ONE–DIMENSIONAL MARGINALS OF OPERATOR STABLE LAWS AND THEIR DOMAINS OF ATTRACTION

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25 March 1998

ABSTRACT: Operator stable laws are the weak limits of affine normalized partial sums of i.i.d. random vectors. It is known that the one–dimensional marginals of operator stable laws need not be stable, or even attracted to a stable law. In this paper we show that for any operator stable law, there exists a basis in which the marginals along every coordinate axis are attracted to a stable or semistable law. This connection between operator stable and semistable laws is new and surprising. We also characterize those operator stable laws whose marginals are stable or semistable. Finally we consider the marginals of random vectors attracted to some operator stable law.

AMS 1991 subject classification: Primary 60E07; Secondary 60F05, 60G50.

Key words and phrases: Operator stable laws, domains of normal attraction, stable laws, semistable laws, marginals.

1. Introduction. The theory of operator stable probability measures on finite dimensional real vectors spaces was begun by Sharpe (1969), see also Jurek and Mason (1993). A probability measure ν on $\mathbb{R}^d$ which is full (that is, not supported on a proper hyperplane) is said to be operator stable if there is a linear operator E on $\mathbb{R}^d$ (called an exponent of ν) and a vector valued function a_t so that for all $t > 0$

$$\nu^t = t^E \nu * \delta(a_t). \tag{1}$$

Here ν is known to be infinitely divisible, so ν^t, the t-th convolution power is well defined. The operator t^E is defined as $\exp(E \log t)$ where exp is the usual exponential mapping for matrices. For any linear operator A the measure $A\nu$ is defined by $A\nu(B) = \nu(A^{-1}(B))$ for Borel sets $B \subset \mathbb{R}^d$, $\delta(a)$ denotes the point mass in $a \in \mathbb{R}^d$ and $*$ denotes convolution.

Operator stable laws are the weak limits of affine normalized partial sums of independent and identically distributed (i.i.d.) random vectors. (See Sharpe (1969).) Let $X, X_1, X_2, \ldots$ be i.i.d. random vectors and let Y be a random vector with a full distribution ν on $\mathbb{R}^d$. If there exist linear operators A_n and nonrandom vectors $s_n \in \mathbb{R}^d$ such that

$$A_n(X_1 + \ldots + X_n) - s_n \Rightarrow Y \tag{2}$$

as $n \to \infty$, we say that X belongs to the *generalized domain of attraction* of Y (resp. ν) and write $X \in \mathrm{GDOA}(Y)$. Here $\Rightarrow$ denotes convergence in distribution. It is shown in Sharpe (1969) that ν is operator stable if and only if $\mathrm{GDOA}(\nu) \neq \emptyset$. Generalized domains of attraction were characterized by Meerschaert (1993) using a multivariate theory of regular variation. In the one–dimensional situation $d = 1$ the operator stable measures are exactly the classical $\alpha-$stable measures and the exponent $E = 1/\alpha$. If (2) holds in this case we say that X belongs to the *domain of attraction* of Y.

In the general case $d \geq 2$ the situation is more complex. Let $\mathcal{E}(\nu)$ denote the collection of all exponents of the operator stable law ν in (1) (for possibly different shift vectors a_t) and $\mathcal{S}(\nu) = \{A : A\nu = \nu * \delta(a) \text{ for some } a \in \mathbb{R}^d\}$ denote the symmetry group of ν which is compact since ν is full. Then Holmes et al. (1982) establish that $\mathcal{E}(\nu) = E + T\mathcal{S}(\nu)$ where $E \in \mathcal{E}(\nu)$ is arbitrary and $T\mathcal{S}(\nu)$ is the tangent space of $\mathcal{S}(\nu)$. Hudson et al. (1986) established the existence of an exponent $E_0 \in \mathcal{E}(\nu)$ which commutes with every element of $\mathcal{S}(\nu)$. Such exponents are called *commuting exponents* and play a central role in deriving a decomposition of the underlying vector space as well as the exponent E in Meerschaert and Veeh (1993) which will be crucial for our work.

