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Fractional conservation of mass

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ABSTRACT

The traditional conservation of mass equation is derived using a first-order Taylor series to represent flux change in a control volume, which is valid strictly for cases of linear changes in flux through the control volume. We show that using higher-order Taylor series approximations for the mass flux results in mass conservation equations that are intractable. We then show that a fractional Taylor series has the advantage of being able to exactly represent non-linear flux in a control volume with only two terms, analogous to using a first-order traditional Taylor series. We replace the integer-order Taylor series approximation for flux with the fractional-order Taylor series approximation, and remove the restriction that the flux has to be linear, or piece-wise linear, and remove the restriction that the control volume must be infinitesimal. As long as the flux can be approximated by a power-law function, the fractional-order conservation of mass equation will be exact when the fractional order of differentiation matches the flux power-law. There are two important distinctions between the traditional mass conservation, and its fractional equivalent. The first is that the divergence term in the fractional mass conservation equation is the fractional divergence, and the second is the appearance of a scaling term in the fractional conservation of mass equation that may eliminate scale effects in parameters (e.g., hydraulic conductivity) that should be scale-invariant.

1. Introduction

The conservation of mass equation for a fluid in a porous medium has its origins in continuum mechanics. The concept is that fluid mass flux going into a control volume is subtracted from the fluid mass flux going out of a control volume. The limit is taken as the control volume shrinks to a point, and the result is the familiar divergence term for the mass flux. This technique works quite well for the materials and processes that it was originally designed to describe, such as fluid flow, or more generally, for any conservative substance or property, such as heat flux in a metal. The success of using the traditional divergence to represent the conservation of a conservative property in a continuum owes to the fact that the continuum is normally a substance that is quite homogenous down to the molecular scale, and the measurement scale is normally many orders of magnitude larger than the scale of heterogeneity.

Applying the divergence term for net fluid mass flux in a porous medium has always been problematic due to the fact that a porous medium is only a continuum down to a scale that is several times larger than the pore scale. For a granular, homogeneous porous medium, the smallest scale of the control volume is on the order of 1 cm^3, a size which in many cases will be larger than the measurement scale [6]. For a heterogeneous porous medium with a finite correlation scale, the effective size of the control volume (or integral scale, in stochastic terms) can easily be on the order of 1–10 m^3. For fractal, or scale-invariant, porous media, the control volume, or integral scale is considered infinite, or at least scale-dependant [16,13], and will be much larger than the measurement scale.

The fundamental assumption of the divergence is that the control volume vanishes in the limit. The reason for this assumption is that the divergence is obtained using a first order Taylor series. By truncating the Taylor series at the first order, we make the implicit assumption that changes in the property being conserved (e.g., fluid mass flux) are small and linear over the control volume. This assumption is only removed by requiring that the control volume vanishes, in which case the first order Taylor series estimate of flux becomes identical to the point estimate. Control volumes for heterogeneous porous media are large, and properties within the control volume (e.g., mass flux, hydraulic conductivity) vary over several orders of magnitude. Hence the use of a first-order Taylor series to approximate flux changes within the control volume may not be appropriate.

In this paper, we first develop the traditional mass conservation equation, emphasizing the assumptions that are important (and suspect) for heterogeneous porous media, especially the use of the Taylor series. We show that an exact form of the fluid mass conservation equation can be written with the infinite Taylor series, although the resulting governing equation is intractable.

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Recent advances in fractional calculus [14] have led to the development of a robust fractional Taylor series. We develop a new form of fractional mass conservation using a first-order fractional Taylor series. We show that the resulting mass conservation governing equation is exact, as long as the fractional order of differentiation matches the order of non-linearity of the fluid flux. We then investigate the nature and properties of the fractional mass conservation equation.

