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SEMI-MARKOV APPROACH TO CONTINUOUS TIME RANDOM WALK LIMIT PROCESSES

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Continuous time random walks (CTRWs) are versatile models for anomalous diffusion processes that have found widespread application in the quantitative sciences. Their scaling limits are typically non-Markovian, and the computation of their finite-dimensional distributions is an important open problem. This paper develops a general semi-Markov theory for CTRW limit processes in $\mathbb{R}^d$ with infinitely many particle jumps (renewals) in finite time intervals. The particle jumps and waiting times can be coupled and vary with space and time. By augmenting the state space to include the scaling limits of renewal times, a CTRW limit process can be embedded in a Markov process. Explicit analytic expressions for the transition kernels of these Markov processes are then derived, which allow the computation of all finite dimensional distributions for CTRW limits. Two examples illustrate the proposed method.

1. Introduction. Continuous time random walks (CTRWs) assume a random waiting time between each successive jump. They are used in physics to model a variety of anomalous diffusion processes (see Metzler and Klafter [34]), and have found applications in numerous other fields (see, e.g., [6, 17, 37, 38]). The scaling limit of the CTRW is a time-changed Markov process in $\mathbb{R}^d$ [31]. The clock process is the hitting time of an increasing Lévy process, which is non-Markovian. The distribution of the scaling limit at one fixed time t is then usually calculated by solving a fractional Fokker–Planck equation [34], that is, a governing equation that involves a fractional derivative in time. The analysis of the joint laws at *multiple times*, however, becomes much more complicated, since the limit process is not Markovian. In fact, the joint distribution of the CTRW limit at two or more different times has yet to be explicitly calculated, even in the simplest cases; see Baule and Friedrich [4] for further discussion.

The main motivation of this paper is to resolve this problem, and our approach is to develop the semi-Markov theory for CTRW scaling limits. CTRWs are renewed after every jump. As it turns out, the discrete set of renewal times of CTRWs converges to a "regenerative set" in the scaling limit, which is not discrete and can be a

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random fractal or a random set of positive Lebesgue measure. This regenerative set allows for the definition of the scaling limit of the previous and next renewal time after a time t. By incorporating these times into the state space, a CTRW limit can become Markovian. Although CTRW scaling limits have appeared in many applications throughout the literature, to our knowledge the renewal property has only been studied for a discrete CTRW. Moreover, CTRW limits are examples for possibly *discontinuous* semi-Markov processes with infinitely many renewals in finite time, and hence the development here complements the literature on continuous semi-Markov processes [15].

It is known [25] that semi-Markov processes can be constructed by assuming a Markov additive process (A_u, D_u) and defining $X_t = A(E_t)$, where E_t is the hitting time of the level t by the process D_u. With this procedure, one also constructs CTRW limit processes. However, such CTRW limits are homogeneous in time, and several applications require time-*in*homogeneous CTRW limit processes [16, 27]. Hence, we will assume that (A_u, D_u) is a diffusion process with jumps (such that D_u is strictly increasing), modeling the cumulative sum of non-i.i.d. jumps and waiting times (see Section 2) which vary with time and space. In this setting, we develop a semi-Markov theory for time-*in*homogeneous CTRW limits.

Coupled CTRW limits, for which waiting times and jumps are not independent, turn out to be particularly interesting. As recently discovered [21, 40], switching the order of waiting time and jump (i.e., jumps precede waiting times) yields a different scaling limit called the overshooting CTRW limit (OCTRW limit). The two processes can have completely different tail behavior [21], and hence provide versatile models for a variety of relaxation behaviors in statistical physics [42]. Both CTRW and OCTRW limit turn out to be semi-Markov processes; however, incorporating the previous renewal time only renders the CTRW limit Markovian and not the OCTRW limit, and the opposite is true when the following renewal time is incorporated. In the uncoupled case, CTRW and OCTRW have the same limit, and hence both approaches yield Markov processes.

This paper gives explicit formulae for the joint transition probabilities of the CTRW limit (resp., OCTRW limit), together with its previous renewal time (resp., following renewal time, see Section 3). These formulae facilitate the calculation of all finite-dimensional distributions for CTRW (and OCTRW) limits. The time-homogeneous case is discussed in Section 4. Finally, Section 5 provides some explicit examples, for problems of current interest in the physics literature.

2. Random walks in space–time.

A continuous time random walk (CTRW) is a random walk in space–time, with positive jumps in time. Let $c > 0$ be a scaling parameter, and let

$$(S_n^c, T_n^c) = (A_0^c, D_0^c) + \sum_{k=1}^{n} (J_k^c, W_k^c)$$

denote a Markov chain on $\mathbb{R}^d \times [0,\infty)$ that tracks the position S_n^c of a randomly selected particle after n jumps, and the time T_n^c the particle arrives at this position. The particle starts at position A_0^c at time D_0^c, $N_t^c = \max\{k \ge 0 : T_k^c \le t\}$ counts the number of jumps by time t, and the CTRW

$$X_t^c = S_{N_t^c}^c$$

is the particle location at time t. The waiting times W_k^c are assumed positive, and when $t < T_0^c$ we define $N_t^c = 0$. The process N_t^c is inverse to T_n^c, in the sense that $N_{T_n^c}^c = n$. Often the sequence (J_k^c, W_k^c) is assumed to be independent and identically distributed, which is the appropriate statistical physics model for particle motions in a heterogeneous medium whose properties are invariant over space and time. The dependence on the time scale $c > 0$ facilitates triangular array convergence schemes, which lead to a variety of interesting limit processes [2, 3, 22, 32]. The CTRW is called *uncoupled* if the waiting time W_k^c is independent of the jump J_k^c; see, for example, [5]. Coupled CTRW models have been applied in physics [34, 39] and finance [29, 36]. If the waiting times are i.i.d. and the jump distribution depends on the current position in space and time, the CTRW limit is a time-changed Markov process governed by a fractional Fokker–Planck equation [16]. A closely related model called the overshooting CTRW (OCTRW) is

$$Y_t^c = S_{N_t^c+1}^c,$$

a particle model for which J_1^c is the random initial location, and each jump J_k^c is *followed* by the waiting time W_k^c. See [23] for applications of OCTRW in finance, where Y_t represents the price at the next available trading time. See [42] for an application of OCTRW to relaxation problems in physics.

In statistical physics applications, it is useful to consider the *diffusion limit* of the (O)CTRW as the time scale $c \to \infty$. To make this mathematically rigorous, let $\mathbb{D}([0,\infty), \mathbb{R}^{d+1})$ denote the space of càdlàg functions $f : [0,\infty) \to \mathbb{R}^{d+1}$ with the Skorokhod J_1 topology, and suppose

$$\left(S_{[cu]}^c, T_{[cu]}^c\right) = \left(A_0^c, D_0^c\right) + \sum_{k=1}^{[cu]} \left(J_k^c, W_k^c\right) \Rightarrow (A_u, D_u), \tag{2.1}$$

where "$\Rightarrow$" denotes the weak convergence of probability measures on $\mathbb{D}([0,\infty), \mathbb{R}^{d+1})$ as $c \to \infty$. Suppose the limit process (A_u, D_u) is a *canonical Feller process* with state space $\mathbb{R}^{d+1}$, in the sense of [35], III Section 2. That is, we assume a stochastic basis $(\Omega, \mathcal{F}_\infty, \mathcal{F}_u, \mathbb{P}^{\chi,\tau})$ in which Ω is the set of right-continuous paths in $\mathbb{R}^{d+1}$ with left-limits and $(A_u(\omega), D_u(\omega)) = \omega(u)$ for all $\omega \in \Omega$. The filtration $\mathcal{F} = \{\mathcal{F}_u\}_{u \ge 0}$ is right continuous and (A_u, D_u) is $\mathcal{F}$-adapted. The laws $\{\mathbb{P}^{\chi,\tau}\}_{(\chi,\tau)\in\mathbb{R}^{d+1}}$ are determined by a Feller semigroup of transition operators $(T_u)_{u\ge 0}$ and are such that $(A_0, D_0) = (\chi, \tau)$, $\mathbb{P}^{\chi,\tau}$-a.s. The σ-fields $\mathcal{F}_\infty$ and $\mathcal{F}_0$ are augmented by the $\mathbb{P}^{\chi,\tau}$-null sets. Expectation with respect to $\mathbb{P}^{\chi,\tau}$ is denoted by $\mathbb{E}^{\chi,\tau}$. The map $(\chi, \tau) \mapsto \mathbb{E}^{\chi,\tau}[Z]$ is Borel-measurable for every $\mathcal{F}_\infty$-measurable random variable Z. If the space–time jumps form an infinitesimal triangular array ([30], Definition 3.2.1), then $(A_u, D_u) - (A_0, D_0)$ is a Lévy process [32]. In the uncoupled case, $A_u - A_0$ and $D_u - D_0$ are independent Lévy processes [31]. If the space–time jump distribution depends on the current position, it was argued in [16, 41] that the limiting process (A_u, D_u) is a jump-diffusion in $\mathbb{R}^{d+1}$.

If (2.1) holds, and if

(2.2) the sample paths $u \mapsto D_u$ are $\mathbb{P}^{\chi,\tau}$-a.s. strictly increasing and unbounded,

then [40], Theorem 3.6, implies that

$$X_t^c \Rightarrow X_t := (A_{E_t-})^+ \quad \text{and} \quad Y_t^c \Rightarrow Y_t := A_{E_t} \quad \text{in } \mathbb{D}([0,\infty), \mathbb{R}^d) \text{ as } c \to \infty, \tag{2.3}$$

where

$$E_t = \inf\{u > 0 : D_u > t\} \tag{2.4}$$

is the first passage time of D_u, so that $E_{D_u} = u$. Then the inverse process (2.4) is defined on all of $\mathbb{R}$ and has a.s. continuous sample paths. The CTRW limit (CTRWL) process X_t in (2.3) is obtained by evaluating the left-hand limit of the outer process A_{u-} at the point $u = E_t$, and then modifying this process to be right-continuous. This changes the value of the process at time points $t > 0$ such that $u = E_t$ is a jump point of the outer process A_u, and $E_{t+\varepsilon} > E_t$ for all $\varepsilon > 0$. If A_u and D_u have no simultaneous jumps, then the CTRW limit X_t equals the OCTRW limit Y_t ([40], Lemma 3.9). However, these two processes can be quite different in the coupled case. For example, if $J_k^c = W_k^c$ form a triangular array in the domain of attraction of a stable subordinator D_u, and if $A_0 = D_0 = 0$, then $A_u = D_u$, and $X_t = D_{E_t-} < t < D_{E_t} = Y_t$ almost surely [7], Theorem III.4. See Example 5.4 for more details.

