MOVING AVERAGES OF RANDOM VECTORS WITH REGULARLY VARYING TAILS

By Mark M. Meerschaert and Hans-Peter Scheffler

University of Nevada and University of Dortmund

First version received October 1998

Abstract. Regular variation is an analytic condition on the tails of a probability distribution which is necessary for an extended central limit theorem to hold, when the tails are too heavy to allow attraction to a normal limit. The limiting distributions which can occur are called operator stable. In this paper we show that moving averages of random vectors with regularly varying tails are in the generalized domain of attraction of an operator stable law. We also prove that the sample autocovariance matrix of these moving averages is in the generalized domain of attraction of an operator stable law on the vector space of symmetric matrices.

AMS 1990 subject classification. Primary 62M10, secondary 62E20, 62F12, 60F05.

Keywords. Moving average, autocovariance matrix, operator stable, regular variation, random matrix.

1. INTRODUCTION

In this paper we establish the basic asymptotic theory for moving averages of random vectors with heavy tails. Heavy tail distributions occur frequently in applications to finance, geology, physics, chemistry, computer and systems engineering. The recent books of Samorodnitsky and Taqqu (1994), Janicki and Weron (1994), Nikias and Shao (1995), and Mittnik and Rachev (1998) review many of these applications. Scalar time series with heavy tails are discussed in Anderson and Meerschaert (1997), Bhansali (1993), Brockwell and Davis (1991), Davis and Resnick (1985a, 1985b, 1986), Jansen and de Vries (1991), Kokoszka and Taqqu (1994, 1996), Loretan and Phillips (1994), Mikosch, Gadrich, Klüppenberg and Adler (1995), and Resnick and Stărică (1995). Modern applications of heavy tail distributions were pioneered by Mandelbrot (1963) and others in connection with problems in finance. When the probability tails of the random fluctuations in a time series model are sufficiently light, the asymptotics are normal. But when the tails are sufficiently heavy that the fourth moments fail to exist, the asymptotics are governed by the extended central limit theorem. For scalar models the limiting distributions are stable. Stable laws are characterized by the fact that sums of independent stable random variables are also stable, and the distribution of the sum can be reduced to that of any summand by an appropriate affine normalization. The normal laws are a special case of stable

0143-9782/00/04 297–328 JOURNAL OF TIME SERIES ANALYSIS Vol. 21, No. 3
© 2000 Blackwell Publishers Ltd., 108 Cowley Road, Oxford OX4 1JF, UK and 350 Main Street, Malden, MA 02148, USA.

laws. Stable stochastic processes are interesting because they provide the most straightforward mechanism for generating random fractals.

For random vectors with heavy tails, the extended central limit theorem yields operator stable limit laws, so called because the affine normalization which reduces the distribution of a sum to that of one summand involves a linear operator. Regular variation is the analytic condition necessary for the extended central limit theorem to hold for heavy tails. See Feller (1971) Chapter XVII for the scalar version and Meerschaert (1993) for the vector version of the extended central limit theorem. In this paper we will show that moving averages of random vectors with regularly varying tail probabilities are asymptotically operator stable. The regular variation arguments at the heart of the proof will appear familiar to any reader who is acquainted with the work of Feller on regular variation. We will also show, using similar regular variation methods, that the sample autocovariance matrix formed from these moving averages is asymptotically operator stable as a random element of the vector space of $d \times d$ symmetric matrices.

2. NOTATION AND PRELIMINARY RESULTS

In this section we present the notion of a regularly varying measure together with the multivariable theory of regular variation necessary in the proofs of our main results. The connection of regularly varying measures to generalized domains of attraction of operator stable laws is also discussed.

Assume that $\{Z_n\}$ are i.i.d. random vectors on $\mathbb{R}^d$ with common distribution μ. A sequence (A_n) of invertible linear operators on $\mathbb{R}^d$ is called regularly varying with index F, where F is a $d \times d$ matrix, if

$$A_{[\lambda n]} A_n^{-1} \to \lambda^F \text{ as } n \to \infty \tag{2.1}$$

where $\lambda^F = \exp(F \log \lambda)$ and exp is the exponential mapping for $d \times d$ matrices. We write $(A_n) \in \mathrm{RV}(F)$ if (2.1) holds. We say that μ is regularly varying with exponent E if there exists a sequence $(A_n) \in \mathrm{RV}(-E)$ such that

$$n(A_n \mu) \to \phi \tag{2.2}$$

where ϕ is some σ-finite Borel measure on $\mathbb{R}^d \backslash \{0\}$ which cannot be supported on any lower dimensional subspace. Note that $t \cdot \phi = (t^E \phi)$ follows. Here $(A\phi)(dx) = \phi(A^{-1}\, dx)$ denotes the image measure. The convergence in (2.2) means that $n\mu(A_n^{-1} S) \to \phi(S)$ for any Borei set S of $\mathbb{R}^d$ which is bounded away from the origin, and whose boundary has ϕ-measure zero. Note that this is the vague convergence on the set $\overline{\mathbb{R}}^d \backslash \{0\}$ where $\overline{\mathbb{R}}^d$ is the one-point compactification of $\mathbb{R}^d$. For more information on multivariable regular variation see Meerschaert (1993), Meerschaert and Scheffler (1999) and Scheffler (1998).