2. Traditional mass conservation

In this section, we develop the traditional fluid mass conservation equation in a porous medium, emphasizing the use of the first-order Taylor series and the consequences of this assumption. Fig. 1 illustrates the standard control volume, with the length of the sides, Δx_1, Δx_2, Δx_3 defined. The inflow component of the fluid mass flux, $F(x_1)$, passing through the $-x_1$ face is

$$F(x_1) = \Delta x_2 \Delta x_3 \rho q_1, \tag{1}$$

where q_1 is the x_1 component of the specific discharge (L/t) passing through the $-x_1$ face of the control volume and ρ is the fluid density (m/L^3).

The mass flux outflow that passes through the $+x_1$ face is obtained by taking the flux passing through the $-x_1$ (1) and adding to it the change that takes place in the mass flux in the x_1-direction multiplied by the distance (Δx_1) over which the change acts

$$F(x_1 + \Delta x_1) = \Delta x_2 \Delta x_3 \rho q_1 + \Delta x_2 \Delta x_3 \frac{\partial \rho q_1}{\partial x_1} \Delta x_1. \tag{2}$$

This step is in virtually every groundwater hydrology text (e.g., [9,8]) that derives mass conservation. What is seldom said is that this is simply the first-order Taylor series approximation for the mass flux expanded about the point x_1

$$F(x_1 + \Delta x_1) = \Delta x_2 \Delta x_3 \left(\rho q_1 + \frac{\partial \rho q_1}{\partial x_1} \Delta x_1 + \frac{\partial^2 \rho q_1}{\partial x_1^2} \frac{\Delta x_1^2}{2} + \frac{\partial^3 \rho q_1}{\partial x_1^3} \frac{\Delta x_1^3}{3!} + \cdots \right). \tag{3}$$

By truncating the Taylor series approximation for the flux at the second-order term, we are making the very important assumption that changes in mass flux that take place within the control volume are linear. If these changes are non-linear, we are further making the assumption that they can be adequately approximated as piece-wise linear. This is the reason that we take the limit as the control volume shrinks to zero. As long as the control volume is vanishingly small (compared to the measurement scale), any non-linear change in flux can be approximated as piece-wise linear.

This critical assumption is illustrated in Fig. 2. Suppose that the actual flux is non-linear. The linear first order Taylor series approximation is shown as the straight line, while the actual non-linear flux is shown by the curved line above it. Only in the area immediately around x (much smaller than Δx) would the piece-wise linear approximation be reasonably accurate.

To complete the traditional derivation of the divergence term in the mass conservation equation, we take the net mass flux in the x_1 direction, $F(x_1) - F(x_1 + \Delta x_1)$, using (1) and (2)

$$\begin{aligned} F(x_1) - F(x_1 + \Delta x_1) &= (\Delta x_2 \Delta x_3 \rho q_1) \\ &\quad - \left(\Delta x_2 \Delta x_3 \rho q_1 + \Delta x_2 \Delta x_3 \frac{\partial \rho q_1}{\partial x_1} \Delta x_1 \right) \\ &= -\Delta V \frac{\partial \rho q_1}{\partial x_1}, \end{aligned} \tag{4}$$

where $\Delta V = \Delta x_1 \Delta x_2 \Delta x_3$ is the volume of the control volume.

Taking the net mass flux in the x_2 and x_3 directions

$$\begin{aligned} F(x_2) - F(x_2 + \Delta x_2) &= (\Delta x_1 \Delta x_3 \rho q_2) - \left(\Delta x_1 \Delta x_3 \rho q_2 + \Delta x_1 \Delta x_3 \frac{\partial \rho q_2}{\partial x_2} \Delta x_2 \right) \\ &= -\Delta V \frac{\partial \rho q_2}{\partial x_2}, \end{aligned} \tag{5}$$

$$\begin{aligned} F(x_3) - F(x_3 + \Delta x_3) &= (\Delta x_1 \Delta x_2 \rho q_3) - \left(\Delta x_1 \Delta x_2 \rho q_3 + \Delta x_1 \Delta x_2 \frac{\partial \rho q_3}{\partial x_3} \Delta x_3 \right) \\ &= -\Delta V \frac{\partial \rho q_3}{\partial x_3} \end{aligned} \tag{6}$$