We assume the Feller semigroup T_u that governs the process (A_u, D_u) acts on the space $C_0(\mathbb{R}^{d+1})$ of continuous real-valued functions on $\mathbb{R}^{d+1}$ that vanish at ∞, and that it admits an infinitesimal generator $\mathcal{A}$ of jump-diffusion form [1], equation (6.42). In light of (2.2), this generator takes the form

$$\begin{aligned}\mathcal{A}f(x,t) &= \sum_{i=1}^d b_i(x,t)\partial_{x_i} f(x,t) + \gamma(x,t)\partial_t f(x,t) \\ &\quad + \frac{1}{2}\sum_{1\le i,j\le d} a_{ij}(x,t)\partial^2_{x_i x_j} f(x,t) \\ &\quad + \int\Bigg[f(x+y, t+w) - f(x,t) \\ &\qquad - \sum_{i=1}^d h_i(y,w)\partial_{x_i} f(x,t)\Bigg] K(x,t;dy,dw),\end{aligned} \tag{2.5}$$

where $(x,t) \in \mathbb{R}^{d+1}$, b_i and γ are real-valued functions, and $A = (a_{ij})$ is a function taking values in the nonnegative definite $d \times d$-matrices. Here, $K(x,t;dy,dw)$ is a jump-kernel from $\mathbb{R}^{d+1}$ to itself, so that for every $(x,t) \in \mathbb{R}^{d+1}$, $C \mapsto K(x,t;C)$ is a measure on $\mathbb{R}^{d+1}$ that is finite on sets bounded away from the origin, and $(x,t) \mapsto K(x,t;C)$ is a measurable function for every Borel set $C \subset \mathbb{R}^{d+1}$. The truncation function $h_i(x,t) = x_i \mathbf{1}\{(x,t) \in [-1,1]^{d+1}\}$. Since the sample paths of D_u are strictly increasing, $\gamma \geq 0$, the diffusive component of D_u is zero, and the measures $K(x,t;dy,dw)$ are supported on $(dy,dw) \in \mathbb{R}^d \times [0,\infty)$. Instead of assuming that $K(x,t;dy,dw)$ integrates $1 \wedge \|(y,w)\|^2$, it then suffices to assume

$$\int [1 \wedge (\|y\|^2 + |w|)] K(x,t;dy,dw) < \infty \qquad \forall (x,t) \in \mathbb{R}^{d+1}. \tag{2.6}$$

The space–time jump kernel K can be interpreted as the joint intensity measure for the long jumps and long waiting times which do not rescale to 0 as $c \to \infty$. If the measures $(dy,dw) \mapsto K(x,t;dy,dw)$ are supported on "the coordinate axes" $(\mathbb{R}^d \times \{0\}) \cup (\{0\} \times [0,\infty))$, then large jumps occur independently of long waiting times, and the CTRWL and OCTRWL are identical ([40], Lemma 3.9). We refer to this as the *uncoupled* case, and to the opposite case as the *coupled* case.

Finally, we assume that the coefficients b_i, γ, a_{ij} and K satisfy Lipschitz and growth conditions as in [1], Section 6.2, so that (A_u, D_u) has an interpretation as the solution to a stochastic differential equation, as well as a semimartingale [19], Section III.2. Then for any canonical Feller process (A_u, D_u) on $\mathbb{R}^{d+1}$, we define the CTRWL process $X_t = (A_{E_t-})^+$, and the OCTRWL process $Y_t = A_{E_t}$, where E_t is given by (2.4). If we set $A_{0-} = A_0$, then E_t, X_t and Y_t are defined for all $t \in \mathbb{R}$.

2.1. *Forward and backward renewal times*. Although the (O)CTRWL is not Markovian, it turns out that it can be embedded in a Markov process on a higher dimensional state space, by incorporating information on the forward/backward renewal times. Define the *regenerative set*

$$\mathbf{M} = \{(t,\omega) \subset \mathbb{R} \times \Omega : t = D_u(\omega) \text{ for some } u \geq 0\},$$

the random set of image points of D_u. These will turn out to be the renewal points of the inverse process E_t defined in (2.4). Since D_u is càdlàg and has a.s. increasing sample paths, for almost all ω the complement of the ω-slice $\mathbf{M}(\omega) := \{t \in \mathbb{R} : (t,\omega) \in \mathbf{M}\}$ in $\mathbb{R}$ is a countable union of intervals of the form $[D_{u-}(\omega), D_u(\omega))$, where $u \geq 0$ ranges over the jump epochs of the process D_u. For example, if D_u is compound Poisson with positive drift, then $\mathbf{M}$ is a.s. a union of intervals $[a,b)$ of positive length. If D_u is a β-stable subordinator with no drift, then $\mathbf{M}$ is a.s. a fractal of dimension β [8].

For any $t \geq 0$, we write G_t, the last time of regeneration before t, and H_t, the next time of regeneration after t, as

$$G_t(\omega) := \sup\{s \leq t : s \in \mathbf{M}(\omega)\} \leq t \leq \inf\{s > t : t \in \mathbf{M}(\omega)\} =: H_t(\omega), \tag{2.7}$$

where for convenience we set $G_t(\omega) = \inf \mathbf{M}(\omega) = \tau$, $\mathbb{P}^{\chi,\tau}$-a.s. whenever the supremum is taken over the empty set. In terms of the CTRW model, the particle has been resting at its current location since time G_t, and will become mobile again at time H_t. It will become clear in the sequel that the future evolution of X_t and Y_t on the time interval $[H_t, \infty)$ depends only the position Y_t at time $t = H_t$, meaning that H_t is a *Markov time* for X_t and Y_t.

Note that G_t and H_t are a.s. defined for all $t \in \mathbb{R}$ and their sample paths are càdlàg. By our assumptions on D_u and the definition (2.4), it is easy to see that

$$G_{t-} = D_{E_t-} \quad \text{and} \quad H_t = D_{E_t}, \qquad \mathbb{P}^{\chi,\tau}\text{-a.s.}$$

The *age process* V_t and the *remaining lifetime* R_t from renewal theory can be defined by

$$V_t := t - G_t \qquad \text{and} \qquad R_t := H_t - t \qquad \text{for all } t \in \mathbb{R}. \tag{2.8}$$

At any time $t > 0$, the particle has been resting at its current location for an interval of time of length V_t, and will move again after an additional time interval of length R_t. We will show below that the processes (X_{t-}, V_{t-}) and (Y_t, R_t) are Markov, and we will compute the joint distribution of these $\mathbb{R}^{d+1}$-valued processes at multiple time points, using the Chapman–Kolmogorov equations. The joint laws of $(X_{t-}, Y_t, V_{t-}, R_t)$ were first calculated in [13, 28], but only in the case where the space–time process (A_u, D_u) is Markov additive (see Section 4) and only for Lebesgue-almost all $t \geq 0$. We now calculate this joint law in our more general time-inhomogeneous setting, for *all* $t \geq 0$. We need the following additional definitions: Let

$$\mathbf{C} = \{(t, \omega) \subset \mathbb{R} \times \Omega : D_{u-}(\omega) = t = D_u(\omega) \text{ for some } u > 0\} \subset \mathbf{M}$$

be the random set of points traversed *continuously* by D_u. The set $\mathbf{C}$ is obtained by removing from the set $\mathbf{M}$ of regenerative points all points t which satisfy $t = D_u > D_{u-}$ for some $u > 0$ (i.e., the right end points of all contiguous intervals). Moreover, since (A_u, D_u) visits each point in $\mathbb{R}^{d+1}$ at most once, it admits a 0-potential, or mean occupation measure, $U^{\chi,\tau}$ defined via

$$\begin{aligned}\int f(x,t) U^{\chi,\tau}(dx, dt) &= \mathbb{E}^{\chi,\tau}\left[\int_0^\infty f(A_u, D_u)\, du\right] = \int_0^\infty T_u f(\chi, \tau)\, du \\ &= \mathbb{E}^{\chi,\tau}\left[\int_0^\infty f(A_{u-}, D_{u-})\, du\right]\end{aligned}$$

for any nonnegative measurable function $f : \mathbb{R}^{d+1} \to [0, \infty)$. The last equality holds because (A_u, D_u) only jumps countably many times. Since (A_u, D_u) has infinite lifetime, $U^{\chi,\tau}$ is an infinite measure. We assume that D_u is transient [11], so that $U^{\chi,\tau}(\mathbb{R}^d \times I) < \infty$ for any compact interval $I \subset [0, \infty)$. For instance, any subordinator is transient [8].

Next we derive the joint law of the Markov process $(X_{t-}, Y_t, V_{t-}, R_t)$. The proof uses sample path arguments, and we consider two cases, starting with the case $\{t \notin \mathbf{C}\}$:

PROPOSITION 2.1. *Fix* $(\chi, \tau) \in \mathbb{R}^{d+1}$ *and* $t \ge \tau$. *Then*

$$\begin{aligned}
&\mathbb{E}^{\chi,\tau}\big[f(X_{t-}, Y_t, V_{t-}, R_t)\mathbf{1}\{t \notin \mathbf{C}\}\big] \\
&\quad = \int_{x\in\mathbb{R}^d}\int_{s\in[\tau,t]} U^{\chi,\tau}(dx, ds) \\
&\qquad \times \int_{y\in\mathbb{R}^d}\int_{w\in[t-s,\infty)} K(x, s; dy, dw) f\big(x, x+y, t-s, w-(t-s)\big)
\end{aligned}\tag{2.9}$$

for all nonnegative measurable f *defined on* $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$.