Now let Y be a random vector with an operator stable distribution ν. For any nonzero $\theta \in \mathbb{R}^d$ we say that $\langle Y, \theta \rangle$ is a one–dimensional marginal of Y. Here $\langle x, y \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^d$. An operator stable law with exponent $E = (1/\alpha)I$, I the identity, is called multivariable stable. Samorodnitsky and Taqqu (1994) show that every one–dimensional marginal of a multivariable stable law is stable with the same index α. However, Marcus (1983) provides an example showing that the converse is not true in general. In fact there exists a probability distribution μ on $\mathbb{R}^2$ whose one–dimensional marginals are all stable with index $0 < \alpha < 1$ but μ is not multivariate α–stable. It follows from Theorem 1 of Giné and Hahn (1983) that μ is not even infinitely divisible even though all one–dimensional marginals are as stable laws of course infinitely divisible.

Meerschaert (1990) gives an example of an operator stable law whose marginals are not stable, or even in the domain of attraction of a stable law. Surprisingly enough these marginals turn out to be *semistable*. This shows a new connection between semistable and operator stable laws.

A nondegenerate probability measure ρ on $\mathbb{R}$ is called semistable if it is infinitely divisible and if there exist a $b > 0$ and $c > 1$ such that

$$\rho^c = b\rho * \delta(s) \tag{3}$$

for some shift $s \in \mathbb{R}$, where $(b\rho)(B) = \rho(b^{-1}B)$. If Z is a random variable with distribution ρ we say that either ρ or Z is (b, c) semistable if (3) holds. We say that U belongs to the *domain of semistable attraction* of a random variable Z with distribution ρ if there exist a sequence k_n of natural numbers tending to infinity with $k_{n+1}/k_n \to c$ as $n \to \infty$, $a_n > 0$ and shifts $s_n \in \mathbb{R}$ such that

$$a_n(U_1 + \ldots + U_{k_n}) - s_n \Rightarrow Z \tag{4}$$

and we write $U \in \mathrm{DOSA}(Z)$. Pillai (1971) shows that $\mathrm{DOSA}(Z) \neq \emptyset$ if and only if Z has a (b, c) semistable distribution. For further information on semistable laws see Kruglov (1972), Shimizu (1970), Pillai (1971), Scheffler (1994) and Meerschaert and Scheffler (1996). Note that if ρ is α-stable then ρ is $(t^{1/\alpha}, t)$ semistable for all $t > 1$, so that stable laws are also semistable.

The main result of this paper is that for any operator stable random vector Y there is a basis $\{\theta_1, \ldots, \theta_d\}$ of $\mathbb{R}^d$ such that every one–dimensional marginal $\langle Y, \theta_i\rangle$ belongs to either the domain of attraction of some stable law, or to the domain of semistable attraction of some semistable law on $\mathbb{R}$. This result also extends to laws belonging to the domain of normal attraction of an operator stable law. That is, the norming operators in (2) are of the special form $A_n = n^{-E}$ for some exponent $E \in \mathcal{E}(\nu)$. We also show that every operator stable law has marginals which are either stable or semistable, and it turns out that these marginals are the only possible limit laws in our main result. Finally we investigate the most general case: one–dimensional marginals of laws in the generalized domain of attraction of some operator stable law. Here a stochastic compactness result is the best obtainable, and again the possible limit laws are the stable or semistable marginals of the operator stable law.

The significance of these results relates to the construction of the norming operators A_n in (2), an important open problem. Hahn and Klass (1985) show that A_n can be constructed by first choosing an appropriate basis (which varies with n), and then applying one–dimensional methods to the marginals in each of these directions. However it is not clear how to choose these bases. Our results show that there is a fixed basis in which one–dimensional methods can be employed, if one allows semistable as well as stable limits.

2. One dimensional marginals of operator stable laws. In order to formulate our first result we first introduce some notation and present the decomposition theorem of Meerschaert and Veeh (1993). Assume that Y is a random vector on $\mathbb{R}^d$ with a full operator stable distribution ν. Let $E \in \mathcal{E}(\nu)$ be a commuting exponent of ν and write $E = S + N$ where S is semisimple and N is nilpotent. Then $SN = NS$. Recall that a linear operator on $\mathbb{R}^d$ is said to be semisimple if its minimal polynomial is the product of distinct prime factors and that N is called nilpotent if $N^k = 0$ for some $k \geq 0$. (See Hoffman and Kunze (1961).) For a linear operator A on $\mathbb{R}^d$ let A^* denote its transpose.