and adding (4)–(6) gives us the net mass flux passing through the control volume

$$\Delta V \left(-\frac{\partial \rho q_1}{\partial x_1} - \frac{\partial \rho q_2}{\partial x_2} - \frac{\partial \rho q_3}{\partial x_3} \right) = -\Delta V \frac{\partial \rho q_i}{\partial x_i} \tag{7}$$

using Einstein's notation, where $\mathbf{q} = \imath_1 q_1 + \imath_2 q_2 + \imath_3 q_3$, $\imath_1$, $\imath_2$, $\imath_3$ are the unit vectors in the x_1, x_2, x_3 directions, respectively.

Eq. (7) is the net mass flux, or divergence, and is then assigned to the accumulation term to provide the standard mass conservation for a fluid in a porous medium

$$-\frac{\partial \rho q_i}{\partial x_i} = \frac{1}{\Delta V} \frac{\partial}{\partial t} (\Delta V n \rho), \tag{8}$$

where n is porosity. A change of variables is normally performed on the right-hand side of (8) so that the dependent variable is either

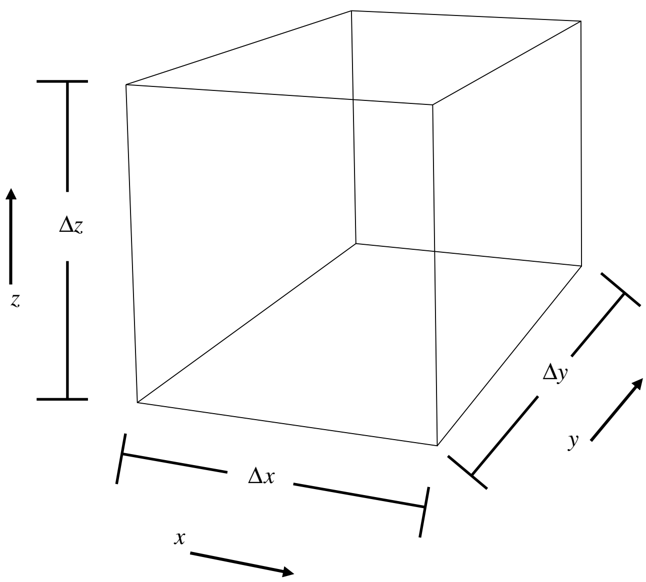

Fig. 1. Definition sketch for control volume.

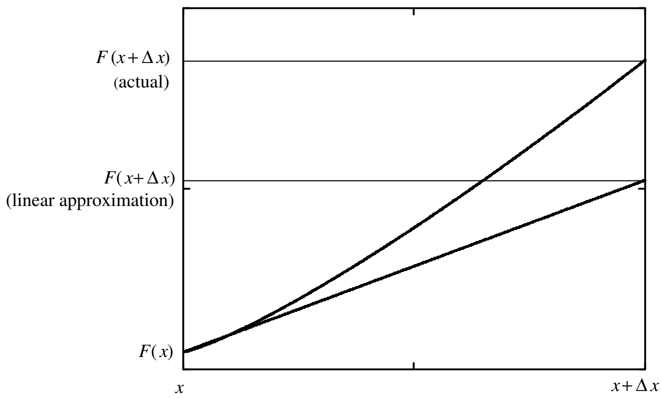

Fig. 2. Non-linear function and the first-order Taylor series approximation.

pressure or head, but we will not concern ourselves with that part of the derivation at this time.

From this discussion, it should be clear that the traditional mass conservation equation (8) is only exact when the change in flux in the control volume is linear, due to the fact that we only used a first-order Taylor series (2) to represent the mass flux change.