Regular variation is an analytic tail condition which is necessary for an extended central limit theorem to apply. We say that $\{Z_n\}$ belongs to the generalized domain of attraction of some full-dimensional limit Y if

© Blackwell Publishers Ltd 2000

$$A_n(Z_1 + \cdots + Z_n - nb_n) \Rightarrow Y \tag{2.3}$$

for some linear operators A_n and nonrandom centering vectors b_n. If $E\|Z_n\|^2 < \infty$ then Y is multivariate normal and we can take $A_n = n^{-1/2}$ and $b_n = EZ_1$. Meerschaert (1993) together with Meerschaert (1994) shows that (2.3) holds with a nonnormal limit if and only if the distribution μ is regularly varying with exponent E, where every eigenvalue of E has real part exceeding $\frac{1}{2}$.

Sharpe (1969) characterized operator stable laws in terms of transforms. The characteristic function of any random vector Y whose probability distribution is infinitely divisible can be written in the form $Ee^{i\langle Y,s\rangle} = e^{\psi(s)}$ where

$$\psi(s) = i\langle a, s\rangle - \frac{1}{2}s'Ms + \int_{x\neq 0} e^{\langle s,x\rangle} - 1 - \frac{i\langle s, x\rangle}{1 + \langle x, x\rangle}\phi(dx).$$

Here $a \in \mathbb{R}^d$, M is a symmetric $d \times d$ matrix, and ϕ is a Lévy measure, i.e. a σ-finite Borel measure on $\mathbb{R}^d\backslash\{0\}$ which assigns finite measure to sets bounded away from the origin and which satisfies $\int \|x\|^2 I(0 < \|x\| \leqslant 1)\phi(dx) < \infty$. We say that Y has Lévy representation $[a, M, \phi]$. If $\phi = 0$ then Y is multivariate normal with covariance matrix M. A nonnormal operator stable law has Lévy representation $[a, 0, \phi]$ where $t\phi = t^E\phi$ for all $t > 0$ and every eigenvalue of the exponent E has real part exceeding $\frac{1}{2}$. If μ varies regularly with exponent E, where every eigenvalue of E has real part exceeding $\frac{1}{2}$, then the limit measure ϕ in (2.2) is also the Lévy measure of Y.

Let μ be regularly varying with exponent E and let $\lambda = \min\{\Re(\alpha)\}$, $\Lambda = \max\{\Re(\alpha)\}$ where α ranges over the eigenvalues of E. Meerschaert (1993) shows that in this case the moment functions

$$U_\zeta(r, \theta) = \int_{|\langle x,\theta\rangle|\leqslant r} |\langle x, \theta\rangle|^\zeta \mu(dx)$$

$$V_\eta(r, \theta) = \int_{|\langle x,\theta\rangle| > r} |\langle x, \theta\rangle|^\eta \mu(dx)$$

are uniformly R–O varying whenever $\eta < 1/\Lambda \leqslant 1/\lambda < \zeta$. A Borel measurable function $R(r)$ is R–O varying if it is real-valued and positive for $r \geqslant A$ and if there exist constants $a > 1$, $0 < m < 1$, $M > 1$ such that $m \leqslant R(tr)/R(r) \leqslant M$ whenever $1 \leqslant t \leqslant a$ and $r \geqslant A$. Then $R(r, \theta)$ is uniformly R–O varying if it is an R–O varying function of r for each θ, and the constants A, a, m, M can be chosen independent of θ. See Seneta (1976) for more information on R–O variation.

In particular it is shown in Meerschaert (1993) (see also Scheffler (1998) for a more general case) that for any $\delta > 0$ there exist real constants m, M, r_0 such that

$$\frac{V_\eta(tr, \theta)}{V_\eta(r, \theta)} \geqslant mt^{\eta - 1/\lambda - \delta}$$

© Blackwell Publishers Ltd 2000

$$\frac{U_\zeta(tr, \theta)}{U_\zeta(r, \theta)} \leqslant M t^{\zeta - 1/\Lambda + \delta}$$

for all $\|\theta\| = 1$, all $t \geqslant 1$ and all $r \geqslant r_0$. A uniform version of Feller (1971) p. 289 (see Scheffler (1998) for a detailed proof) yields that for some positive real constants A, B, t_0 we have

$$A \leqslant \frac{t^{\zeta-\eta} V_\eta(t, \theta)}{U_\zeta(t, \theta)} \leqslant B$$

for all $\|\theta\| = 1$ and all $t \geqslant t_0$.