PROOF. The complement of the section set $\mathbf{C}(\omega)$ in $\mathbb{R}$ is a.s. a countable union of closed intervals $[D_{u-}, D_u]$, where u is a jump epoch of D_u. Hence, for $t \notin \mathbf{C}$ we have $G_{t-} \le t \le H_t$ and $G_{t-} < H_t$, hence $\Delta D_{E_t} = H_t - G_{t-} > 0$. In the complementary case $\{t \in \mathbf{C}\}$, the sample path of E_t is left-increasing at t, and hence the $\mathcal{F}$-optional time E_t is announced by the optional times $E_{t-1/n}$. Hence, $E_t^* := E_t \cdot \mathbf{1}\{t \in \mathbf{C}\} + \infty \cdot \mathbf{1}\{t \notin \mathbf{C}\}$ is $\mathcal{F}$-predictable ([24], page 410), and since in our setting (A_u, D_u) is a canonical Feller process, it is quasi left-continuous ([24], Proposition 22.20), and $\Delta(A, D)_{E_t} = (0, 0)$ a.s. Writing $\mathcal{J} = \{(u, \omega) \in \mathbb{R}^+ \times \Omega : \Delta(A, D)_u \ne (0, 0)\}$ for the random set of jump epochs of (A_u, D_u), we hence find that

$$\begin{aligned}
&f(X_{t-}, Y_t, V_{t-}, R_t)\mathbf{1}\{t \notin \mathbf{C}\} \\
&\quad = \sum_{u\in\mathcal{J}} f(A_{u-}, A_u, t - D_{u-}, D_u - t)\mathbf{1}\{D_{u-} \le t \le D_u\} \\
&\quad = \sum_{u\in\mathcal{J}} f(A_{u-}, A_{u-} + \Delta A_u, t - D_{u-}, D_{u-} + \Delta D_u - t) \\
&\qquad\qquad \times \mathbf{1}\{D_{u-} \le t \le D_{u-} + \Delta D_u\},
\end{aligned}$$

noting that all members of the sum except exactly one ($u = E_t$) equal 0. The last expression equals $\int W(\omega, u; x, s)\mu(\omega, du; dy, dw)$ for the optional random measure

$$\mu(\omega, du; dy, dw) = \sum_{v\ge 0} \mathbf{1}_{\mathcal{J}}(v, \omega)\delta_{(v,\Delta(A_v(\omega), D_v(\omega)))}(du; dy, dw) \tag{2.10}$$

on $du \times (dy, dw) \in \mathbb{R}^+ \times \mathbb{R}^{d+1}$ associated with the jumps of (A_u, D_u), and the predictable integrand

$$\begin{aligned}
W(\omega, u; y, w) :=\ & f\big(A_{u-}(\omega), A_{u-}(\omega) + y, t - D_{u-}(\omega), D_{u-}(\omega) + w - t\big) \\
& \times \mathbf{1}\big\{D_{u-}(\omega) \le t \le D_{u-}(\omega) + w\big\}.
\end{aligned}$$

The compensator μ^p of μ equals [19], page 155

$$\mu^p(\omega; du, dy, dw) = K\big(A_{u-}(\omega), D_{u-}(\omega); dy, dw\big)\, du. \tag{2.11}$$

Then the compensation formula [19], II.1.8, implies that

$$\begin{aligned}
&\mathbb{E}^{\chi,\tau}\big[f(X_{t-}, Y_t, V_{t-}, R_t)\mathbf{1}\{t \notin \mathbf{C}\}\big] \\
&\quad = \mathbb{E}^{\chi,\tau}\left[\int W(\omega, u; y, w)\mu^p(\omega, du; dy, dw)\right] \\
&\quad = \mathbb{E}^{\chi,\tau}\bigg[\int_{u=0}^{\infty}\int_{y\in\mathbb{R}^d}\int_{w=0}^{\infty} f\big(A_{u-}(\omega), A_{u-}(\omega)+y, t-D_{u-}(\omega), \\
&\qquad\qquad\qquad D_{u-}(\omega)+w-t\big) \\
&\qquad\qquad \times \mathbf{1}\big\{D_{u-}(\omega) \le t \le D_{u-}(\omega)+w\big\} \\
&\qquad\qquad \times K\big(A_{u-}(\omega), D_{u-}(\omega); dy, dw\big)\,du\bigg] \\
&\quad = \int_{x\in\mathbb{R}^d}\int_{s=\tau}^{\infty}\int_{y\in\mathbb{R}^d}\int_{w=0}^{\infty} f(x, x+y, t-s, s+w-t) \\
&\qquad\qquad \times \mathbf{1}\{s \le t \le s+w\}K(x, s; dy, dw)U^{\chi,\tau}(dx, ds),
\end{aligned}$$

which is equivalent to (2.9). □

The following proposition handles the case $\{t \in \mathbf{C}\}$.

PROPOSITION 2.2. *Fix $(\chi, \tau) \in \mathbb{R}^{d+1}$ and $t \ge \tau$. Suppose that the temporal drift γ is bounded and continuous, and assume that the mean occupation measure $U^{\chi,\tau}(dx, dt)$ is Lebesgue-absolutely continuous with a continuous density $u^{\chi,\tau}(x, t)$. Then*

$$\mathbb{E}^{\chi,\tau}\big[f(Y_t)\mathbf{1}\{t \in \mathbf{C}\}\big] = \int_{x\in\mathbb{R}^d} f(x)\gamma(x, t)u^{\chi,\tau}(x, t)\,dx \tag{2.12}$$

for all bounded measurable f. Also (2.12) remains true if Y_t is replaced by X_{t-}, Y_{t-} or X_t.

PROOF. Similarly to the proof in [28], D_u admits a decomposition into a continuous and a discontinuous part via

$$D_u^c = \int_0^u \gamma(A_s, D_s)\,ds, \qquad D_u^d = \sum_{0\le s\le u} \Delta D_s, \qquad t \ge 0.$$

To see this, we first note that (A_u, D_u) is a semimartingale, and hence D_u allows the decomposition

$$D_u = \sum_{s\le u} \Delta D_s \mathbf{1}\{\Delta D_s > 1\} + B_u + M_u, \tag{2.13}$$

where B_u is a predictable process of finite variation (the first characteristic of D_u) and M_u is a local martingale. Due to [19], IX Section 4a, and (2.5), $B_u =$

$\int_0^u \tilde{\gamma}(A_s, D_s)\,ds$ where $\tilde{\gamma}(x,t) = \gamma(x,t) + \int s\mathbf{1}\{\|(y,s)\| \le 1\}K(x,t;dy,ds)$. Since D_u has no diffusive part, M_u is purely discontinuous and equal to $M_u = \sum_{s\le u} \Delta D_s \mathbf{1}\{\Delta D_s \le 1\} - \int_0^u \int w\mathbf{1}\{\|(y,w)\| \le 1\}K(A_s, D_s; dy, dw)\,ds$. But then (2.13) reads $D_u = D_u^d + D_u^c$.

For fixed ω, the paths of D, D^c and D^d are nondecreasing and define Lebesgue–Stieltjes measures dD, dD^c and dD^d on $[0,\infty)$. Then for any bounded measurable f and g, we have

$$\int_0^\infty f(A_u)g(D_u)\gamma(A_u, D_u)\,du = \int_0^\infty f(A_u)g(D_u)\,dD_u^c. \tag{2.14}$$

The continuous measure dD^c does not charge the countable set $\{u : \Delta D_u \neq 0\}$ of discontinuities of D_u and coincides with dD on the complement $\{u : \Delta D_u = 0\}$. Hence the right-hand side of (2.14) can be written as

$$\int_0^\infty f(A_u)g(D_u)\mathbf{1}\{u : \Delta D_u = 0\}\,dD_u. \tag{2.15}$$

The following substitution formula holds for all right-continuous, unbounded and strictly increasing $F : [0,\infty) \to [0,\infty)$, the inverse $F^{-1}(t) = \inf\{u : F(u) > t\}$ and measurable $h : [0,\infty) \to [0,\infty)$:

$$\int_0^\infty h(u)\,dF_u = \int_0^\infty h\big(F^{-1}(t)\big)\,dt.$$

To see this, first show the statement for h an indicator function of an interval $(a,b] \subset [0,\infty)$ and then for a function taking finitely many values. The statement for positive h then follows by approximation via a sequence of finitely valued functions from below, and for general h by a decomposition into positive and negative part. Applying the substitution formula to (2.15) with $F(u) = D_u$, the right-hand side of (2.14) reduces to

$$\int_0^\infty f(Y_t)g(H_t)\mathbf{1}\{t : \Delta D_{E_t} = 0\}\,dt.$$

Now note that $\Delta D_{E_t} = 0$ is equivalent to $t \in \mathbf{C}$ and implies $H_t = t$. Hence, the above lines show that the left-hand side of (2.14) equals

$$\int_0^\infty f(Y_t)g(t)\mathbf{1}\{t \in \mathbf{C}\}\,dt.$$

Take expectations and apply Tonelli's theorem to get

$$\int_{\mathbb{R}^{d+1}} f(x)\gamma(x,t)g(t)u^{\chi,\tau}(x,t)\,dx\,dt = \int_0^\infty \mathbb{E}^{\chi,\tau}\big[f(Y_t)\mathbf{1}\{t \in \mathbf{C}\}\big]g(t)\,dt.$$

Since g is an arbitrary nonnegative bounded measurable function, this yields (2.12) for almost every t. By our assumption that D_u is transient, $U^{\chi,\tau}(\mathbb{R}^d \times I) < \infty$ for compact $I \subset [0,\infty)$, and then it can be seen that the continuous function $u^{\chi,\tau}(x,t)$ must be bounded on $\mathbb{R}^d \times I$. Let I contain t and apply dominated convergence to see that the right-hand side of (2.12) is continuous in t. We have already noted in the proof of Proposition 2.1 that $\Delta(A, D)_{E_t} = (0, 0)$ on $\{t \in \mathbf{C}\}$, which shows the continuity of the left-hand side. This shows the equality for *all* $t \geq 0$, and also that $X_t - X_{t-} = 0 = Y_t - Y_{t-}$ on $\{t \in \mathbf{C}\}$. $\square$

We can now characterize the joint law of $(X_{t-}, Y_t, V_{t-}, R_t)$:

THEOREM 2.3. *Fix $(\chi, \tau) \in \mathbb{R}^{d+1}$ and $t \geq \tau$. If γ does not vanish, then suppose that the mean occupation measure $U^{\chi,\tau}(dx, dt)$ has a continuous Lebesgue density $u^{\chi,\tau}(x, t)$, and if $\gamma \equiv 0$, let $u^{\chi,\tau}(x, t) \equiv 0$. Then*

$$\begin{aligned}
&\mathbb{E}^{\chi,\tau}\big[f(X_{t-}, Y_t, V_{t-}, R_t)\big] \\
&\quad = \int_{x\in\mathbb{R}^d} f(x, x, 0, 0)\gamma(x, t)u^{\chi,\tau}(x, t)\,dx \\
&\qquad + \int_{x\in\mathbb{R}^d}\int_{s\in[\tau,t]} U^{\chi,\tau}(dx, ds) \\
&\qquad \times \int_{y\in\mathbb{R}^d}\int_{w\in[t-s,\infty)} K(x, s; dy, dw) f\big(x, x+y, t-s, w-(t-s)\big)
\end{aligned}$$

for all bounded measurable f. Moreover, $X_\tau = Y_\tau = \chi$ and $V_\tau = R_\tau = 0$, $\mathbb{P}^{\chi,\tau}$-almost surely.