Let us suppose that the change in mass flux is parabolic

$$f(y) = q + (y - x)^p, \tag{9}$$

where $q = f(x)$ is the flux at the point x, the function $f(y)$ represents flux at the point $y = x + \Delta x$, and the power law exponent $p = 2$. By adding the second-order term to the Taylor series

$$F(x_1 + \Delta x_1) = \Delta x_2 \Delta x_3 \left(\rho q_1 + \frac{\partial \rho q_1}{\partial x_1} \Delta x_1 + \frac{\partial^2 \rho q_1}{\partial x_1^2} \frac{\Delta x_1^2}{2} \right) \tag{10}$$

we can derive a mass conservation equation that is exact for parabolic non-linear fluid fluxes. The net mass flux in the x_1 direction is

$$\begin{aligned} F(x_1) & \\ - F(x_1 + \Delta x_1) &= (\Delta x_2 \Delta x_3 \rho q_1) \\ &- \left(\Delta x_2 \Delta x_3 \rho q_1 + \Delta x_2 \Delta x_3 \frac{\partial \rho q_1}{\partial x_1} \Delta x_1 + \Delta x_2 \Delta x_3 \frac{\partial^2 \rho q_1}{\partial x_1^2} \frac{\Delta x_1^2}{2} \right) \\ &= -\Delta V \left(\frac{\partial \rho q_1}{\partial x_1} + \frac{\partial^2 \rho q_1}{\partial x_1^2} \frac{\Delta x_1}{2} \right). \end{aligned} \tag{11}$$

Taking the net flux in the x_2 and x_3 directions, and adding these to (11) and assigning the result to the net accumulation term gives us the mass conservation equation for the control volume

$$-\frac{\partial \rho q_i}{\partial x_i} - \frac{\Delta V}{2} \left(\Delta x_i \frac{\partial^2 \rho q_i}{\partial^2 x_i} \right) = \frac{1}{\Delta V} \frac{\partial}{\partial t} (\Delta V n \rho). \tag{12}$$

Eq. (12) is the exact mass conservation equation for the case in which the mass flux is non-linear and proportional to the square of the travel distance (9).

Although we expect fluid mass flux to be non-linear, there is no reason to expect that it would follow (9) in general, much less for a specific case. So we can generalize (12) by including all the terms in the Taylor series. The derivation is straightforward, and follows along the same lines as (10)–(12), and results in

$$-\sum_{n=1}^{\infty} \frac{(\Delta x_i)^{n-1}}{n!} \frac{\partial^n \rho q_i}{\partial x_i^n} = \frac{1}{\Delta V} \frac{\partial}{\partial t} (\Delta V n \rho). \tag{13}$$

Eq. (13) is the most general form of the fluid mass conservation in a porous medium, which is exact for any linear or non-linear changes in fluid flux. Although it is relatively easy to derive (13), it is intractable analytically or numerically, for even the most simple cases.

A recent breakthrough in fractional calculus has given us a tool that is extremely useful in solving this problem: the fractional Taylor series [14].

3. The fractional Taylor series

A good introduction to the use and meaning of fractional derivatives in physical and biological systems can be found in Metzler and Klafter [12]. The fractional Taylor series is a generalization of the Taylor series for fractional derivatives, where α is the fractional order of differentiation, $0 < \alpha < 1$. The fractional Taylor series at the point $y = x + \Delta x$ is defined by Odibat and Shawagfeh [14]

$$\begin{aligned} F(y) = F(x) + D_x^{\alpha} F(x+) \frac{(y-x)^{\alpha}}{\Gamma(\alpha+1)} + D_x^{\alpha} D_x^{\alpha} F(x+) \\ \times \frac{(y-x)^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots, \end{aligned} \tag{14}$$

where $\Gamma(x)$ is the gamma function, and D_x^{α} is the Caputo fractional derivative of order $0 < \alpha < 1$ with base point x, which is defined for $0 < \alpha < 1$ by

$$D_x^{\alpha} F(y) = \frac{1}{\Gamma(1-\alpha)} \int_x^y F'(y-u)(u-x)^{-\alpha} \mathrm{d}u.$$