If μ is regularly varying with index E then Meerschaert and Scheffler (1997) show that there exists a unique direct sum decomposition $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ and real numbers $0 < \lambda = a_1 < \cdots < a_p = \Lambda$ with the following properties: the subspaces $V_1 \cdots V_p$ are mutually orthogonal; for any nonzero vector $\theta \in V_i$ the marginal absolute moment $E|\langle Z_n, \theta\rangle|^\rho$ exists for $\rho < 1/a_i$ and this moment is infinite for $\rho > 1/a_i$; $P_i A_n = A_n P_i$ and $P_i E = E P_i$ where P_i is orthogonal projection onto V_i; every eigenvalue of $P_i E$ has real part equal to a_i; for any nonzero vector $x \in \mathbb{R}^d$ we have $\log\|P_i A_n x\|/\log n \to -a_i$ (uniformly on compact subsets); and $\log\|P_i A_n\|/\log n \to -a_i$. This is called the spectral decomposition.

The spectral decomposition implies that the distribution $P_i\mu$ of the random vector $P_i Z_n$ on V_i varies regularly with exponent $P_i E$. Since every eigenvalue of the exponent $P_i E$ has real part equal to a_i, we can also apply the above R–O variation results whenever $\theta \in V_i$ and $\eta < 1/a_i < \zeta$. For a linear operator A let A^t denote its transpose.

LEMMA 2.1. *Assume that the distribution μ of Z is regularly varying with index E; let $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ be the spectral decomposition of $\mathbb{R}^d$ and let $0 < a_1 < \ldots < a_p$ denote the corresponding real parts of the eigenvalues of E. Then for any $i = 1, \ldots, p$, given $0 < \eta < 1/a_i < \zeta$ there exists n_0 and constants $K_1, \ldots, K_4 > 0$ such that*

$$\begin{aligned}
&(i) \quad nE|\langle A_n Z, \theta\rangle|^\zeta I(|\langle A_n Z, \theta\rangle| \leqslant 1) < K_1 \\
&(ii) \quad nE|\langle A_n Z, \theta\rangle|^\zeta I(\|A_n Z\| \leqslant 1) < K_2 \\
&(iii) \quad nE|\langle A_n Z, \theta\rangle|^\eta I(|\langle A_n Z, \theta\rangle| > 1) < K_3 \\
&(iv) \quad nE|\langle A_n Z, \theta\rangle|^\eta I(\|A_n Z\| > 1) < K_4
\end{aligned} \tag{2.4}$$

for all $n \geqslant n_0$ and all $\|\theta\| = 1$ in V_i.

PROOF. Suppose we are given an arbitrary sequence of unit vectors θ_n in V_i, and write $r_n x_n = A_n^t \theta_n$ where $r_n > 0$ and $\|x_n\| = 1$. Then we have

© Blackwell Publishers Ltd 2000

$$
\begin{aligned}
nE|\langle A_nZ,\,\theta_n\rangle|^{\zeta} I(|\langle A_nZ,\,\theta_n\rangle| \leqslant 1) &= nE|\langle Z,\,A_n^t\theta_n\rangle|^{\zeta} I(|\langle Z,\,A_n^t\theta_n\rangle| \leqslant 1)\\
&= nE|\langle Z,\,r_nx_n\rangle|^{\zeta} I(|\langle Z,\,r_nx_n\rangle| \leqslant 1)\\
&= nr_n^{\zeta} E|\langle Z,\,x_n\rangle|^{\zeta} I(|\langle Z,\,x_n\rangle| \leqslant r_n^{-1})\\
&= nr_n^{\zeta} U_{\zeta}(r_n^{-1},\,x_n)
\end{aligned}
$$

Since $\zeta > 1/a_i$ both $U_{\zeta}(r,\,\theta)$ and $V_0(r,\,\theta)$ are uniformly R–O varying on compact subsets of $\theta \neq 0$ in V_i. Then for some m, M, t_0 we have

$$
m \leqslant \frac{t^{\zeta} V_0(t,\,\theta)}{U_{\zeta}(t,\,\theta)} \leqslant M \tag{2.5}
$$

for all $\|\theta\| = 1$ in V_i and all $t \geqslant t_0$. Since (2.2) holds where ϕ cannot be supported on any lower dimensional subspace we must have $\|A_n\| \to 0$. Then $\|A_n\theta\| \to 0$ uniformly on compact subsets of $\mathbb{R}^d \backslash \{0\}$, and so for some n_0 we have $\|A_n\theta\| \leqslant t_0^{-1}$ for all $\|\theta\| = 1$ and all $n \geqslant n_0$. Then for all $n \geqslant n_0$ we have