Then by Theorem 3.2 of Meerschaert and Veeh (1993) there exists a direct sum decomposition $\mathbb{R}^d = U_1 \oplus \cdots \oplus U_s$, $s \geq 1$, into subspaces invariant under E and N (and hence under S) such that

$$\begin{aligned} N &= N_1 \oplus \cdots \oplus N_s \\ S &= S_1 \oplus \cdots \oplus S_s \end{aligned} \tag{4}$$

where N_i is nilpotent and

$$\begin{aligned} &S_i = a_i I \quad \text{or} \\ &S_i = \begin{pmatrix} B & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B \end{pmatrix}, \quad \text{where } B = \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix} \end{aligned} \tag{5}$$

for some $a_i \geq \frac{1}{2}$ and $b_i > 0$. Note that by Sharpe (1969) all the real parts of the eigenvalues of E are necessarily $\geq \frac{1}{2}$ and that the first case of (5) is the case of a real eigenvalue whereas the second case of (5) corresponds to a pair of conjugate complex eigenvalues with real part a_i. Our first result shows that every operator stable law has one–dimensional marginals which are either stable or semistable. This result seems to be the first known connection between operator stable and semistable laws.

THEOREM 1. Let Y be a random vector with a full operator stable distribution ν and $E \in \mathcal{E}(\nu)$ be a commuting exponent. Then, using the decomposition of $\mathbb{R}^d$ and E in (4) and (5) above we have: For $i = 1, \ldots, s$ and $\theta_0 \in \text{Kern} N_i^*$ we have either
(a) $\langle Y, \theta_0 \rangle$ is stable with index $1/a_i$ if $S_i = a_i I$
or
(b) $\langle Y, \theta_0 \rangle$ is $(e^{2\pi a_i/b_i}, e^{2\pi 1/b_i})$ semistable if $S_i = \text{diag}(B, \ldots, B)$,
where $B = \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}$.

PROOF. Since $\theta_0 \in \mathrm{Kern} N_i^*$ we have $N_i^*\theta_0 = 0$ and hence $t^{N_i^*}\theta_0 = \theta_0$ for all $t > 0$. Then $t^{E^*}\theta_0 = t^{E_i^*}\theta_0 = t^{S_i^*+N_i^*}\theta_0 = t^{S_i^*}t^{N_i^*}\theta_0 = t^{S_i^*}\theta_0$.

Now if $S_i = a_i I$ then $t^{E^*}\theta_0 = t^{a_i}\theta_0$ for all $t > 0$. Let $T_0(x) = \langle x, \theta_0\rangle$ denote a homomorphism from $\mathbb{R}^d$ to $\mathbb{R}$ and let $\nu_0 = T_0(\nu)$ denote the image measure. Then (1) implies

$$T_0(\nu^t) = T_0(\nu)^t = \nu_0^t = T_0(t^E\nu) * \delta(T_0(a_t)) \tag{6}$$

for all $t > 0$. But if $\hat{\rho}$ denotes the Fourier transform of a probability measure ρ on $\mathbb{R}$ we get

$$\begin{aligned} T_0(t^E\nu)\widehat{}(s) &= \int_{R^d} e^{isT_0(x)} d(t^E\nu)(x) = \int_{\mathbb{R}^d} e^{is\langle x, t^{E^*}\theta_0\rangle} d\nu(x) \\ &= \int_{R^d} e^{ist^{a_i}T_0(x)} d\nu(x) = T_0(\nu)\widehat{}(st^{a_i}) \\ &= \hat{\nu}_0(st^{a_i}) = (t^{a_i}\nu)\widehat{}(s) \end{aligned}$$

showing by the uniqueness theorem of the Fourier transform that $T_0(t^E\nu) = (t^{a_i}\nu_0)$. Hence by (6) we have

$$\nu_0^t = (t^{a_i}\nu_0) * \delta(s_t) \quad \text{for all } t > 0$$

and some $s_t \in \mathbb{R}$. Since ν is full it follows that ν_0 is nondegenerate and therefore stable with index $1/a_i$.