PROOF. On the set $\{t \in \mathbf{C}\}$, $V_{t-} = 0 = R_t$. The above formula then follows from Propositions 2.2 and 2.1. Assumption (2.2) and the right-continuity of D yields $V_\tau = R_\tau = 0$. The sample paths of E are then seen to be right-increasing at τ and $E_t > E_\tau = 0$ for $t > \tau$. The right-continuity of A together with $A_0 = \chi$, $\mathbb{P}^{\chi,\tau}$-a.s. yields $X_\tau = Y_\tau = \chi$. $\square$

3. The Markov embedding. In this section, we establish the Markov property of the processes (Y_t, R_t) and (X_{t-}, V_{t-}). Since $\{E_t \leq u\} = \{D_u \geq t\}$, $\mathbb{P}^{\chi,\tau}$-a.s. for every $(\chi, \tau) \in \mathbb{R}^{d+1}$ [31], equation (3.2), we see that E_t is an $\mathcal{F}$-optional time for every t. We introduce the filtration $\mathcal{H} = \{\mathcal{H}_t\}_{t\in\mathbb{R}}$ where $\mathcal{H}_t = \mathcal{F}_{E_t}$ and note that (Y_t, R_t) is adapted to $\mathcal{H}$. Moreover, if T is $\mathcal{H}$-optional, then $E_T : \omega \mapsto E_{T(\omega)}(\omega)$ is $\mathcal{F}$-optional (see Lemma A.1). We define the family of operators $\{Q_{s,t}\}_{s\leq t}$ acting on the space $B_b(\mathbb{R}^d \times [0, \infty))$ of real-valued bounded measurable functions f defined on $\mathbb{R}^d \times [0, \infty)$ as follows:

$$\begin{aligned}
&Q_{s,t} f(y, 0) = \mathbb{E}^{y,s}\big[f(Y_t, R_t)\big], \\
&Q_{s,t} f(y, r) = \mathbf{1}\{r > t-s\} f\big(y, r-(t-s)\big) \\
&\qquad\qquad\qquad + \mathbf{1}\{0 \leq r \leq t-s\} Q_{s+r,t} f(y, 0).
\end{aligned} \tag{3.1}$$

The dynamics of $Q_{s,t}$ can be interpreted as follows: If the process (Y_t, R_t) starts at (y, r), the position in space y does not change while the remaining lifetime R_t decreases linearly to 0. When $r = 0$, the process continues with the dynamics given by (Y_t, R_t) started at location y at time $s + r$. Note that $Q_{s,t} f(y, r)$ is measurable in (s, t, y, r), for every bounded measurable f, by the construction of the probability measures $\mathbb{P}^{\chi,\tau}$. We can now state the strong Markov property of (Y_t, R_t) with respect to $\mathcal{H}$ and $Q_{s,t}$.

THEOREM 3.1. *Suppose that the operators $Q_{s,t}$ are given by* (3.1). *Then*:

(i) *The operators $Q_{s,t}$ satisfy the Chapman–Kolmogorov equations*:

$$Q_{q,s} Q_{s,t} f = Q_{q,t} f, \qquad q \le s \le t,$$

and moreover, $Q_{s,t}\mathbf{1} = \mathbf{1}$.

(ii) *Let* $(\chi, \tau) \in \mathbb{R}^{d+1}$, $t \ge 0$ *and let T be a $\mathcal{H}$-optional time. Then*

$$\mathbb{E}^{\chi,\tau}\big[f(Y_{T+t}, R_{T+t})|\mathcal{H}_T\big] = Q_{T,T+t} f(Y_T, R_T), \qquad \mathbb{P}^{\chi,\tau}\text{-almost surely}$$

for every real-valued bounded measurable f.

(iii) *The process $t \mapsto (Y_t, R_t)$ is quasi-left-continuous with respect to $\mathcal{H}$.*

Hence, (Y_t, R_t) *is a Hunt process with respect to $\mathcal{H}$ and transition operators $Q_{s,t}$.*

PROOF. A proof is given in the Appendix. □

We define the filtration $\mathcal{G} = \{\mathcal{G}_t\}_{t\in\mathbb{R}}$ via $\mathcal{G}_t = \mathcal{F}_{E_t-}$, the σ-field of all $\mathcal{F}$-events strictly before E_t. Evidently, the left-continuous process (X_{t-}, V_{t-}) is adapted to $\mathcal{G}$. The main idea behind the Markov property of (X_{t-}, V_{t-}) is that, knowing the current state $(x, v) = (X_{t-}, V_{t-})$ and the joint distribution of the next space–time increment given by the kernel $K(x, v; dy, dw)$ in (2.5), one can calculate the distribution of the next renewal time H_t and the position Y_t at that time. Then the probability of events after the renewal point H_t can be calculated starting at the point (Y_t, H_t) in space–time. We introduce the following notation: Define the family of probability kernels $\{K_v\}_{v\ge 0}$ on $\mathbb{R}^{d+1}$

$$\begin{aligned} &K_v(x, t; C) = \frac{K(x, t; C \cap (\mathbb{R}^d \times [v, \infty)))}{K(x, t; \mathbb{R}^d \times [v, \infty))}, \\ &\qquad\qquad v > 0, (x, t) \in \mathbb{R}^{d+1}, C \subset \mathbb{R}^{d+1}, \\ &K_0(x, t; C) = \delta_{(0,0)}(C), \end{aligned} \tag{3.2}$$

where C is a Borel set. For $v > 0$, $K_v(x, t; dy, dw)$ is the conditional probability distribution of a space–time jump (y, w) (a jump-waiting time pair), given that a time-jump (a waiting time) greater than or equal to v occurs. Should the denominator $K(x, t; \mathbb{R}^d \times [v, \infty))$ equal 0, we set $K_v(x, t; C) = 0$. If $v = 0$, then K_0 is the Dirac-measure concentrated at $(0, 0) \in \mathbb{R}^{d+1}$. Since $v \mapsto K(x, t; C \cap R^d \times [v, \infty))$ is decreasing, and hence measurable, it follows that $v \mapsto K_v(x, t; C)$ is measurable for every $(x, t) \in \mathbb{R}^{d+1}$ and Borel $C \subset \mathbb{R}^{d+1}$.

We now define the family of operators $\{P_{s,t}\}_{s\le t}$ acting on the space $B_b(\mathbb{R}^d \times [0,\infty))$ of real-valued bounded measurable functions defined on $\mathbb{R}^d \times [0,\infty)$:

$$
\begin{aligned}
&P_{s,t} f(x,0) = \mathbb{E}^{x,s}\big[f(X_{t-}, V_{t-})\big],\\
&P_{s,t} f(x,v)\\
&\quad = f(x, v+t-s) K_v\big(x, s-v; \mathbb{R}^d \times [v+t-s,\infty)\big)\\
&\qquad + \int_{y\in\mathbb{R}^d} \int_{w\in[v,v+t-s)} P_{s+w-v,t} f(x+y,0) K_v(x, s-v; dy, dw).
\end{aligned}
\tag{3.3}
$$

The dynamics given by $P_{s,t}$ can be interpreted as follows. With probability $K_v(x, s-v; \mathbb{R}^d \times (v+t-s,\infty))$, the process remains at x and the age increases by $t-s$. This is the probability that the size of a jump of D whose base point is at $(x, s-v)$ exceeds $v+t-s$, given that it exceeds v. The remaining probability mass for the jump of (A, D) is spread on the set $(y,w) \in \mathbb{R}^d \times [v, v+t-s)$, and the starting point is updated from x to $x+y$ at the time $s-v+w$.

THEOREM 3.2. *Let $P_{s,t}$ be the operators defined by* (3.3). *Then*:

(i) *The operators $(P_{s,t})$ satisfy the Chapman–Kolmogorov property*:

$$P_{q,s} P_{s,t} f = P_{q,t} f, \qquad q \le s \le t,$$

and moreover, $P_{s,t}\mathbf{1} = \mathbf{1}$.

(ii) *The process (X_{t-}, V_{t-}) satisfies the simple Markov property with respect to $\mathcal{G}$ and $P_{s,t}$*:

$$\mathbb{E}^{\chi,\tau}\big[f(X_{t-}, V_{t-}) | \mathcal{G}_s\big] = P_{s,t} f(X_{s-}, V_{s-}), \qquad \mathbb{P}^{\chi,\tau}\text{-a.s.}$$

for all $(\chi,\tau) \in \mathbb{R}^{d+1}$, $\tau \le s \le t$ and real-valued bounded measurable f.

PROOF. A proof is given in the Appendix. □

REMARK 3.3. It would be interesting to investigate whether the *moderate* Markov property (e.g., see Chung and Glover [10]) holds for (X_{t-}, V_{t-}). An application of the compensation formula to the process (X_{t-}, G_{t-}) might yield a proof, but this would require the semimartingale characteristics of (X_{t-}, G_{t-}), which we have not been able to calculate.

4. The time-homogeneous case. If the coefficients $b(x,t), \gamma(x,t), a(x,t)$ and $K(x,t; dy, dw)$ of the generator $\mathcal{A}$ in (2.5) do not depend on $t \in \mathbb{R}$, then we say that (A_u, D_u) is a *Markov additive process*. This means that the future of (A_u, D_u) only depends on the current state of A_u; see, for example, [12].