$$
\begin{aligned}
nr_n^{\zeta} U_{\zeta}(r_n^{-1},\,x_n) &\leqslant m^{-1} nV_0(r_n^{-1},\,x_n)\\
&= m^{-1} nP(|\langle Z,\,x_n\rangle| > r_n^{-1})\\
&= m^{-1} nP(|\langle Z,\,r_nx_n\rangle| > 1)\\
&= m^{-1} nP(|\langle Z,\,A_n^t\theta_n\rangle| > 1)\\
&= m^{-1} nP(|\langle A_nZ,\,\theta_n\rangle| > 1)\\
&\leqslant m^{-1} nP(\|A_nZ\| > 1)
\end{aligned}
$$

as $n \to \infty$, and this upper bound holds independent of our choice of the sequence θ_n. By (2.2) we have $nP(\|A_nZ\| > 1) = nA_n\mu\{z\colon \|z\| > 1\} \to \phi\{z\colon \|z\| > 1\}$ (if $\{z\colon \|z\| > 1\}$ is not a continuity set of ϕ then we can use the upper bound $nP(\|A_nZ\| > r)$ instead, where $0 < r < 1$). Then the sequence of real numbers $nP(\|A_nZ\| > 1)$ is bounded above by some $K > 0$, and assertion (i) of (2.4) holds with $K_1 = m^{-1}K$. Since

$$
nE|\langle A_nZ,\,\theta_n\rangle|^{\zeta} I(\|A_nZ\| \leqslant 1) \leqslant nE|\langle A_nZ,\,\theta_n\rangle|^{\zeta} I(|\langle A_nZ,\,\theta_n\rangle| \leqslant 1)
$$

we immediately obtain assertion (ii) with $K_2 = K_1$. Now write

© Blackwell Publishers Ltd 2000

$$\begin{aligned} nE|\langle A_n Z, \theta_n\rangle|^\eta I(|\langle A_n Z, \theta_n\rangle| > 1) &= nE|\langle Z, A_n^t \theta_n\rangle|^\eta I(|\langle Z, A_n^t\theta_n\rangle| > 1) \\ &= nE|\langle Z, r_n x_n\rangle|^\eta I(|\langle Z, r_n x_n\rangle| > 1) \\ &= n r_n^\eta E|\langle Z, x_n\rangle|^\eta I(|\langle Z, x_n\rangle| > r_n^{-1}) \\ &= n r_n^\eta V_\eta(r_n^{-1}, x_n) \end{aligned}$$

Since $\eta < 1/a_i$ we also have $V_\eta(r, \theta)$ uniformly R–O varying on compact subsets of $\theta \neq 0$ in V_i. Then for some m', M', t_0 we have

$$m' \leqslant \frac{t^{\zeta-\eta} V_\eta(t, \theta)}{U_\zeta(t, \theta)} \leqslant M' \tag{2.6}$$

for all $\|\theta\| = 1$ in V_i and all $t \geqslant t_0$. Choosing t_0 large enough so that both (2.5) and (2.6) hold whenever $t \geqslant t_0$ and $\|\theta\| = 1$ in V_i, and then choosing $n_0 = n_0(t_0)$ such that $\|A_n\theta\| \leqslant t_0^{-1}$ for all $\|\theta\| = 1$ whenever $n \geqslant n_0$, we have for all $n \geqslant n_0$ that

$$r_n^\eta V_\eta(r_n^{-1}, x_n) \leqslant M' n r_n^\zeta U_\zeta(r_n^{-1}, x_n)$$

and so assertion (iii) of (2.4) holds with $K_3 = M' K_1$. Finally write

$$\begin{aligned} nE|\langle A_n Z, \theta_n\rangle|^\eta I(\|A_n Z\| > 1) &= nE|\langle A_n Z, \theta_n\rangle|^\eta I(|\langle A_n Z, \theta_n\rangle| > 1) \\ &\quad + nE|\langle A_n Z, \theta_n\rangle|^\eta I(\|A_n Z\| > 1 \text{ and } |\langle A_n Z, \theta_n\rangle| \leqslant 1) \\ &\leqslant nE|\langle A_n Z, \theta_n\rangle|^\eta I(|\langle A_n Z, \theta_n\rangle| > 1) + nP(\|A_n Z\| > 1) \end{aligned}$$

so that assertion (iv) of (2.4) holds with $K_4 = K_3 + K$. This concludes the proof of Lemma 2.1.