Here F' is the usual first derivative, and the notation $x+$ in (14) indicates the limit as we approach x from the right. Note that there are other definitions of the fractional derivative, but the fractional Taylor series is only valid for the Caputo form. The main distinguishing feature of the Caputo fractional derivative is that, like the integer order derivative, the Caputo fractional derivative of a constant is zero. This property is critical for a fractional Taylor series. Note also that the third term in (14) involves the α fractional derivative of the α fractional derivative, which is not the same as the 2α fractional derivative. This is to ensure that the α fractional derivative of the function (9) is a constant when $p = \alpha$, and the α fractional derivative of that constant is zero. Then the coefficients of the fractional Taylor series can be found in the usual way, by repeated differentiation. The traditional integer-order Taylor series is recovered from (14) when $\alpha = 1$, using the well-known property of the Gamma function: $\Gamma(n+1) = n!$.

The fractional Taylor series is extremely useful for approximating non-integer power law functions. We will use (9), a non-linear power law function, to illustrate this point.

The traditional integer order Taylor series approximation for (9) with $p = 2$, expanded about x and truncated at the second order term, is

$$f(y) = f(x) + f'(x)(y-x) + f''(x)\frac{(y-x)^2}{2!}. \tag{15}$$

Since $q = f(x)$, and $f'(x) = 2 \cdot (y-x)^1|_{y=x} = 0$, and $f''(x) = 2 \cdot 1 \cdot (y-x)^0|_{y=x} = 2$, Eq. (15) becomes

$$f(y) = q + (y-x)^2. \tag{16}$$

Hence, the second-order Taylor series approximation of $f(y)$ is exact, because the order of non-linearity of the function matches the order of the Taylor series approximation. If $p = 3$, a third-order Taylor series would provide an exact approximation.

However, if p is a non-integer real number, no finite integer order Taylor series can give an exact match between the value of the function and its Taylor series approximation.

Now let us examine the fractional Taylor series approximation of (9), when $p > 0$ is some real number. The Caputo fractional derivative of the function (9) is

$$D_x^{\alpha} f(y) = D_x^{\alpha}[q + (y-x)^p] = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} (y-x)^{p-\alpha}. \tag{17}$$

This well-known formula (17) from fractional calculus is not hard to check, using the standard formula for the Beta integral

$$\int_A^B (s-A)^{a-1}(B-s)^{b-1} \mathrm{d}s = (B-A)^{a+b-1} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

which is valid for any real numbers $A < B$ and positive real numbers a, b. The first order fractional Taylor series for (9) expanded about x is exact when $\alpha = p$: the first term $f(x) = q$. For the second term, use (17) along with the fact that the Caputo fractional derivative of the constant q is zero to write

$$D_x^{\alpha} f(y) = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} (y-x)^{p-\alpha}.$$

Then for any $y > x$ we have

$$D_x^{\alpha} f(y) = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} \cdot 1. \tag{18}$$

The last equality follows from the fact that, since $p = \alpha$, the exponent $p - \alpha = 0$, so the term $(y - x)^{p-\alpha} = 1$ for any $y > x$. Then of course the limit as $y \to x+$ is

$$D_x^\alpha f(x+) = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} \cdot 1.$$

Hence second term in the fractional Taylor series is

$$[D_x^\alpha f(x+)] \cdot \frac{(y-x)^\alpha}{\Gamma(\alpha+1)} = \left[\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} \cdot 1\right] \cdot \frac{(y-x)^\alpha}{\Gamma(\alpha+1)} = (y-x)^p.$$

Here we have used $p = \alpha$ and $\Gamma(1) = 1$. Since $D_x^\alpha f(y)$ is a constant, the remaining higher order Caputo fractional derivatives are all zero, and so the two-term fractional Taylor series approximation is exact

$$f(y) = f(x) + D_x^\alpha f(x+) \frac{(y-x)^\alpha}{\Gamma(\alpha+1)} + D_x^\alpha D_x^\alpha f(x+) \frac{(y-x)^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots$$
$$= q + (y-x)^p + 0 + \cdots$$

This is a very important result. It tells us that if we match the order of the fractional Taylor series approximation to the exponent in the power law function we are trying to approximate (9), then the two-term fractional Taylor series approximation to this function is exact.