THEOREM 4.1. *If the space–time random walk limit process* (A_u, D_u) *in* (2.1) *is Markov additive, then the Markov processes* (X_{t-}, V_{t-}) *and* (Y_t, R_t) *are time-homogeneous. Writing* $K_r(x; dy, ds) := K_r(x, t; dy, ds)$ *and* $\mathbb{P}^x = \mathbb{P}^{x,0}$, *the transition semigroup* $Q_{t-s} := Q_{s,t}$ *of the Markov process* (Y_t, R_t) *is given by*

$$
\begin{aligned}
&Q_t f(y, 0) = \mathbb{E}^y\big[f(Y_t, R_t)\big],\\
&Q_t f(y, r) = \mathbf{1}\{0 \le t < r\} f(y, r - t) + \mathbf{1}\{0 \le r \le t\} Q_{t-r} f(y, 0)
\end{aligned}
\tag{4.1}
$$

and the transition semigroup $P_{t-s} := P_{s,t}$ *of the Markov process* (X_{t-}, V_{t-}) *is given by*

$$
\begin{aligned}
&P_t f(x, 0) = \mathbb{E}^x\big[f(X_{t-}, V_{t-})\big],\\
&P_t f(x, v) = f(x, v + t) K_v\big(x; \mathbb{R}^d \times [v + t, \infty)\big)\\
&\qquad\qquad + \int_{\mathbb{R}^d \times [v, v+t)} P_{v+t-w} f(x + y, 0) K_v(x; dw, dy),
\end{aligned}
\tag{4.2}
$$

acting on the bounded measurable functions defined on $[0, \infty) \times \mathbb{R}^d$.

PROOF. Since (A_u, D_u) is Markov additive, we have $\vartheta_s \mathcal{A} f = \mathcal{A} \vartheta_s f$ for all $s \in \mathbb{R}$, where the shift operator $\vartheta_s f(x, t) = f(x, t + s)$. It follows that the resolvents $(\lambda - \mathcal{A})^{-1}$, the semigroup T_u and the kernel $Uf(\chi, \tau) = U^{\chi,\tau}(f)$ commute with ϑ_s. Then $\mathbb{E}^{\chi,\tau}[f(A_u, D_u)] = \mathbb{E}^{\chi,0}[f(A_u, \tau + D_u)]$ for all u and measurable f, and hence it suffices to work with the laws $\mathbb{P}^{\chi,0}$. Now in Theorem 2.3,writing $U^{\chi,\tau}(dx, dt) = U^{\chi}(dx, dt - \tau)$, we have

$$
\mathbb{E}^{\chi,\tau}\big[f(X_{t-}, Y_t, V_{t-}, R_t)\big] = \mathbb{E}^{\chi,0}\big[f(X_{(t-\tau)-}, Y_{t-\tau}, V_{(t-\tau)-}, R_{t-\tau})\big],
$$

where $\tau = 0$ without loss of generality. It follows that (4.1) and (4.2) are semigroups acting on the bounded measurable functions defined on $[0, \infty) \times \mathbb{R}^d$, compare [18], equations (19) and (31). □

REMARK 4.2. Under the assumptions of Theorem 2.3, a simple substitution yields the formulation of P_t and Q_t in terms of transition probabilities: For P_t, we find

$$
\begin{aligned}
&P_t(x_0, 0; dx, dv)\\
&\quad = \gamma(x, t) u^{x_0}(x, t)\, dx\, \delta_0(dv)\\
&\qquad + K\big(x_0; \mathbb{R}^d \times [v, \infty)\big) U^{x_0}(dx, t - dv) \mathbf{1}\{0 \le v \le t\},\\
&P_t(x_0, v_0; dx, dv)\\
&\quad = \delta_{x_0}(dx) \delta_{v_0+t}(dv) K_{v_0}\big(x_0; \mathbb{R}^d \times [v_0 + t, \infty)\big)\\
&\qquad + \int_{y \in \mathbb{R}^d} \int_{w \in [v_0, v_0+t)} P_{v_0+t-w}(x_0 + y, 0; dx, dv) K_{v_0}(x_0; dy, dw),
\end{aligned}
\tag{4.3}
$$

and for Q_t we have

$$
\begin{aligned}
&Q_t(y_0, 0; dy, dr)\\
&\quad = \gamma(y,t)u^{y_0}(y,t)\,dy\delta_0(dr)\\
&\qquad + \int_{x\in\mathbb{R}^d}\int_{w\in[0,t]} U^{y_0}(dx,dw)K(x; dy - x, dr + t - w),\\
&Q_t(y_0, r_0; dy, dr)\\
&\quad = \mathbf{1}\{0 < t < r_0\}\delta_{y_0}(dy)\delta_{r_0-t}(dr) + \mathbf{1}\{0 \le r_0 \le t\}Q_{t-r_0}(y_0, 0; dy, dr).
\end{aligned} \tag{4.4}
$$

5. Finite-dimensional distributions. In this section, we provide two examples to illustrate the explicit computation of finite dimensional distributions for the CTRWL process X_t and the OCTRWL process Y_t.

EXAMPLE 5.1 (The inverse stable subordinator). A very simple CTRW model takes deterministic jumps $J_n^c = c^{-1}$ and waiting times W_k^c in the domain of attraction of a standard β-stable subordinator $\bar{D}_u$ such that $\mathbb{E}[e^{-s\bar{D}_u}] = e^{-us^\beta}$. Setting $(A_0, D_0) = (\chi, \tau)$, (2.1) holds with $(A_u, D_u) = (\chi + u, \tau + \bar{D}_u)$, where $\bar{D}_u$ is a β-stable subordinator. Here, the CTRWL and the OCTRWL coincide, since A_u has no jumps. If $(\chi, \tau) = (0, 0)$, then in (2.3) we have $X_t = Y_t = E_t$, the inverse β-stable subordinator. Now we will compute the joint distributions of this first passage time process. The joint Laplace transform of these finite-dimensional distributions was computed by Bingham [9] but to our knowledge, the distributions themselves have not been reported in the literature.

The space–time limit (A_u, D_u) is a canonical Feller process on $\mathbb{R}^{d+1}$ with generator $\mathcal{A}$ given by (2.5) with $d = 1$, $b_1 \equiv 1$, $\gamma \equiv 0$, $a_{11} \equiv 0$, and jump kernel $K(x, t; dy, dw) = \delta_0(dy)\Phi(w)\,dw$ by [33], Proposition 3.10, where the Lévy measure $\Phi(w)\,dw = \beta w^{-\beta-1}\,dw/\Gamma(1-\beta)$. The stable Lévy process $\bar{D}_u$ has a smooth density $g(t, u)$ so that $\mathbb{P}^{0,0}(\bar{D}_u \in dt) = g(t, u)\,dt$ for every $u > 0$ by [20], Theorem 4.10.2. The underlying process (A_u, D_u) is Markov additive, hence (X_{t-}, V_{t-}) and (Y_t, R_t) are time-homogeneous Markov processes. In [40], Lemma 4.2, it was shown that (X_t, V_t), has no fixed discontinuities, hence (X_{t-}, V_{t-}) has the same law as (X_t, V_t). One checks that the 0-potential of (A_u, D_u) is absolutely continuous with density

$$u^{\chi,\tau}(x, t) = g(t - \tau, x - \chi)\mathbf{1}\{t > \tau, x > \chi\}. \tag{5.1}$$

Then it follows from (4.3) that the transition semigroup of (X_{t-}, V_{t-}) is given by

$$
\begin{aligned}
&P_t(x_0, 0; dx, dv)\\
&\quad = g(t - v, x - x_0)\Phi(v, \infty)\,dx\,dv,
\end{aligned} \tag{5.2}
$$

$$
\begin{aligned}
&P_t(x_0, v_0; dx, dv)\\
&\quad = \delta_{x_0}(dx)\delta_{v_0+t}(dv)\left(\frac{v_0+t}{v_0}\right)^{-\beta}\mathbf{1}\{v_0>0\}\\
&\qquad + \left(\frac{v_0}{v}\right)^{\beta}\int_{s=v_0}^{v_0+t-v} g\big((t-v)-(s-v_0), x-x_0\big)\\
&\qquad\qquad \times \frac{\beta s^{-1-\beta}\,ds}{\Gamma(1-\beta)}\mathbf{1}\{x>x_0, 0<v<t\}\,dx\,dv.
\end{aligned}
$$

Hence, for $0<t_1<t_2$, the joint distribution of $(E_{t_1}, V_{t_1}, E_{t_2}, V_{t_2})$ is

$$
\begin{aligned}
&\mathbb{P}^{0,0}(E_{t_1}\in dx, V_{t_1}\in dv, E_{t_2}\in dy, V_{t_2}\in dw)\\
&\quad = P_{t_2-t_1}(x, v; dy, dw)P_{t_1}(0,0; dx, dv)\\
&\quad = g(t_1-v, x)\Phi(v,\infty)\mathbf{1}\{x>0, 0<v<t_1\}\,dx\,dv\\
&\qquad \times\Bigg[\delta_x(dy)\delta_{v+t_2-t_1}(dw)\left(\frac{v+t_2-t_1}{v}\right)^{-\beta}\\
&\qquad\quad + \int_{s=v}^{v+t_2-t_1-w} g\big((t_2-t_1-w)-(s-v), y-x\big)\\
&\qquad\qquad \times \frac{\beta s^{-\beta-1}\,ds}{\Gamma(1-\beta)}\left(\frac{v}{w}\right)^{\beta} dy\,dw\,\mathbf{1}\{y>x, 0<w<t_2-t_1\}\Bigg]
\end{aligned}
$$

since $E_0=V_0=0$ for the physical starting point $(0,0)$. Integrating out the backward renewal times V_{t_1} and V_{t_2}, it follows that the joint distribution of (E_{t_1}, E_{t_2}) is

$$
\begin{aligned}
&\mathbb{P}(E_{t_1}\in dx, E_{t_2}\in dy)\\
&\quad = \mathbf{1}\{x>0\}\delta_x(dy)\int_{v=0}^{t_1} g(t_1-v, x)\frac{(v+t_2-t_1)^{-\beta}}{\Gamma(1-\beta)}\,dv\\
&\qquad + \int_{v=0}^{t_1}\int_{w=0}^{t_2-t_1}\int_{s=v}^{v+t_2-t_1-w} g\big((t_2-t_1-w)-(s-v), y-x\big)\,dy\,\mathbf{1}\{y>x\}\\
&\qquad\qquad \times\frac{\beta s^{-\beta-1}\,ds}{\Gamma(1-\beta)}\left(\frac{v}{w}\right)^{\beta} dw\,dv.
\end{aligned}
\tag{5.3}
$$

REMARK 5.2. The joint distribution of (E_{t_1}, E_{t_2}) can also be computed from the OCTRW embedding, but the computation appears to be simpler using the CTRWL embedding.