Now assume that $S_i = \mathrm{diag}(B, \ldots, B)$ where $B = \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}$. Then $t^{S_i^*} = t^{a_i}R(b_i \log t)$ for all $t > 0$, where $R(s)$ is a rotation of angle s in the U_i^* space. That is R is an orthogonal operator with $R(s + 2\pi) = R(s)$, R is continuous and $R(0) = I$. Hence, as before $t^{E^*}\theta_0 = t^{S_i^*}\theta_0 = t^{a_i}R(b_i \log t)\theta_0$. Define $\nu_0 = T_0(\nu)$ as above. Then (6) holds for all $t > 0$. But

$$T_0(t^E\nu)\widehat{}(s) = \int_{\mathbb{R}^d} e^{ist^{a_i}\langle x, R(b_i \log t)\theta_0\rangle} d\nu(x).$$

If we set $t_0 = \exp(2\pi/b_i)$ we get $R(b_i \log t_0) = I$ and hence

$$\begin{aligned} T_0(t_0^E\nu)\widehat{}(s) &= \int_{\mathbb{R}^d} e^{ist_0^{a_i}T_0(x)} d\nu(x) \\ &= \hat{\nu}_0(st_0^{a_i}) = (t_0^{a_i}\nu_0)\widehat{}(s). \end{aligned}$$

Therefore

$$\nu_0^{t_0} = (t_0^{a_i}\nu_0) * \delta(T_0(a_{t_0}))$$

showing that ν_0 is $(t_0^{a_i}, t_0)$ semistable. This concludes the proof.

REMARK. The following example shows that the marginals of an operator stable law are not necessarily semistable. Hence the result of Theorem 1 is in some sense the best possible. Suppose that $d = 4$ and $E = B_1 \oplus B_2$ where B_i are of the form (5) with $a_1 = a_2 = a$ and b_1/b_2 irrational. Let $e_1, \ldots, e_4$ denote the standard basis for R^4 and let $V_1 = \mathrm{Span}\{e_1, e_2\}$, $V_2 = \mathrm{Span}\{e_3, e_4\}$. Take Y_i independent operator stable with exponent B_i on V_i and let $Y = (Y_1, Y_2)$ so that Y is operator stable on $\mathbb{R}^4$ with exponent E, with two independent 2–dimensional components. Suppose Y_i has semistable but not stable marginals, for example we can take the Lévy measure of Y_i concentrated on one orbit. Then $Z_1 = \langle Y, e_1 \rangle$ is $(e^{2\pi a/b_1}, e^{2\pi 1/b_1})$ semistable and $Z_2 = \langle Y, e_3 \rangle$ is $(e^{2\pi a/b_2}, e^{2\pi 1/b_2})$ semistable. Then the Lévy measure ϕ_i of Z_i satisfies $\phi_i(t, \infty) = t^{-1/a}\, h_i(\log t)$ where h_i is periodic with period $\log(e^{2\pi a/b_i}) = 2\pi a/b_i$. But then $Z = Z_1 + Z_2 = \langle Y, e_1 + e_3 \rangle$ has Lévy measure $\phi = \phi_1 + \phi_2$ and so $\phi(t, \infty) = t^{-1/a}\, h(\log t)$ where h is not periodic, hence Theorem 1 of Kruglov (1972) shows that Z is not semistable.

3. Domains of normal attraction. In this section we will investigate the one–dimensional marginals of laws in the domain of normal attraction of an operator stable law. Let $X, X_1, X_2, \ldots$ be i.i.d. random vectors with common distribution μ and let ν be a full operator stable law. If for some $E \in \mathcal{E}(\nu)$ there exist nonrandom vectors $s_n \in \mathbb{R}^d$ such that

$$n^{-E}(X_1 + \ldots + X_n) - s_n \Rightarrow Y \tag{7}$$

as $n \to \infty$, we say that X belongs to the *domain of normal attraction* of Y (resp. ν) and write $X \in \mathrm{DONA}(Y)$. Domains of normal attraction were characterized by Jurek (1980) who also showed that $\mathrm{DONA}(Y)$ does not depend on the particular choice of the exponent $E \in \mathcal{E}(\nu)$.

Let ρ be a nondegenerate α-stable probability measure on $\mathbb{R}$ and assume that $U, U_1, U_2, \ldots$ are i.i.d. random variables. We say that U belongs to the *domain of attraction* of a random variable Z with distribution ρ, if there exist $a_n > 0$ and shifts $s_n \in \mathbb{R}$ such that

$$a_n(U_1 + \ldots + U_n) - s_n \Rightarrow Z. \tag{8}$$

In this case we write $U \in \mathrm{DOA}(Z)$. It is a classical result that $\mathrm{DOA}(Z) \neq \emptyset$ if and only if Z has an α-stable distribution.

