}\gamma e^{-ikvt}\hat{h}_-(-k,Dt)\,dt=\frac{\gamma}{D}\bar{h}_-\left(-k,\frac{\omega+kv}{D}\right)=\frac{\gamma}{D}\left(\frac{D}{i\omega+ikv}\right)\frac{\psi\left(\frac{\omega+kv}{D}\right)}{-ik+\psi\left(\frac{\omega+kv}{D}\right)},$$

rearrange and use the definition of $\psi(\omega)$ from Appendix C to get

$$[p'(i\omega+ikv)^\gamma+q'(-i\omega-ikv)^\gamma]\bar{C}(k,\omega)=-ik\tilde{D}\bar{C}(k,\omega)+\bar{r}(k,\omega),$$

where $\tilde{D}=D^\gamma/D'$ and $\bar{r}(k,\omega)=\gamma p'(i\omega+ikv)^{\gamma-1}-\gamma q'(-i\omega-ikv)^{\gamma-1}$. Use (29) with $s=i\omega$ to get

$$\left(-\frac{\partial}{\partial t}-v\frac{\partial}{\partial x}\right)^\gamma f(x,t)\mapsto(-i\omega-ikv)^\gamma\bar{f}(k,\omega)$$

and then (34) follows. Use the FT formula $t^{-\gamma}H(t)/\Gamma(1-\gamma)\mapsto(i\omega)^{\gamma-1}$ to see that

$$\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-i\omega t}e^{-ikx}\delta(x-vt)\frac{t^{-\gamma}H(t)}{\Gamma(1-\gamma)}\,dx\,dt=\int_0^\infty e^{-i\omega t}e^{-ikvt}\frac{t^{-\gamma}}{\Gamma(1-\gamma)}\,dt=(i\omega+ikv)^{\gamma-1}$$

and similarly, use the FT formula $(-t)^{-\gamma}H(-t)/\Gamma(1-\gamma)\mapsto(-i\omega)^{\gamma-1}$ to get

$$\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-i\omega t}e^{-ikx}\delta(x-vt)\frac{(-t)^{-\gamma}H(-t)}{\Gamma(1-\gamma)}\,dx\,dt=(-i\omega-ikv)^{\gamma-1},$$

which leads directly to (35).

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References

Baeumer, B., M. M. Meerschaert, and E. Nane (2009), Space–time duality for fractional diffusion, *J. Appl. Probab.*, *46*, 1100–1115.
Bear, J. (1972), *Dynamics of Fluids in Porous Media*, Elsevier, New York.
Benson, D. A., S. W. Wheatcraft, and M. M. Meerschaert (2000a), The fractional-order governing equation of Lévy motion, *Water Resour. Res.*, *36*(6), 1413–1423.
Benson, D. A., S. W. Wheatcraft, and M. M. Meerschaert (2000b), Application of a fractional advection-dispersion equation, *Water Resour. Res.*, *36*(6), 1403–1412.
Benson, D. A., R. Schumer, M. M. Meerschaert, and S. W. Wheatcraft (2001), Fractional dispersion, Lévy motion, and the MADE tracer tests, *Transp. Porous Media*, *42*(1/2), 211–240.
Benson, D. A., M. M. Meerschaert, and J. Revielle (2013), Fractional calculus in hydrologic modeling: A numerical perspective, *Adv. Water Resour.*, *51*, 479–497.
Berkowitz, B., A. Cortis, M. Dentz, and H. Scher (2006), Modeling non-Fickian transport in geological formations as a continuous time random walk, *Rev. Geophys.*, *44*, RG2003, doi:10.1029/2005RG000178.
Bingham, N. H. (1971), Limit theorems for occupation times of Markov processes, *Z. Wahrscheinlichkeitstheorie Verw. Gebiete*, *17*, 1–22.
Brown, J. W., R. V. Churchill, and M. Lapidus (1996), *Complex Variables and Applications*, vol. 7, McGraw-Hill, New York.
Das, S. (2011), *Functional Fractional Calculus*, Springer, New York.
Deng, Z., L. Bengtsson, and V. P. Singh (2006), Parameter estimation for fractional dispersion model for rivers, *Environ. Fluid Mech.*, *6*(5), 451–475.