3. MOVING AVERAGES

Suppose that $\{Z_j\}_{j=-\infty}^{\infty}$ is a double sequence of i.i.d. random vectors with common distribution μ varying regularly with index E, i.e. (2.2) holds. Define the moving average process

$$X_t = \sum_{j=-\infty}^{\infty} C_j Z_{t-j} \tag{3.1}$$

where C_j are $d \times d$ real matrices such that for each j either $C_j = 0$ or else C_j^{-1} exists and $A_n C_j = C_j A_n$ for all n. The spectrum of the exponent of regular variation E is connected with the tail behavior of the random vectors Z_n. Let $\Lambda = \max\{\Re(\alpha)\}$ and $\lambda = \min\{\Re(\alpha)\}$ where α ranges over the eigenvalues of E as before. Lemma 2 of Meerschaert (1993) or Scheffler (1998), Corollary 4.21 implies that $E\|Z_n\|^\rho$ exists for $0 < \rho < 1/\Lambda$ and is infinite for $\rho > 1/\lambda$. Then the following lemma implies that the moving average (3.1) is well-defined as long as

© Blackwell Publishers Ltd 2000

$$\sum_{j=-\infty}^{\infty} \|C_j\|^{\delta} < \infty \tag{3.2}$$

for some $\delta < 1/\Lambda$ with $\delta \leqslant 1$. Here $\|A\|$ denotes the operator norm of a linear operator A on $\mathbb{R}^d$. For the remainder of the paper we will assume that this is the case, so that the moving average (3.1) is well-defined.

LEMMA 3.1. *Suppose that* $E\|Z_n\|^{\alpha} < \infty \, \forall \alpha < \alpha_0$ *and that (3.2) holds for some* $\delta < \alpha_0$, $\delta \leqslant 1$. *Then (3.1) exists almost surely.*

PROOF. First suppose $\alpha_0 > 1$ so that $E\|Z_n\| < \infty$. Then $\sum \|C_j\|^{\delta} < \infty$ implies that $\sum \|C_j\| < \infty$ because $\|C_j\|^{\delta} \to 0$ since the series converges, so $\|C_j\| < 1$ for all large j, and for such j we have $\|C_j\| \leqslant \|C_j\|^{\delta}$.

Then we have $\|X_t\| = \|\sum_j C_j Z_{t-j}\| \leqslant \sum \|C_j Z_{t-j}\| \leqslant \sum \|C_j\| \cdot \|Z_{t-j}\|$ so that $E\|X_t\| \leqslant E\sum \|C_j\| \, \|Z_{t-j}\| = \sum E\|C_j\| \, \|Z_{t-j}\| = \sum \|C_j\| E\|Z_{t-j}\| = E\|Z_1\| \sum \|C_j\| < \infty$ by monotone convergence and then X_t exists almost surely. To see this, note that $\tilde{X} = \sum \|C_j\| \, \|Z_{t-j}\|$ is a well-defined random variable which is nonnegative and has a finite mean and so it is almost surely finite, i.e. the sum in (3.1) converges absolutely with probability one.

If $\alpha_0 \leqslant 1$ then $\sum \|C_j\|^{\delta} < \infty$ for some $\delta < \alpha_0$ and $E\|Z_1\|^{\delta} < \infty$ as well. Also $|x+y|^{\delta} \leqslant |x|^{\delta} + |y|^{\delta} \, \forall x, \ y \geqslant 0$ so

$$\begin{aligned} E\|X_t\|^{\delta} &= E\left[\left\| \sum_j C_j Z_{t-j} \right\|^{\delta}\right] \\ &\leqslant E\left[\left(\sum_j \|C_j\| \, \|Z_{t-j}\| \right)^{\delta}\right] \\ &\leqslant E\left[\left(\sum_j \|C_j\|^{\delta} \|Z_{t-j}\|^{\delta} \right)\right] \\ &= \sum_j \|C_j\|^{\delta} E\|Z_1\|^{\delta} < \infty \end{aligned}$$

so as before X_t converges absolutely almost surely.

THEOREM 3.2. *Suppose that* X_t *is a moving average defined by (3.1) where* $\{Z_n\}$ *are i.i.d. with common distribution* μ *on* $\mathbb{R}^d$. *Suppose that* μ *varies regularly with exponent E, where every eigenvalue of E has real part exceeding* $\frac{1}{2}$. *Then for the norming operators* A_n *in (2.2) we have*

$$A_n(X_1 + \cdots + X_n - na_n) \Rightarrow U \tag{3.3}$$

where $a_n \in \mathbb{R}^d$ *and U is full-dimensional and operator stable.*

© Blackwell Publishers Ltd 2000

REMARK 3.3. *In the proof of Theorem 3.2 we show that we can take the centering constants* $a_n = \sum_j C_j b_n$ *in (3.3) where* b_n *are the centering constants from (2.3) above. The operator stable limit* $U = \sum_j C_j Y$ *in (3.3) is nonnormal with Lévy measure* $\sum_j C_j \phi$.