A very important consequence for mass conservation of this result is that we can replace the integer-order Taylor series approximation for flux with the fractional-order Taylor series approximation, and remove the restriction that the flux has to be linear, or piece-wise linear, and remove the restriction that the control volume must be infinitesimal. As long as the flux can be approximated by a power-law function, the fractional-order conservation of mass equation will be exact. In the next section, we derive the fractional conservation of mass equation using the fractional order Taylor series approximation for the flux term.

4. Fractional conservation of mass equation

Using Fig. 1, we define the component of flux passing through the $-x_1$ face, which is the same as Eq. (1)

$$F(x_1) = \Delta x_2 \Delta x_3 \rho q_1. \tag{19}$$

Now we compute the component of flux passing through the $+x_1$ face using the two-term αth order fractional Taylor series expanded about x_1

$$F(x_1 + \Delta x_1) = \Delta x_2 \Delta x_3 \rho q_1 + \Delta x_2 \Delta x_3 \frac{\partial^\alpha \rho q_1}{\partial x_1^\alpha} \frac{\Delta x_1^\alpha}{\Gamma(\alpha+1)}. \tag{20}$$

The net mass flux in the x_1-direction is obtained by subtracting (19) from (20)

$$F(x_1) - F(x_1 + \Delta x_1) = -\Delta x_2 \Delta x_3 \frac{\partial^\alpha \rho q_1}{\partial x_1^\alpha} \frac{\Delta x_1^\alpha}{\Gamma(\alpha+1)}. \tag{21}$$

We assume here that the medium is heterogeneous, but isotropic. Hence α is the same in all directions. Further, it should be stated that this model of heterogeneity is also non-stationary.

The net mass fluxes through the x_2 and x_3 faces are

$$F(x_2) - F(x_2 + \Delta x_2) = -\Delta x_1 \Delta x_3 \frac{\partial^\alpha \rho q_2}{\partial x_2^\alpha} \frac{\Delta x_2^\alpha}{\Gamma(\alpha+1)} \tag{22}$$

and

$$F(x_3) - F(x_3 + \Delta x_3) = -\Delta x_1 \Delta x_2 \frac{\partial^\alpha \rho q_3}{\partial x_3^\alpha} \frac{\Delta x_3^\alpha}{\Gamma(\alpha+1)}. \tag{23}$$

The net mass flux, ΔF through the control volume is obtained by summing (21)–(23)

$$\Delta F = -\frac{\Delta x_1^\alpha \Delta x_2 \Delta x_3}{\Gamma(\alpha+1)} \frac{\partial^\alpha \rho q_1}{\partial x_1^\alpha} - \frac{\Delta x_1 \Delta x_2^\alpha \Delta x_3}{\Gamma(\alpha+1)} \frac{\partial^\alpha \rho q_2}{\partial x_2^\alpha} - \frac{\Delta x_1 \Delta x_2 \Delta x_3^\alpha}{\Gamma(\alpha+1)} \times \frac{\partial^\alpha \rho q_3}{\partial x_3^\alpha}. \tag{24}$$