Similarly, let ρ be a nondegenerate (b, c) semistable distribution for some $c > 1$. We say that U belongs to the *domain of semistable attraction* of a random variable Z with distribution ρ, if there exist a sequence k_n of natural numbers tending to infinity with $k_{n+1}/k_n \to c$ as $n \to \infty$, $a_n > 0$ and shifts $s_n \in \mathbb{R}$ such that

$$a_n(U_1 + \ldots + U_{k_n}) - s_n \Rightarrow Z. \tag{9}$$

We write $U \in \mathrm{DOSA}(Z)$. It is shown in Pillai (1971) that $\mathrm{DOSA}(Z) \neq \emptyset$ if and only if Z has a (b, c) semistable distribution.

Now we come to the main result of this paper. Given an operator stable random vector Y with distribution ν, we construct a basis in which, for any $X \in \mathrm{DONA}(Y)$, there is a complete set of one–dimensional marginals along the coordinate axes, each of which is attracted to some stable or semistable law on $\mathbb{R}$. This basis does not depend on which X we choose, and the limiting stable distributions are themselves one–dimensional marginals of Y as identified in Theorem 1. Since $Y \in \mathrm{DONA}(Y)$ this result also applies *a fortiori* to operator stable laws.

THEOREM 2. Let Y be a random vector with a full operator stable distribution ν and let $E \in \mathcal{E}(\nu)$ be a commuting exponent. Let $X \in \mathrm{DONA}(Y)$. Then, using the decomposition of $\mathbb{R}^d$ and E in (4) and (5) above we have: For $i = 1, \dots, s$ and nonzero $\theta_0 \in U_i^*$ we have either
(a) $\langle X, \theta_0 \rangle \in \mathrm{DOA}(\langle Y, \bar{\theta}_0 \rangle)$ for some unit vector $\bar{\theta}_0 \in \mathrm{Kern} N_i^*$ if $S_i = a_i I$;
or
(b) $\langle X, \theta_0 \rangle \in \mathrm{DOSA}(\langle Y, \bar{\theta}_0 \rangle)$ for some unit vector $\bar{\theta}_0 \in \mathrm{Kern} N_i^*$,
if $S_i = \mathrm{diag}(B, \dots, B)$, where $B = \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}$. Here $\langle Y, \bar{\theta}_0 \rangle$ is $(e^{2\pi a_i / b_i}, e^{2\pi / b_i})$ semistable as in Theorem 1.

PROOF. Assume first that $S_i = a_i I$ for some $a_i \geq 1/2$. Since $E_i^* = S_i^* + N_i^*$ and S_i^* and N_i^* commute we get $n^{E^*} \theta_0 = n^{a_i} n^{N_i^*} \theta_0$. Choose $j \geq 1$ such that $N_i^{*(j-1)} \theta_0 \neq 0$ but $N_i^{*j} \theta_0 = 0$. Then

$$n^{N_i^*} \theta_0 = \theta_0 + (\log n) N_i^* \theta_0 + \ldots + \frac{1}{(j-1)!} (\log n)^{j-1} N_i^{*(j-1)} \theta_0$$

and hence

$$\frac{n^{N_i^*} \theta_0}{(\log n)^{j-1}} \to \frac{1}{(j-1)!} N_i^{*(j-1)} \theta_0 = \bar{r}_0 \bar{\theta}_0 \tag{10}$$

for some unit vector $\bar{\theta}_0$ and some $\bar{r}_0 > 0$. Note that since $N_i^* \bar{\theta}_0 = 1/(\bar{r}_0 (j-1)!) N_i^{*j} \theta_0 = 0$ we have $\bar{\theta}_0 \in \mathrm{Kern} N_i^*$. Now write $n^{E^*} \theta_0 = n^{a_i} n^{N_i^*} \theta_0 = r_n^{-1} \theta_n$ for some $r_n > 0$ and $\|\theta_n\| = 1$. If we write $n^{N_i^*} \theta_0 = \rho_n \theta_n$ we get from (10) that $\theta_n \to \bar{\theta}_0$. Furthermore $r_n^{-1} = n^{a_i} \rho_n$. Then (7) implies

$$\begin{aligned} & r_n \Big(\langle X_1, \theta_0 \rangle + \ldots + \langle X_n, \theta_0 \rangle \Big) - r_n \langle s_n, n^{E^*} \theta_0 \rangle \\ =& r_n \big(\langle X_1 + \ldots + X_n, \theta_0 \rangle - \langle s_n, n^{E^*} \theta_0 \rangle \big) \\ =& r_n \big\langle n^{-E} (X_1 + \ldots + X_n) - s_n, n^{E^*} \theta_0 \big\rangle \\ =& \big\langle n^{-E} (X_1 + \ldots + X_n) - s_n, \theta_n \big\rangle \Rightarrow \langle Y, \bar{\theta}_0 \rangle \end{aligned}$$

using Billingsley (1968), Theorem 5.5. Note that by Theorem 1 (a) $\langle Y, \bar{\theta}_0 \rangle$ is nondegenerate stable with index $1/a_i$.