Before we proof Theorem 3.2 we need an additional result that shows that under the general assumptions on the moving averages parameter matrices C_j the spectral decomposition remains invariant under C_j.

LEMMA 3.4. *Suppose the* μ *is regularly varying with index E and spectral decomposition* $\mathbb{R}^d = V_1 \oplus \ldots \oplus V_p$ *and norming operators* A_n *as in (2.2). If* C_j *is invertible and* $C_j A_n = A_n C_j$ *for all n then* $C_j V_i = V_i$ *for all* $i = 1 \cdots p$.

PROOF. Let $L_i = V_i \oplus \cdots \oplus V_p$ and $\overline{L}_i = V_1 \oplus \cdots \oplus V_i$. Meerschaert and Scheffler (1998) show that for any $x \in L_i \backslash L_{i+1}$ we have $\log\|A_n x\|/\log n \to -a_i$. This convergence is uniform on compact subsets of $x \in L_i \backslash L_{i+1}$. For any $x \in \overline{L}_i \backslash \overline{L}_{i-1}$ we have $\log\|(A_n^t)^{-1}x\|/\log n \to a_i$ (uniformly on compact sets). Since C_j is invertible there exist $c_j > 0$ and $b_j < \infty$ such that $c_j\|x\| \leqslant \|C_j x\| \leqslant b_j\|x\|$ for all x. Then $\log\|C_j A_n x\|/\log n \to -a_i$ if and only if $x \in L_i \backslash L_{i+1}$, and $\log\|A_n C_j x\|/\log n \to -a_i$ if and only if $C_j x \in L_i \backslash L_{i+1}$. Since $A_n C_j = C_j A_n$ this implies that $C_j^{-1}(L_i \backslash L_{i+1}) = L_i \backslash L_{i+1}$, and so $C_j(L_i \backslash L_{i+1}) = L_i \backslash L_{i+1}$. Similarly we have $\log\|(A_n^t)^{-1}x\|/\log n \to a_i$ if and only if $x \in \overline{L}_i \backslash \overline{L}_{i-1}$ as well as $\log\|(A_n^t)^{-1}(C_j^t)^{-1}x\|/\log n \to a_i$ if and only if $(C_j^t)^{-1}x \in \overline{L}_i \backslash \overline{L}_{i-1}$. Then $(C_j^t)^{-1}(\overline{L}_i \backslash \overline{L}_{i-1}) = \overline{L}_i \backslash \overline{L}_{i-1}$ and since the subspaces V_i are mutually orthogonal it follows that $C_j(\overline{L}_i \backslash \overline{L}_{i-1}) = \overline{L}_i \backslash \overline{L}_{i-1}$. Then $C_j(L_i \cap \overline{L}_i) = L_i \cap \overline{L}_i$ and $L_i \cap \overline{L}_i = V_i$. This concludes the proof of Lemma 3.4.

PROOF OF THEOREM 3.2. Suppose that (2.3) holds and define

$$X_t^{(m)} = \sum_{|j| \leqslant m} C_j Z_{t-j}$$

$$a_n^{(m)} = \sum_{|j| \leqslant m} C_j b_n$$

$$U^{(m)} = \sum_{|j| \leqslant m} C_j Y$$

for $m \geqslant 1$, where Y is the limit in (2.3). From (2.3) it follows that

$$\left(A_n \sum_{t=1}^{n} (Z_{t-j} - b_n) \colon |j| \leqslant m \right) \Rightarrow (Y, \ldots, Y)$$

since we know that the convergence holds for the $j = 0$ component, and since the difference between this component and any other component (at most $2m$

© Blackwell Publishers Ltd 2000

terms) tends to zero in probability. Now by the continuous mapping theorem we obtain

$$A_n(X_1^{(m)} + \cdots + X_n^{(m)} - na_n^{(m)}) \Rightarrow U^{(m)}$$

and since $U^{(m)} \to U = \sum_j C_j Y$ almost surely it suffices to show (e.g. by Billingsley (1968) Theorem 4.2) that

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(\|T\| > \varepsilon) = 0$$

where

$$T = \sum_{|j|>m} \sum_{t=1}^{n} A_n C_j (Z_{t-j} - b_n).$$

For this it suffices to show that

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(|\langle T, x\rangle| > \varepsilon) = 0 \tag{3.4}$$

for any unit vector $x \in V_i$ where $1 \leqslant i \leqslant p$. Decompose $T = A + B - C$ where

$$A = \sum_{|j|>m} \sum_{t=1}^{n} A_n C_j Z_{t-j} I(\|A_n Z_{t-j}\| \leqslant 1)$$

$$B = \sum_{|j|>m} \sum_{t=1}^{n} A_n C_j Z_{t-j} I(\|A_n Z_{t-j}\| > 1)$$

$$C = \sum_{|j|>m} n A_n C_j b_n.$$

First suppose that $a_i > 1$, so that assertion (ii) of Lemma 2.1 holds with $\zeta = 1$. Write $C_j^t x = r_j x_j$ where $r_j > 0$ and $\|x_j\| = 1$. Note that $x_j \in V_i$ by Lemma 3.4. Then