REMARK 5.3. Baule and Friedrich [4] compute the Laplace transform of the joint distribution function $H(x,y,s,t)$ of $x=E_s$ and $y=E_t$ and show that

$$
(\partial_x+\partial_y)H(x,y,s,t) = -(\partial_s+\partial_t)^{\beta}H(x,y,s,t)
$$

on $0 < s < t$ and $0 < x < y$. Equation (5.3) provides an explicit solution to this governing equation, which solves an open problem in [4]. The finite dimensional laws of any uncoupled CTRW limit can easily be calculated from the finite dimensional laws of E_t, given the law of the process A_u. This follows from a simple conditioning argument; see, for example, [31].

EXAMPLE 5.4. Kotulski [26] considered a CTRW with jumps equal to the waiting times $J_n^c = W_k^c$, in the domain of attraction of a standard β-stable subordinator $\bar{D}_u$ such that $\mathbb{E}[e^{-s\bar{D}_u}] = e^{-us^\beta}$. Equation (2.1) holds with $(A_u, D_u) = (A_0 + \bar{D}_u, D_0 + \bar{D}_u)$. The space–time limit (A_u, D_u) is a canonical Feller process on $\mathbb{R}^{d+1}$ with generator $\mathcal{A}$ given by (2.5) with $d = 1$, $\gamma \equiv 0$ and $K(x, t; dy, dw) = \delta_w(dy)\Phi(dw)$, where $\Phi(dw) = \phi(w)\,dw = \beta w^{-\beta-1}\,dw/\Gamma(1-\beta)$. The stable Lévy process $\bar{D}_u$ has a smooth density $g(t, u)$ so that $\mathbb{P}^{0,0}(\bar{D}_u \in dt) = g(t, u)\,dt$ for every $u > 0$. Since the Markov process (A_u, D_u) is Markov additive, we need only compute the potential for $\tau = 0$:

$$U^{\chi,0}(dx, dt) = \delta_{\chi+t}(dx) \int_{u=0}^{\infty} g(t, u)\,du\,dt. \tag{5.4}$$

Next, one sees that

$$\int_{u=0}^{\infty} g(t, u)\,du = \frac{t^{\beta-1}}{\Gamma(\beta)} \tag{5.5}$$

by taking Laplace transforms on both sides (also see [33], Example 2.9). The 0-potential hence equals

$$U^{\chi,0}(dx, dt) = \delta_{\chi+t}(dx) \frac{t^{\beta-1}}{\Gamma(\beta)}\,dt. \tag{5.6}$$

With $\Phi([v, \infty)) = v^{-\beta}/\Gamma(1-\beta)$, (4.3) reads

$$\begin{aligned}
P_t(x_0, 0; dx, dv) &= \frac{v^{-\beta}}{\Gamma(1-\beta)} \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} \delta_{x_0+t-v}(dx)\,dv\mathbf{1}\{0 < v < t\},\\
P_t(x_0, v_0; dx, dv) &= \delta_{x_0}(dx)\delta_{v_0+t}(dv)\left(\frac{v_0+t}{v_0}\right)^{-\beta}\\
&\quad + \int_{s=v_0}^{v_0+t} \left(\frac{v}{v_0}\right)^{-\beta} \delta_{x_0+v_0+t-v}(dx) \frac{(v_0+t-s-v)^{\beta-1}}{\Gamma(\beta)}\\
&\qquad \times \mathbf{1}\{0 < v < v_0 + t - s\} \frac{\beta s^{-\beta-1}}{\Gamma(1-\beta)}\,ds\,dv.
\end{aligned}$$

Note that the above formulae extend Example 5.5 in [5], which calculates the law of X_{t-}. The joint distribution of $\{(X_{t_i-}, V_{t_i-}) : 0 \le i \le n\}$ can now be computed by a simple conditioning argument. Similarly, the semigroup for (Y_t, R_t) reads

$$
\begin{aligned}
&Q_t(y_0, r_0; dy, dr) \\
&\quad = \delta_{y_0}(dy)\delta_{r_0-t}(dr)\mathbf{1}\{r_0 \ge t\} + Q_{t-r_0}(dy - y_0, dr)\mathbf{1}\{0 < r_0 < t\} \\
&\quad = \delta_{y_0}(dy)\delta_{r_0-t}(dr)\mathbf{1}\{r_0 \ge t\} \\
&\qquad + \mathbf{1}\{0 < r_0 < t\}\delta_{r+t-r_0+y_0}(dy) \\
&\qquad \times \int_{w=0}^{t-r_0} \frac{w^{\beta-1}}{\Gamma(\beta)} \frac{\beta(t - r_0 + r - w)^{-\beta-1}}{\Gamma(1-\beta)}\, dw\, dr.
\end{aligned}
$$

The joint distributions of X_{t-}, Y_t lead directly to the joint distribution of CTRWL, OCTRWL, respectively, for a wide variety of coupled models; see [21].

APPENDIX: PROOFS

LEMMA A.1. *Let T be $\mathcal{H}$-optional. Then $E_T : \omega \mapsto E_{T(\omega)}(\omega)$ is $\mathcal{F}$-optional.*

PROOF. We first assume that T is single valued. That is, fix $t > 0$ and $U \in \mathcal{H}_t$, and let $T(\omega) = t \cdot \mathbf{1}\{\omega \in U\} + \infty \cdot \mathbf{1}\{\omega \notin U\}$. It is easy to check that T is indeed $\mathcal{H}$-optional. Now $\{E_T \le u\} = \{E_t \le u\} \cap U$, and the right-hand side lies in $\mathcal{F}_u$, which follows from $U \in \mathcal{H}_t = \mathcal{F}_{E(t)}$ and the definition of the stopped σ-algebra $\mathcal{F}_{E(t)}$. Now consider an $\mathcal{H}$-optional time T with countably many values t_n, so that $\Omega = \bigcup_{n\in\mathbb{N}}\{\omega : T(\omega) = t_n\}$. Then due to the a.s. nondecreasing sample paths of E, we have $E(\inf T_n) = \inf E(T_n)$, and an application of [24], Lemma 6.3/4, together with the right-continuity of the filtrations $\mathcal{F}$ and $\mathcal{H}$ shows that E_T is $\mathcal{H}$-optional. $\square$

Stopping times allows for a decomposition into a predictable and totally inaccessible part [24]. The following lemma gives an interpretation for stopping times of the form E_T.

LEMMA A.2. *Let $T > 0$ be an $\mathcal{H}$-predictable stopping time. Then the $\mathcal{F}$-stopping time E_T is predictable on the set $\{\omega : E_{T-\varepsilon}(\omega) < E_T(\omega)\ \forall \varepsilon > 0\} = \{V_{T-} = 0\}$ and totally inaccessible on the complement $\{\omega : \exists \varepsilon > 0, E_{T-\varepsilon}(\omega) = E_T(\omega)\} = \{V_{T-} > 0\}$. Moreover, $\Delta(A, D)_{E_T} = (0, 0)$ on $\{V_{T-} = 0\}$ and $\Delta D_{E_T} > 0$ on $\{V_{T-} > 0\}$, $\mathbb{P}^{\chi,\tau}$-a.s.*

PROOF. Let T_n be an announcing sequence ([24], page 410), for T, that is T_n are $\mathcal{H}$-stopping times, $T_n < T$, $T_n \uparrow T$ a.s. Then due to the a.s. continuity of sample paths of E, the sequence E_{T_n} announces E_T on the set $\{V_{T-} = 0\}$, that is E_T is predictable on this set. As a canonical Feller process, (A, D) is quasi-left-continuous, and all its jump times are totally inaccessible ([24], Proposition 22.20), hence $\Delta(A, D)_{E_T} = (0, 0)$, $\mathbb{P}^{\chi,\tau}$-a.s. on $\{V_{T-} = 0\}$. On the complementary set $\{V_{T-} > 0\}$, we have $0 < H_T - G_{T-} = \Delta D_{E_T}$, and hence the process D jumps at E_T. $\square$

PROOF OF THEOREM 3.1. We first prove (ii). Consider the set of ω such that $H_T(\omega) > t$. In this case, $\mathbf{M}_\omega \cap (T, t) = \varnothing$, and hence $E_T = E_t$, so $(Y_t, H_t) = (Y_T, H_T)$, which implies that

$$\begin{aligned}\mathbb{E}^{\chi,\tau}\big[f(Y_t, R_t)\mathbf{1}_{\{H_T>t\}}|\mathcal{H}_T\big] &= f(Y_T, H_T - t)\mathbf{1}_{\{H_T>t\}}\\ &= f\big(Y_T, R_T - (t - T)\big)\mathbf{1}_{\{H_T>t\}}.\end{aligned}\tag{A.1}$$

This corresponds to the first case in (3.1). Turning to the second case, $H_T(\omega) \le t$, consider the shift operators θ_t acting on Ω, which are defined as usually by $(\theta_t\omega)(u) = \omega(t + u)$, or equivalently

$$(A, D)_u(\theta_t\omega) = (A, D)_{t+u}(\omega),\tag{A.2}$$

since (A, D) is canonical for Ω. Then from the definition of the inverse process E, we find

$$\begin{aligned}E_t(\theta_{E_T}\omega) &= \inf\{u \ge 0 : D_u(\theta_{E_T}\omega) > t\} = \inf\{u \ge 0 : D_{u+E_T}(\omega) > t\}\\ &= \inf\{u : u - E_T(\omega) \ge 0, D_u(\omega) > t\} - E_T(\omega)\\ &= E_t(\omega) - E_T(\omega),\end{aligned}\tag{A.3}$$

where $\theta_{E_T}\omega = \theta_u\omega$ if $E_T(\omega) = u$. Now observe that $(A, D)_{E_t}$ is the point in $\mathbb{R}^{d+1}$ where the process (A, D) enters the set $\mathbb{R}^d \times (t, \infty)$. This point will be the same for the space–time path started at the earlier time E_T, that is,

$$(A, D)_{E_t} \circ \theta_{E_T} = (A, D)_{E_t}.\tag{A.4}$$

In fact, using (A.2) and (A.3) we find

$$\begin{aligned}(A, D)_{E_t}(\theta_{E_T}\omega) &= (A, D)_{E_t(\theta_{E_T}\omega)}(\theta_{E_T}\omega) = (A, D)_{E_T(\omega)+E_t(\theta_{E_T}\omega)}(\omega)\\ &= (A, D)_{E_t(\omega)}(\omega) = (A, D)_{E_t}(\omega)\end{aligned}$$

for all $\omega \in \Omega$. Hence, we have shown that

$$H_t(\theta_{E_T}\omega) = H_t(\omega), \qquad Y_t(\theta_{E_T}\omega) = Y_t(\omega)$$

on the set $\{H_T \le t\}$. This yields

$$\begin{aligned}\mathbb{E}^{\chi,\tau}\big[f(Y_t, R_t)\mathbf{1}_{\{H_T\le t\}}|\mathcal{H}_T\big] &= \mathbb{E}^{\chi,\tau}\big[f(Y_t, R_t) \circ \theta_{E_T}\mathbf{1}_{\{H_T\le t\}}|\mathcal{H}_T\big]\\ &= \mathbb{E}^{\chi,\tau}\big[f(Y_t, R_t) \circ \theta_{E_T}|\mathcal{F}_{E_T}\big]\mathbf{1}_{\{H_T\le t\}}\\ &= \mathbb{E}^{(A,D)_{E_T}}\big[f(Y_t, R_t)\big]\mathbf{1}_{\{H_T\le t\}}\\ &= \mathbb{E}^{Y_T,H_T}\big[f(Y_t, R_t)\big]\mathbf{1}_{\{H_T\le t\}}\end{aligned}\tag{A.5}$$

$\mathbb{P}^{\chi,\tau}$-almost surely, using the strong Markov property of (A, D) at the stopping time E_T. Then (ii) follows by adding equations (A.1) and (A.5).