Assume now that $S_i = \mathrm{diag}(B, \ldots, B)$ where $B = \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}$ for some $a_i \geq 1/2$ and $b_i > 0$. Note that in this case

$$n^{E^*}\theta_0 = n^{E_i^*}\theta_0 = n^{a_i}R(b_i \log n)n^{N_i^*}\theta_0$$

where $R(\cdot)$ is a rotation of the U_i space as in the proof of Theorem 1.

For $n \geq 1$ define

$$k_n = \inf\{k \geq 1 : b_i \log k \geq 2\pi n\}.$$

Then it follows that $k_{n+1}/k_n \to \exp(2\pi/b_i) = c > 1$. Furthermore, if we write $b_i \log k_n = 2\pi n + \delta_n$ for some $\delta_n \geq 0$, we get from the definition of k_n that

$$\begin{aligned} 0 \leq \delta_n &= b_i \log k_n - 2\pi n < b_i \log k_n - b_i \log(k_n - 1) \\ &= b_i \log \frac{k_n}{k_n - 1} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence $R(b_i \log k_n) = R(\delta_n) \to R(0) = I$ as $n \to \infty$.

Write $k_n^{E^*}\theta_0 = k_n^{a_i}R(b_i \log k_n)k_n^{N_i^*}\theta_0 = r_n^{-1}\theta_n$ for some $r_n > 0$ and unit vectors θ_n. Furthermore, if we set $k_n^{N_i^*}\theta_0 = \rho_n\omega_n$ for some $\rho_n > 0$ and $\|\omega_n\| = 1$ we get as in the proof of the first case that $\omega_n \to \bar{\theta}_0$ for some unit vector $\bar{\theta}_0 \in \mathrm{Kern}N_i^*$. But $r_n^{-1} = k_n^{a_i}\rho_n$ and $\theta_n = R(b_i \log k_n)\omega_n \to \bar{\theta}_0$ as $n \to \infty$.

Then, by (7) and Theorem 5.5 of Billingsley (1968) we get

$$\begin{aligned} & r_n\big(\langle X_1, \theta_0\rangle + \ldots + \langle X_{k_n}, \theta_0\rangle\big) - r_n\langle s_{k_n}, k_n^{E^*}\theta_0\rangle \\ =& r_n\big(\langle X_1 + \ldots + X_{k_n}, \theta_0\rangle - \langle s_{k_n}, k_n^{E^*}\theta_0\rangle\big) \\ =& r_n\big\langle k_n^{-E}(X_1 + \ldots + X_{k_n}) - s_{k_n}, k_n^{E^*}\theta_0\big\rangle \\ =& \big\langle k_n^{-E}(X_1 + \ldots + X_{k_n} - s_{k_n}, \theta_n\big\rangle \Rightarrow \langle Y, \bar{\theta}_0\rangle \end{aligned}$$

as $n \to \infty$ showing that $\langle X, \theta_0\rangle \in \mathrm{DOSA}(\langle Y, \bar{\theta}_0\rangle)$. Note that by Theorem 1(b) $\langle Y, \bar{\theta}_0\rangle$ is (b, c) semistable with $c = \exp(2\pi/b_i) > 1$. This concludes the proof.

REMARK: Since $Y \in \mathrm{DONA}(Y)$ for any full operator stable distribution Theorem 2 applies to Y showing that $\langle Y, \theta_0\rangle$ for any nonzero $\theta_0 \in U_i^*$ and any $i = 1, \ldots, s$ belongs to some domain of (semistable) attraction.

COROLLARY 1. In the situation of Theorem 2 there exists a basis $\{\theta_1, \ldots, \theta_d\}$ of $\mathbb{R}^d$ such that $\langle X, \theta_i\rangle$ belongs to the domain of (semistable) attraction of some non-degenerate (semi) stable law on $\mathbb{R}$.