© Blackwell Publishers Ltd 2000

$$\begin{aligned}P[|\langle A, x\rangle| > \varepsilon/3] &\leqslant 3\varepsilon^{-1}E|\langle A, x\rangle| \\ &\leqslant 3\varepsilon^{-1}\sum_{|j|>m} nE|\langle A_nC_jZ_1, x\rangle|I(\|A_nZ_1\| \leqslant 1) \\ &= 3\varepsilon^{-1}\sum_{|j|>m} nE|\langle C_jA_nZ_1, x\rangle|I(\|A_nZ_1\| \leqslant 1) \\ &= 3\varepsilon^{-1}\sum_{|j|>m} nE|\langle A_nZ_1, C_j^t x\rangle|I(\|A_nZ_1\| \leqslant 1) \\ &= 3\varepsilon^{-1}\sum_{|j|>m} r_j nE|\langle A_nZ_1, x_j\rangle|I(\|A_nZ_1\| \leqslant 1) \\ &\leqslant 3\varepsilon^{-1}K_2\sum_{|j|>m}\|C_j\|\end{aligned}$$

which tends to zero as $m \to \infty$ in view of the fact that (3.2) holds with $\delta < 1/a_p \leqslant 1/a_i < 1$. Choosing $0 < \delta < 1/a_p$ so that (3.2) holds, we have that assertion (iv) of Lemma 2.1 holds with $\eta = \delta$. Note also that $|x+y|^\delta \leqslant |x|^\delta + |y|^\delta$ since $\delta < 1/a_i < 1$. Then

$$\begin{aligned}P[|\langle B, x\rangle| > \varepsilon/3] &\leqslant 3\varepsilon^{-\delta}E|\langle B, x\rangle|^\delta \\ &\leqslant 3\varepsilon^{-\delta}\sum_{|j|>m} nE|\langle A_nC_jZ_1, x\rangle|^\delta I(\|A_nZ_1\| > 1) \\ &= 3\varepsilon^{-\delta}\sum_{|j|>m} nE|\langle C_jA_nZ_1, x\rangle|^\delta I(\|A_nZ_1\| > 1) \\ &= 3\varepsilon^{-\delta}\sum_{|j|>m} nE|\langle A_nZ_1, C_j^t x\rangle|^\delta I(\|A_nZ_1\| > 1) \\ &= 3\varepsilon^{-\delta}\sum_{|j|>m} r_j^\delta nE|\langle A_nZ_1, x_j\rangle|^\delta I(\|A_nZ_1\| > 1) \\ &\leqslant 3\varepsilon^{-\delta}K_4\sum_{|j|>m}\|C_j\|^\delta\end{aligned}$$

which tends to zero as $m \to \infty$ in view of the fact that (3.2) holds. The standard convergence criteria for triangular arrays (e.g. Araujo and Giné (1980)) shows that we can take $b_n = EZ_1I(\|A_nZ_1\| \leqslant 1)$ in (2.3). Then for all $n \geqslant n_0$ we have

© Blackwell Publishers Ltd 2000

$$
\begin{aligned}
|\langle C, x\rangle| &\leqslant \sum_{|j|>m} n|\langle A_n C_j b_n, x\rangle| \\
&= \sum_{|j|>m} n|\langle A_n C_j E Z_1 I(\|A_n Z_1\| \leqslant 1), x\rangle| \\
&\leqslant \sum_{|j|>m} nE|\langle A_n C_j Z_1, x\rangle| I(\|A_n Z_1\| \leqslant 1) \\
&\leqslant K_2 \sum_{|j|>m} \|C_j\|
\end{aligned}
$$

which tends to zero as $m \to \infty$ as in the argument for A. Since

$$
P(|\langle T, x\rangle| > \varepsilon) \leqslant P(|\langle A, x\rangle| > \varepsilon/3) + P(|\langle B, x\rangle| > \varepsilon/3) + P(|\langle C, x\rangle| > \varepsilon/3)
$$

it follows that (3.4) holds when x is a unit vector in V_i and $a_i > 1$.