As for (i), let $(y_0, r_0) \in \mathbb{R}^d \times [0, \infty)$. Then $\mathbb{P}^{y_0, q+r_0}(Y_r = y_0, R_q = r_0) = 1$, and hence by nested conditional expectations and the above calculations we have

$$\begin{aligned} Q_{q,t} f(y_0, r_0) &= \mathbb{E}^{y_0,q+r_0}[Q_{q,t} f(Y_q, R_q)] \\ &= \mathbb{E}^{y_0,q+r_0}[\mathbb{E}^{y_0,q+r_0}[f(Y_t, R_t)|\mathcal{H}_q]] \\ &= \mathbb{E}^{y_0,q+r_0}[\mathbb{E}^{y_0,q+r_0}[\mathbb{E}^{y_0,q+r_0}[f(Y_t, R_t)|\mathcal{H}_s]|\mathcal{H}_q]] \\ &= \mathbb{E}^{y_0,q+r_0}[\mathbb{E}^{y_0,q+r_0}[Q_{s,t} f(Y_s, R_s)|\mathcal{H}_q]] \\ &= \mathbb{E}^{y_0,q+r_0}[Q_{q,s} Q_{s,t} f(Y_q, R_q)] = Q_{q,s} Q_{s,t} f(y_0, r_0). \end{aligned}$$

We turn to the remaining case (iii). By definition of R_t, it suffices to show that if T is a $\mathcal{H}$-predictable time, then $(Y, H)_{T-} = (Y, H)_T$, $\mathbb{P}^{\chi,\tau}$-a.s. for every $(\chi, \tau) \in \mathbb{R}^{d+1}$. Hence let $T_n < T$, $T_n \uparrow T$ be a sequence of $\mathcal{H}$-optional times announcing T. As in Lemma A.2, we check the two cases in which the $\mathcal{F}$-stopping time E_T is predictable or totally inaccessible.

On the set $\{\omega : E_{T-\varepsilon}(\omega) < E_T(\omega), \forall \varepsilon > 0\}$, the process E is left-increasing at T, continuous, and $E_{T_n} \uparrow E_T$, $E_{T_n} < E_T$ if $T_n \uparrow T$, $T_n < T$. Moreover, $\Delta(A, D)_{E_T} = (0, 0)$ a.s. (Lemma A.2). Hence,

$$(H, Y)_{T-} = (A, D)_{E_{T-}} = (A, D)_{E_T -} = (A, D)_{E_T} = (H, Y)_T.$$

On the set $\{\omega : \exists \varepsilon > 0 : E_{T-\varepsilon}(\omega) = E_T(\omega)\}$, E is left-constant at T. Hence, $E_{T_n} = E_T$ for large n, and

$$(H, Y)_{T-} = \lim (H, Y)_{T_n} = \lim (A, D)_{E_{T_n}} = (A, D)_{E_T} = (H, Y)_T.$$

The two cases together imply that $(H, Y)_{T-} = (H, Y)_T$ a.s. $\square$

For the proof of Theorem 3.2, we will need the following lemma.

LEMMA A.3. *Let $(\chi, \tau) \in \mathbb{R}^{d+1}$, and let $t \geq \tau$. Then for every bounded measurable f defined on $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$, we have $\mathbb{P}^{\chi,\tau}$-a.s.:*

$$\begin{aligned} &\mathbb{E}^{\chi,\tau}[f(X_{t-}, G_{t-}; \Delta(A, D)_{E(t)})|\mathcal{G}_t] \\ &\quad = \int_{\mathbb{R}^{d+1}} K_{V_{t-}}(X_{t-}, G_{t-}; dx \times dz) f(X_{t-}, G_{t-}; x, z). \end{aligned}$$

PROOF. Since (X_{t-}, G_{t-}) are $\mathcal{G}_t$-measurable, by a monotone class argument and dominated convergence, it suffices to prove the formula

$$\mathbb{E}^{\chi,\tau}[f(\Delta(A, D)_{E(t)})|\mathcal{G}_t] = \int_{\mathbb{R}^{d+1}} K_{V_{t-}}(X_{t-}, G_{t-}; f) \tag{A.6}$$

for all bounded measurable f defined on $\mathbb{R}^{d+1}$. As in Lemma A.2, we consider the two cases $\{V_{t-}=0\}$ and $\{V_{t-}>0\}$. On $\{V_{t-}=0\}$, we have $\Delta(A,D)_{E_t}=(0,0)$, $\mathbb{P}^{\chi,\tau}$-a.s., and hence

$$\begin{aligned}(A.7)\qquad &\mathbb{E}^{\chi,\tau}\big[f(\Delta(A,D)_{E(t)}\mathbf{1}\{V_{t-}=0\}|\mathcal{G}_t\big]\\ &\quad= f(0,0)=\delta_{(0,0)}(f)=K_{V_{t-}}(X_{t-},G_{t-};f)\mathbf{1}\{V_{t-}=0\}.\end{aligned}$$

On $\{V_{t-}>0\}$, the process D jumps at E_t (Lemma A.2), and since D has increasing sample paths this is equivalent to

(A.8) "there exists a unique number $s>0$ such that $D_{s-}<t\le D_s$."

We rewrite the restriction of (A.6) to $\{V_{t-}>0\}$ in integral form:

$$\begin{aligned}&\mathbb{E}^{\chi,\tau}\big[f\big(\Delta(A,D)_{E(t)}\big)\mathbf{1}_C\mathbf{1}\{V_{t-}>0\}\big]\\ &\quad=\mathbb{E}^{\chi,\tau}\big[K_{V_{t-}}(X_{t-},G_{t-};f)\mathbf{1}_C\mathbf{1}\{V_{t-}>0\}\big],\qquad C\in\mathcal{G}_t,\end{aligned}$$

where $\mathbf{1}_C(\omega)=1$ iff $\omega\in C$. Now we invoke [14], Theorem IV.67(b), which says that there exists an $\mathcal{F}$-adapted *predictable* process Z such that $\mathbf{1}_C=Z_{E(t)}$. Then it suffices to show that for every $\mathcal{F}$-adapted predictable process Z, the following two random variables have the same expectation with respect to $\mathbb{P}^{\chi,\tau}$:

$$\begin{aligned}(A.9)\qquad &f\big(\Delta(A,D)_{E(t)}\big)Z_{E(t)}\mathbf{1}\{V_{t-}>0\},\\ &K_{V_{t-}}f(X_{t-},G_{t-})Z_{E(t)}\mathbf{1}\{V_{t-}>0\}.\end{aligned}$$

We begin on the right-hand side and find, using (A.8) and $X_{t-}=A_{E_t-}$, $G_{t-}=D_{E_t-}$

$$\begin{aligned}&\mathbb{E}^{\chi,\tau}\big[K_{V_{t-}}(X_{t-},G_{t-};f)Z_{E_t}\mathbf{1}\{V_{t-}>0\}\big]\\ &\quad=\mathbb{E}^{\chi,\tau}\big[K_{t-D_{E_t-}}(A_{E_t-},D_{E_t-};f)Z_{E_t}\mathbf{1}\{D_{E_t-}<t\}\big]\\ &\quad=\mathbb{E}^{\chi,\tau}\Big[\sum_{s>0}K_{t-D_{s-}}(A_{s-},D_{s-};f)Z_s\mathbf{1}\{D_{s-}<t\le D_s\}\Big]\\ &\quad=\mathbb{E}^{\chi,\tau}\Big[\sum_{s>0}K_{t-D_{s-}}(A_{s-},D_{s-};f)Z_s\mathbf{1}\{D_{s-}<t\}\mathbf{1}\{\Delta D_s\ge t-D_{s-}\}\Big]\\ &\quad=\mathbb{E}^{\chi,\tau}\big[W(\cdot,s;y,w)\mu(\cdot,ds;dy,dw)\big]=\cdots,\end{aligned}$$

where the optional random measure μ is as in (2.10) and

$$\begin{aligned}W(\omega,s;y,w)&=K_{t-D_{s-}(\omega)}\big(A_{s-}(\omega),D_{s-}(\omega);f\big)Z(s,\omega)\\ &\qquad\times\mathbf{1}\big\{D_{s-}(\omega)<t\big\}\mathbf{1}\big\{w\ge t-D_{s-}(\omega)\big\}\end{aligned}$$

is a predictable integrand. The compensation formula [19], II.1.8, and (2.11) then yield

$$\begin{aligned}\cdots &= \mathbb{E}^{\chi,\tau}\big[W(\cdot,s;y,w)\mu^p(\cdot,ds;dy,dw)\big]\\ &= \mathbb{E}^{\chi,\tau}\bigg[\int_0^\infty K_{t-D_{s-}}(A_{s-},D_{s-};f)Z_s\mathbf{1}\{D_{s-}<t\}\\ &\qquad\qquad \times K\big(A_{s-},D_{s-};\mathbb{R}^d\times(t-D_{s-},\infty)\big)\,ds\bigg].\end{aligned}$$