Deng, Z.-Q., V. P. Singh, and L. Bengtsson (2004), Numerical solution of fractional advection-dispersion equation, *J. Hydraul. Eng.*, *130*(5), 422–431.

Friedrich, R., F. Jenko, A. Baule, and S. Eule (2006), Anomalous diffusion of inertial, weakly damped particles, *Phys. Rev. Lett.*, *96*(23), 230601.

Heaviside, O. (2008), *Electromagnetic Theory*, vol. 3, Cosimo, Inc., New York.

Howell, K. B. (2001), *Principles of Fourier Analysis*, Chapman and Hall, Boca Raton, Fla.

Hunt, B. (2006), Asymptotic solutions for one-dimensional dispersion in rivers, *J. Hydraul. Eng.*, *132*(1), 87–93.

Kim, S., and M. L. Kavvas (2006), Generalized Fick's law and fractional ADE for pollution transport in a river: Detailed derivation, *J. Hydraul. Eng.*, *11*(1), 80–83.

Mainardi, F., and R. Gorenflo (2000) On Mittag-Leffler-type functions in fractional evolution processes, *J. Comput. Appl. Math.*, *118*, 283–299.

Meerschaert, M., and H.-P. Scheffler (2001), *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*, John Wiley, New York.

Meerschaert, M., and H.-P. Scheffler (2004), Limit theorems for continuous time random walks with infinite mean waiting times, *J. Appl. Probab.*, *41*, 623–638.

Meerschaert, M. M. (2012), Fractional calculus, anomalous diffusion, and probability, in *Fractional Dynamics*, edited by R. Metzler and J. Klafter, pp. 265–284, World Sci., Singapore.

Meerschaert, M. M., and A. Sikorskii (2012), *Stochastic Models for Fractional Calculus*, Walter de Gruyter, Berlin.

Meerschaert, M. M., D. A. Benson, H.-P. Scheffler, and P. Becker-Kern (2002), Governing equations and solutions of anomalous random walk limits, *Phys. Rev. E*, *66*(6), 060102.

Metzler, R., and J. Klafter (2004), The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics, *J. Phys. A*, *37*(31), R161.

Montroll, E. W., and G. H. Weiss (1965), Random walks on lattices. II, *J. Math. Phys.*, *6*(2), 167–181.

Morse, P. M., and H. Feshbach (1953), *Methods of Theoretical Physics (Part 1)*, McGraw-Hill, New York.

Nolan, J. P. (1997), Numerical calculation of stable densities and distribution functions, *Commun. Statist. Stochastic Models*, *13*(4), 759–774.

Samorodnitsky, G., and M. S. Taqqu (1994), *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman and Hall, New York.

Sokolov, I. M., and R. Metzler (2003), Towards deterministic equations for Lévy walks: The fractional material derivative, *Phys. Rev. E*, *67*(1), 010101.

Zhang, X. X., J. W. Crawford, L. K. Deeks, M. I. Sutter, A. G. Bengough, and I. M. Young (2005), A mass balance based numerical method for the fractional advection-dispersion equation: Theory and application, *Water Resour. Res.*, *41*, W07029, doi:10.1029/2004WR003818.

Zhang, Y., D. A. Benson, M. M. Meerschaert, and E. M. LaBolle (2007), Space-fractional advection-dispersion equations with variable parameters: Diverse formulas, numerical solutions, and application to the Macrodispersion Experiment site data, *Water Resour. Res.*, *43*, W05439, doi:10.1029/2006WR004912.

Zhang, Y., D. A. Benson, and D. M. Reeves (2009), Time and space nonlocalities underlying fractional-derivative models: Distinction and literature review of field applications, *Adv. Water Res.*, *32*(4), 561–581.

Zolotarev, V. (1961), Expression of the density of a stable distribution with exponent α greater than one by means of a density with exponent $1/\alpha$, *Dokl. Akad. Nauk SSSR*, *98*, 735–738.

Zolotarev, V. (1986) *One-Dimensional Stable Distributions*, *Transl. Math. Monogr. 65*, Am. Math. Soc., Providence, R. I.