PROOF. For $i = 1, \ldots, s$ let $\{\theta_1^{(i)}, \ldots, \theta_{d_i}^{(i)}\}$, $d_i = \dim U_i^*$ be a basis of U_i^*. Then the union of these basis vectors form a basis of $\mathbb{R}^d$ which by Theorem 2 has the desired property.

REMARK: The example in Marcus (1983) shows that as for multivariate stable laws in our general situation we can not expect to characterize operator stable laws by their one–dimensional marginals.

We next consider the special case of operator stable laws with semisimple exponents E. It is shown in Theorem 2.1 of Meerschaert and Veeh (1993) that the nilpotent part of every exponent $E \in \mathcal{E}(\nu)$ is the same. (The statement of that theorem contains an obvious typographical error.) Hence, if the nilpotent part of an exponent $E \in \mathcal{E}(\nu)$ is zero then every exponent of ν is semisimple. In that case, when we reduce to one–dimensional methods by projecting onto the coordinate axes in the appropriate basis, the limit laws are obtained by projecting the limiting random vector Y onto these same coordinate axes.

COROLLARY 2. Let Y be a random vector with a full operator stable distribution ν and let $E = S \in \mathcal{E}(\nu)$ be a semisimple commuting exponent. Let $X \in \mathrm{DONA}(Y)$. Then, using the decomposition of $\mathbb{R}^d$ and E in (4) and (5) above we have: For $i = 1, \dots, s$ and nonzero $\theta_0 \in U_i^*$ we have either
(a) $\langle X, \theta_0\rangle \in \mathrm{DOA}(\langle Y, \theta_0\rangle)$ if $S_i = a_i I$, where $\langle Y, \theta_0\rangle$ has a nondegenerate $1/a_i$-stable law.
or
(b) $\langle X, \theta_0\rangle \in \mathrm{DOSA}(\langle Y, \theta_0\rangle)$ if $S_i = \mathrm{diag}(B, \dots, B)$, where $B = \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}$. Here $\langle Y, \theta_0\rangle$ is nondegenerate $(e^{2\pi a_i/b_i}, e^{2\pi/b_i})$ semistable.

PROOF. Since $N = 0$, $\mathrm{Kern} N_i^* = U_i^*$ for $i = 1, \dots, s$ so by Theorem 1 $\langle Y, \theta_0\rangle$ is nondegenerate and either stable with index $1/a_i$ or $(e^{2\pi a_i/b_i}, e^{2\pi/b_i})$ semistable.

For $X \in \mathrm{DONA}(Y)$ we get (10) for $j = 1$ and hence $\bar{r}_0 = 1$ and $\bar{\theta}_0 = \theta_0 = \theta_n$ for all $n \geq 1$ in the proof of the first part of Theorem 2. Then the argument of that part shows that $\langle X, \theta_0\rangle \in \mathrm{DOA}(\langle Y, \theta_0\rangle)$.

In the proof of the second part of Theorem 2 we get in our present situation that $\rho_n = 1$ and $\omega_n = \theta_0$ for all $n \geq 1$. Hence $r_n^{-1} = k_n^{a_i}$ and $\theta_n = R(b_i \log k_n)\theta_0 \to \theta_0$ as $n \to \infty$. Then as before it follows that $\langle X, \theta_0\rangle \in \mathrm{DOSA}(\langle Y, \theta_0\rangle)$. This concludes the proof.

4. Generalized domains of attraction. We now investigate the one–dimensional marginals of laws belonging to the generalized domain of attraction of a full operator stable law. In this most general situation, all we can show is a stochastic compactness result. We say that a sequence of random variables $(Z_n)_n$ is *stochastically compact* if the laws of Z_n are weakly relatively compact and all limit laws are nondegenerate.

THEOREM 3. Let Y be a random vector with a full operator stable distribution ν. Let $X \in \mathrm{GDOA}(\nu)$ and assume that $X_1, X_2, \dots$ are i.i.d. as X. Then for all nonzero $\theta \in \mathbb{R}^d$ there exist $r_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\left(r_n \sum_{i=1}^{n} \langle X_i, \theta\rangle - b_n \right)_{n \geq 1}$$

is stochastically compact. Moreover, the limit set is contained in the set $\{\langle Y, \theta_0\rangle : \theta_0 \neq 0\}$ of all one–dimensional marginals of Y.