Using the definition of K_v (3.2), this equals

$$\begin{aligned}&= \mathbb{E}^{\chi,\tau}\bigg[\int_0^\infty\int_{\mathbb{R}^d\times[t-D_{s-},\infty)} K(A_{s-},D_{s-};dy,dw)f(y,w)Z_s\\ &\qquad\qquad \times\mathbf{1}\{D_{s-}<t\}\,ds\bigg]\\ &= \mathbb{E}^{\chi,\tau}\bigg[\int_0^\infty\int_{\mathbb{R}^{d+1}} K(A_{s-},D_{s-};dy,dw)f(y,w)Z_s\\ &\qquad\qquad \times\mathbf{1}\{D_{s-}<t\le D_{s-}+w\}\,ds\bigg].\end{aligned}\tag{A.10}$$

Proceeding similarly with the left-hand side of (A.9), we find

$$\begin{aligned}&\mathbb{E}^{\chi,\tau}\big[f\big(\Delta(A,D)_{E(t)}\big)Z_{E(t)}\mathbf{1}\{V_{t-}>0\}\big],\\ &\quad= \mathbb{E}^{\chi,\tau}\bigg[\sum_{s>0} f\big(\Delta(A,D)_s\big)Z_s\mathbf{1}\{D_{s-}<t\le D_{s-}+\Delta D_s\}\bigg]\\ &\quad= \mathbb{E}^{\chi,\tau}\big[\tilde W(\cdot,s;y,w)\mu(\cdot,ds;dy,dw)\big]\\ &\quad= \mathbb{E}^{\chi,\tau}\big[\tilde W(\cdot,s;y,w)\mu^p(\cdot,ds;dy,dw)\big],\end{aligned}\tag{A.11}$$

where $\tilde W(\omega,s;y,w)=f(y,w)Z_s(\omega)\mathbf{1}\{D_{s-}(\omega)<t\le D_{s-}(\omega)+w\}$. We check that (A.11) and (A.10) are equal. Hence, we have shown

$$\begin{aligned}&\mathbb{E}^{\chi,\tau}\big[f\big(\Delta(A,D)_{E(t)}\big)\mathbf{1}\{V_{t-}>0\}|\mathcal{G}_t\big]\\ &\qquad = K_{V_{t-}}f(X_{t-},G_{t-})\mathbf{1}\{V_{t-}>0\},\end{aligned}\tag{A.12}$$

and adding equations (A.7) and (A.12) yields (A.6). $\square$

For later use, we note the formula

$$\begin{aligned}&K_{v+t}(x,z;C)K_v\big(x,z;\mathbb{R}^d\times[v+t,\infty)\big)\\ &\qquad = K_v(x,z;C)\qquad (x,z)\in\mathbb{R}^{d+1}, v,t\ge 0,\end{aligned}\tag{A.13}$$

valid for all Borel-sets $C\subset\mathbb{R}^d\times[v+t,\infty)$.

PROOF OF THEOREM 3.2. We begin with statement (ii). We consider the two cases $H_s \geq t$ and $H_s < t$. On the set $\{H_s \geq t\}$, E is constant on the interval $[s, t]$, and hence we have $(G, X)_{t-} = (G, X)_{s-}$. Using $G_{s-} + \Delta D_{E_s} = H_s$ and Lemma A.3, we calculate

$$
\begin{aligned}
&\mathbb{E}^{\chi,\tau}\left[f(X_{t-}, V_{t-})\mathbf{1}\{H_s \geq t\}|\mathcal{G}_s\right] \\
&\quad = f(X_{s-}, t - G_{s-})\mathbb{P}^{\chi,\tau}(H_s \geq t|\mathcal{G}_s) \\
&\quad = f(X_{s-}, t - s + V_{s-})\mathbb{P}^{\chi,\tau}(\Delta D_{E_s} \geq t - G_{s-}|\mathcal{G}_s) \qquad \text{(A.14)}\\
&\quad = f(X_{s-}, t - s + V_{s-})K_{V_{s-}}\big(X_{s-}, G_{s-}; [t - G_{s-}, \infty) \times \mathbb{R}^d\big) \\
&\quad = f(X_{s-}, V_{s-} + t - s)K_{V_{s-}}\big(X_{s-}, s - V_{s-}; [t - s + V_{s-}, \infty) \times \mathbb{R}^d\big),
\end{aligned}
$$

which corresponds to the first summand in (3.3).

We now turn to the case $H_s < t$, and recall the shift operators θ_t from (A.2). For the left-continuous version of (A, D), we can write

$$(A, D)_{t-}(\theta_s\omega) = (A, D)_{s+t-}(\omega), \qquad s \geq 0, t > 0.$$

Note that we had to assume $t > 0$ above, for the left-hand limit to be defined. We find now, similarly to (A.4),

$$(A, D)_{E_t-} \circ \theta_{E_s} = (A, D)_{E_t-}$$

on $\{H_s < t\}$. Indeed, by (A.3), $E_t(\omega) = E_s(\omega) + E_t(\theta_{E_s}\omega)$, and so

$$
\begin{aligned}
(A, D)_{E_t-}(\theta_{E_s}\omega) &= (A, D)_{E_t(\theta_{E_s}\omega)-}(\theta_{E_s}\omega) = (A, D)_{E_s(\omega)+E_t(\theta_{E_s}\omega)-}(\omega) \\
&= (A, D)_{E_t(\omega)-}(\omega) = (A, D)_{E_t-}(\omega).
\end{aligned}
$$

If $t > 0$ and $H_s(\omega) < t$, then by (A.3) $E_t(\theta_{E_s}\omega) = E_t(\omega) - E_s(\omega) > 0$, and the left-hand limit is well defined. Thus, we have shown that on the set $\{H_s < t\}$ we have $(X_{t-}, V_{t-}) = (X_{t-}, V_{t-}) \circ \theta_{E_s}$. We will use the strong Markov property of (A, D) in the following form:

$$\mathbb{E}^{\chi,\tau}[F \circ \theta_T|\mathcal{F}_T] = \mathbb{E}^{A_T,D_T}[F], \qquad \mathbb{P}^{\chi,\tau}\text{-a.s.},$$

valid for all $\mathcal{F}$-stopping times T and random variables F on $(\Omega, \mathcal{F}_\infty, \mathbb{P}^{\chi,\tau})$. Using Lemma A.3 and the strong Markov property at E_s, we then calculate

$$
\begin{aligned}
&\mathbb{E}^{\chi,\tau}\left[f(X_{t-}, V_{t-})\mathbf{1}\{H_s < t\}|\mathcal{G}_s\right] \\
&\quad = \mathbb{E}^{\chi,\tau}\left[\mathbb{E}^{\chi,\tau}\left[f(X_{t-}, V_{t-}) \circ \theta_{E_s}|\mathcal{H}_s\right]\mathbf{1}\{H_s < t\}|\mathcal{G}_s\right] \\
&\quad = \mathbb{E}^{\chi,\tau}\left[\mathbb{E}^{Y_s,H_s}\left[f(X_{t-}, V_{t-})\right]\mathbf{1}\{H_s < t\}|\mathcal{G}_s\right] \\
&\quad = \mathbb{E}^{\chi,\tau}\left[\mathbb{E}^{(X,G)_{s-}+\Delta(A,D)_{E_s}}\left[f(X_{t-}, V_{t-})\right]\mathbf{1}\{G_{s-} + \Delta D_{E_s} < t\}|\mathcal{G}_s\right] \\
&\quad = \int_{\mathbb{R}^{d+1}} K_{V_{s-}}(X_{s-}, G_{s-}; dy \times dw) \qquad \text{(A.15)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \mathbb{E}^{(X,G)_{s-}+(y,w)}\big[f(X_{t-},V_{t-})\big]\mathbf{1}\{G_{s-}+w<t\}\\
&=\int_{\mathbb{R}^d\times[V_{s-},V_{s-}+t-s)} K_{V_{s-}}(X_{s-},s-V_{s-};dy\times dw)\\
&\quad\times \mathbb{E}^{X_{s-}+y,s-V_{s-}+w}\big[f(X_{t-},V_{t-})\big],
\end{aligned}
$$

which corresponds to the second summand in (3.3). Adding equations (A.14) and (A.15) yields statement (ii). For statement (i), we calculate

$$
\begin{aligned}
&P_{r,s}P_{s,t}f(x,v)\\
&\quad= P_{s,t}f(x,v+s-r)K_v\big(x,r-v;\mathbb{R}^d\times[v+s-r,\infty)\big)\\
&\qquad+\int_{\mathbb{R}^d\times[v,v+s-r)} K_v(x,r-v;dy\times dw)\mathbb{E}^{x+y,r-v+w}\big[P_{s,t}f(X_{s-},V_{s-})\big]\\
&\quad= K_v\big(x,r-v;\mathbb{R}^d\times[v+s-r,\infty)\big)\\
&\qquad\times\bigg\{f(x,v+t-r)K_{v+s-r}\big(x,r-s;\mathbb{R}^d\times[v+t-r,\infty)\big)\\
&\qquad\qquad+\int_{\mathbb{R}^d\times[v+s-r,v+t-r)} K_{v+s-r}(x,r-v;dy\times dw)\\
&\qquad\qquad\qquad\times\mathbb{E}^{x+y,r-v+w}\big[f(X_{t-},V_{t-})\big]\bigg\}\\
&\qquad+\int_{\mathbb{R}^d\times[v,v+s-r)} K_v(x,r-v;dy\times dw)\mathbb{E}^{x+y,r-v+w}\big[P_{s,t}f(X_{s-},V_{s-})\big]\\
&\quad=\cdots.
\end{aligned}
$$

Using (A.13) and applying the statement (ii) with $(\chi,\tau)=(x+y,r-v+w)$ yields

$$
\begin{aligned}
\cdots &= f(x,v+t-r)K_v\big(x,r-v;\mathbb{R}^d\times[v+t-r,\infty)\big)\\
&\quad+\int_{\mathbb{R}^d\times[v+s-r,v+t-r)} K_v(x,r-v;dy\times dw)\mathbb{E}^{x+y,r-v+w}\big[f(X_{t-},V_{t-})\big]\\
&\quad+\int_{\mathbb{R}^d\times[v,v+s-r)} K_v(x,r-v;dy\times dw)\\
&\qquad\qquad\times\mathbb{E}^{x+y,r-v+w}\big[\mathbb{E}^{x+y,r-v+w}\big[f(X_{t-},V_{t-})|\mathcal{G}_s\big]\big]\\
&= P_{r,t}f(x,v),
\end{aligned}
$$

which is statement (i). $\square$

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