We then assign the right hand side of (24) to the accumulation term as in (8)

$$-\Delta x_1^\alpha \Delta x_2 \Delta x_3 \frac{\partial^\alpha \rho q_1}{\partial x_1^\alpha} - \Delta x_1 \Delta x_2^\alpha \Delta x_3 \frac{\partial^\alpha \rho q_2}{\partial x_2^\alpha} - \Delta x_1 \Delta x_2 \Delta x_3^\alpha \frac{\partial^\alpha \rho q_3}{\partial x_3^\alpha} = \Gamma(\alpha+1) \frac{\partial}{\partial t}(\Delta x_1 \Delta x_2 \Delta x_3 n \rho). \tag{25}$$

By assuming the control volume is a cube, $\Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta x$, (25) simplifies to

$$-\frac{\partial^\alpha \rho q_i}{\partial x_i^\alpha} = \frac{\Gamma(\alpha+1)}{\Delta x^{\alpha+2}} \frac{\partial}{\partial t}(\Delta x^3 n \rho). \tag{26}$$

Eq. (26) is the most general form of the fractional conservation of mass equation. Note that when $\alpha = 1$, $\Gamma(2) = 1$ and we recover the usual integer-order conservation of mass equation, (8).

Now let us turn our attention to the right hand side of (26), the accumulation term. We will follow the usual convention and assume that the control volume can vary in size (over time) in the vertical dimension, Δx_3, only, so that

$$\frac{\partial}{\partial t}(\Delta x^3 n \rho) = \Delta x^2 \frac{\partial}{\partial t}(\Delta x_3 n \rho). \tag{27}$$

Combining (26) and (27)

$$\frac{\partial^\alpha \rho q_i}{\partial x_i^\alpha} = -\frac{\Gamma(\alpha+1)}{\Delta x^\alpha} \frac{\partial}{\partial t}(\Delta x_3 n \rho). \tag{28}$$

Since Δx_3, n and ρ all depend on pressure (p), we can make a change of variables and use the chain rule to get

$$\frac{\partial}{\partial t}(\Delta x_3 n \rho) = n\rho \frac{\partial \Delta x_3}{\partial p} \frac{\partial p}{\partial t} + \Delta x_3 \rho \frac{\partial n}{\partial p} \frac{\partial p}{\partial t} + \Delta x_3 n \frac{\partial \rho}{\partial p} \frac{\partial p}{\partial t}. \tag{29}$$

Using the standard linear relationships [9]

$$\frac{\partial \Delta x_3}{\partial p} = \beta_s \Delta x_3, \tag{30}$$

where β_s is the coefficient of compressibility for the porous medium

$$\frac{\partial n}{\partial p} = \beta_s(1-n), \tag{31}$$

$$\frac{\partial \rho}{\partial p} = \rho \beta_w, \tag{32}$$

where β_w is the coefficient of compressibility of the water. Substituting (30)–(32) into (29)

$$\frac{\partial}{\partial t}(\Delta x_3 n \rho) = \Delta x_3 \rho (\beta_s + n\beta_w) \frac{\partial p}{\partial t}. \tag{33}$$

Substituting (33) into (28), we get

$$-\frac{\partial^\alpha \rho q_i}{\partial x_i^\alpha} = \Gamma(\alpha+1) \Delta x^{1-\alpha} \rho (\beta_s + n\beta_w) \frac{\partial p}{\partial t}. \tag{34}$$

Eq. (34) is the αth order fractional conservation of mass equation allowing for vertical compressibility of the control volume. Again, if $\alpha = 1$, we recover the usual integer order conservation of mass equation

$$-\frac{\partial \rho q_i}{\partial x_i} = \rho (\beta_s + n\beta_w) \frac{\partial p}{\partial t}. \tag{35}$$

There are three differences between (34) and (35), two of which are significant. The first difference is that (34) has $\Gamma(\alpha+1)$. This is just a constant which is easily computed knowing the value of α. Previous applications of fractional calculus to flow and transport in porous media [3,5,11] suggest that α reflects the degree of heterogeneity in the porous medium. When $\alpha \to 1$, the porous medium is essentially homogeneous and (34) → (35); i.e., the conventional mass conservation equation is recovered from the fractional mass conservation equation. As α gets smaller (than 1), the porous medium becomes increasingly heterogeneous.