PROOF. Fix any nonzero $\theta \in \mathbb{R}^d$ and write $(A_n^*)^{-1}\theta = r_n^{-1}\theta_n$ for some $\|\theta_n\| = 1$ and some $r_n > 0$ where A_n are the norming operators for X in (2). Using the compactness of the unit sphere in $\mathbb{R}^d$ any sequence (n') contains a further sequence $(n'') \subset (n')$ such that $\theta_n \to \theta_0$ along (n''). Since

$$\langle X_i, \theta\rangle = \langle A_n X_i, (A_n^*)^{-1}\theta\rangle = r_n^{-1}\langle A_n X_i, \theta_n\rangle$$

we get if we let $b_n = \langle s_n, \theta_n\rangle$ that

$$\begin{aligned} r_n \sum_{i=1}^n \langle X_i, \theta\rangle - b_n &= \sum_{i=1}^n \langle A_n X_i, \theta_n\rangle - \langle s_n, \theta_n\rangle \\ &= \Big\langle A_n \sum_{i=1}^n X_i - s_n, \theta_n \Big\rangle \Rightarrow \langle Y, \theta_0\rangle \end{aligned}$$

along (n'') using Billingsley (1968), Theorem 5.5. Since Y is full, $\langle Y, \theta_0\rangle$ is nondegenerate for all $\theta_0 \neq 0$. This concludes the proof.

REFERENCES

Billingsley, P. (1968) *Convergence of Probability measures,* Wiley, New York.

Giné, E. and M.G. Hahn (1983) On Stability of Probability Laws with Univariate Stable Marginals, *Z. Wahrsch. verw. Geb.*, **64**, 157–165.

Hahn, M. and M. Klass (1985) Affine Normability of Partial Sums of i.i.d. Random Vectors: A Characterization, *Z. Wahrsch. verw. Geb.*, **69**, 479–505.

Hoffman, K. and R. Kunze (1961) *Linear Algebra*, Prentice Hall, New Jersey.

Holmes, J., W. Hudson and J.D. Mason (1982) Operator stable laws: multiple exponents and elliptical symmetry, *Ann. Probab.* **10**, 602–612.

Hudson, W., Z. Jurek, and J.A Veeh (1986) The Symmetry Group and Exponents of Operator Stable Probability Measures, *Ann. Probab.* **14**, 1014–1023.

Jurek, Z.J. (1980) Domains of normal attraction of operator-stable measures on Euclidean spaces, *Bull. Acad. Polon. Sci. Ser. Sci. Math.* **28**, 397–409.

Jurek, Z. and J. D. Mason (1993) *Operator–Limit Distributions in Probability Theory*, Wiley, New York.

Kruglov, V.M. (1972) On the extension of the class of stable distributions, *Theory Probab. Appl.* **17**, 685–694.

Marcus, D.J. (1983) Non-Stable Laws with all Projections Stable, *Z. Wahrsch. verw. Geb.*, **64**, 139–156.

Meerschaert, M.M. (1990) Moments of random vectors which belong to some domain of normal attraction, *Ann. Probab.*, **18**, 870–876.

Meerschaert, M.M. (1991) Spectral decomposition for generalized domains of attraction, *Ann. Probab.* **19**, 875–892.

Meerschaert, M.M. and J.A. Veeh (1993) The Structure of the Exponents and Symmetries of an Operator Stable Measure, *J. Theoretical Probab.* **6**, 713–726.

Meerschaert, M.M. and H.P. Scheffler (1996) Series representation for semistable laws and their domains of semistable attraction, *J. Theoretical Probab.* **9**, 931–959.

Pillai, R. (1971) Semi stable laws as limit distributions, *Ann. Math. Stat.* **42**, 780–783.

Samorodnitsky, G. and M.S. Taqqu (1994) *Stable non-Gaussian Random Processes,* Chapman and Hall, New York.

Scheffler, H.P. (1994) Domains of semi-stable attraction of nonnormal semi-stable laws, *J. Multivariate Analysis* **51**, 432–444.

Sharpe, M. (1969) Operator-Stable Probability Distributions on Vector Groups, *Trans. Amer. Math. Soc.* **136**, 51–65.

Shimizu, R. (1970) On the domain of partial attraction of semi-stable distributions, *Ann. Inst. Statist. Math.* **22**, 245–255.

Zolotarev, V.M. (1986) *One–Dimensional Stable Distributions,* Vol. 65 of "Translations of mathematical monographs", AMS.