The second difference is that the divergence term on the left-hand side of (34) is the fractional derivative of the flux, which in three dimensions is the fractional divergence [11]. The fractional divergence is defined using Einstein's notation

$$\mathrm{div}^{\alpha}(\mathbf{q}) = \nabla^{\alpha} \cdot \mathbf{q} = \frac{\partial^{\alpha} q_i}{\partial x_i^{\alpha}}.$$

Now Eq. (34) can be written in vector form

$$-\rho(\nabla^{\alpha} \cdot \mathbf{q}) = \Gamma(\alpha + 1)\Delta x^{1-\alpha}\rho(\beta_s + n\beta_w)\frac{\partial p}{\partial t}. \tag{36}$$

As α gets smaller than 1, the fractional divergence provides more global information about the changes in flux. This is because the fractional derivative, unlike the integer order derivative, is a non-local operator [7].

The third difference is that the right hand side of (34) contains the term $\Delta x^{1-\alpha}$. We believe that the retention of this term in the fractional mass conservation equation amounts to a built-in scale effect. This makes sense because when $\alpha < 1$, the medium heterogeneity is scale-dependant with long-range or infinite auto-correlation. The scale effect goes away as $\alpha \to 1$, $\Delta x^{1-\alpha} \to \Delta x^0 \to 1$, as it should for a homogeneous medium.

The fractional advection dispersion equation has proven useful in modeling contaminant flow in heterogeneous porous media [1–3]. The fractional advection dispersion equation is known to be a special case of a general transport equation with convolution flux [7] and a limit case of the continuous time random walk with power-law particle jumps [4,10]. It is a simple matter to derive the fractional advection dispersion equation from the fractional conservation of mass equation (34) using a moving coordinate system at the plume center of mass, in exactly the same way that the usual advection dispersion equation follows from the traditional conservation of mass equation (8), see [11]. This approach validates the utility of the fractional advection dispersion equation and related theories, by highlighting the scaling factor that renders the fractional equation scale invariant. We believe that this scaling captures the fractal nature of the porous medium [16].

5. Conclusions

A fractional conservation of mass equation captures power law variations of flux in fractal porous media. The fractional equation becomes scale invariant by incorporating a power law correction for the finite size control volume. The fundamental limitations of the traditional mass conservation equation are: (1) it is valid for flow fields in which changes in flux (within the control volume) are small and linear, or piece-wise linear; and (2) the control volume (or measurement scale) must be large compared to the scale of heterogeneity. Both of these limitations are required due to the fact that flux changes in the control volume are approximated by a first-order Taylor series. We have shown that the fractional Taylor series can be an exact representation of non-linear (power law) flux using only the first two terms. By using the fractional Taylor series to represent flux change through the control volume, we develop fractional conservation of mass equations (26) and (34) that remove the aforementioned limitations of the traditional mass conservation equation.

There are two important distinctions between the traditional mass conservation, (35), and its fractional equivalent, (34). The first is that the divergence term in the fractional mass conservation equation is the fractional divergence, and is equivalent to the definition of fractional divergence provided by Meerschaert et al. [11]. The traditional integer order divergence is only able to move mass into/out of adjacent control volumes [15] since it is based on traditional Brownian motion. The fractional divergence is able to move mass between adjacent and further away control volumes, depending on the nature of the non-linear flux, because the fractional space derivatives in the fractional divergence term contain global information about flux changes.

The second distinction is the appearance of the term $\Delta\, x^{1-\alpha}$ in the accumulation term of (34), which introduces a scale effect. In traditional mass conservation, scale effects are seen as scale-dependant changes in parameters (e.g. hydraulic conductivity, storage) that should be scale-independent. These parameters should be scale-invariant in the fractional conservation of mass equation since the scale effects will be handled by the fractional derivatives and the $\Delta x^{1-\alpha}$ term.

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