Suppose then that $\frac{1}{2} < a_i \leqslant 1$ and note that (2.4) part (ii) holds with $\zeta = 2$. Since $C = EA$ we have by Chebyshev's inequality that

$$
\begin{aligned}
P[|\langle A - C, x\rangle| > \varepsilon/2] &\leqslant 2\varepsilon^{-2} \operatorname{var}(\langle A, x\rangle) \\
&= 2\varepsilon^{-2} \sum_{|j|>m} n \operatorname{var}[\langle A_n C_j Z_1, x\rangle I(\|A_n Z_1\| \leqslant 1)] \\
&\leqslant 2\varepsilon^{-2} \sum_{|j|>m} nE[\langle A_n C_j Z_1, x\rangle^2 I(\|A_n Z_1\| \leqslant 1)] \\
&= 2\varepsilon^{-2} \sum_{|j|>m} n r_j^2 E[\langle A_n Z_1, x_j\rangle^2 I(\|A_n Z_1\| \leqslant 1)] \\
&\leqslant 2\varepsilon^{-2} K_2 \sum_{|j|>m} \|C_j\|^2
\end{aligned}
$$

which tends to zero as $m \to \infty$ in view of the fact that (3.2) holds with $\delta \leqslant 1$. If $\frac{1}{2} < a_i < 1$ then (2.4) part (iv) holds with $\eta = 1$ and so

© Blackwell Publishers Ltd 2000

$$\begin{aligned} P[|\langle B, x\rangle| > \varepsilon/2] &\leqslant 2\varepsilon^{-1} E|\langle B, x\rangle| \\ &\leqslant 2\varepsilon^{-1} \sum_{|j|>m} nE|\langle A_n C_j Z_1, x\rangle| I(\|A_n Z_1\| > 1) \\ &= 2\varepsilon^{-1} \sum_{|j|>m} r_j nE|\langle A_n Z_1, x_j\rangle| I(\|A_n Z_1\| > 1) \\ &\leqslant 2\varepsilon^{-1} K_4 \sum_{|j|>m} \|C_j\| \end{aligned}$$

which tends to zero as $m \to \infty$ in view of the fact that (3.2) holds with $\delta \leqslant 1$. Finally if $a_i = 1$ then (2.4) part (iv) holds with $\eta = \delta$ where we choose $\delta < 1$ such that (3.2) holds. Then argue as before that

$$\begin{aligned} P[|\langle B, x\rangle| > \varepsilon/2] &\leqslant 2\varepsilon^{-\delta} E|\langle B, x\rangle|^{\delta} \\ &\leqslant 2\varepsilon^{-\delta} \sum_{|j|>m} nE|\langle A_n C_j Z_1, x\rangle|^{\delta} I(\|A_n Z_1\| > 1) \\ &= 2\varepsilon^{-\delta} \sum_{|j|>m} r_j^{\delta} nE|\langle A_n Z_1, x_j\rangle|^{\delta} I(\|A_n Z_1\| > 1) \\ &\leqslant 2\varepsilon^{-\delta} K_4 \sum_{|j|>m} \|C_j\|^{\delta} \end{aligned}$$

which tends to zero as $m \to \infty$ in view of the fact that (3.2) holds. Since

$$P(|\langle T, x\rangle| > \varepsilon) \leqslant P(|\langle A - C, x\rangle| > \varepsilon/2) + P(|\langle B, x\rangle| > \varepsilon/2)$$

it follows that (3.4) holds when x is a unit vector in V_i and $a_i \leqslant 1$, which concludes the proof of Theorem 3.2.

REMARK 3.5. The assumption that $A_n C_j = C_j A_n$ is somewhat restrictive, but necessary for our method of proof. It may be possible to relax this restriction. For example, it is not hard to check that for finite moving averages $X_t = C_0 Z_t + \cdots + C_q Z_{t-q}$ we have

$$(CA_n C^{-1})(X_1 + \cdots + X_n - na_n) \Rightarrow CY$$

where $C = C_0 + \cdots + C_q$. We conjecture that the same is true in general, i.e. that (3.3) holds even with $A_n C_j \neq C_j A_n$ if we replace A_n by $CA_n C^{-1}$, but we have not been able to prove this.

© Blackwell Publishers Ltd 2000

4. SAMPLE AUTOCOVARIANCE MATRIX FOR MOVING AVERAGES

In the previous section we proved that moving averages of random variables with regularly varying tail probabilities are asymptotically operator stable. In this section we prove that the sample autocovariance matrix formed from these moving averages is also asymptotically operator stable as a random element of the vector space of $d \times d$ symmetric matrices. The sample covariance matrix at lag h of the moving average X_t is defined by

$$\hat{\Gamma}_n(h) = \frac{1}{n}\sum_{t=1}^{n} X_t X'_{t+h}. \tag{4.1}$$

Meerschaert and Scheffler (1999) consider the asymptotic behavior of the sample covariance matrix of an i.i.d. sequence of random vectors with regularly varying tails. Suppose that $\{Z_n\}$ are i.i.d. random vectors on $\mathbb{R}^d$ with common distribution μ, where μ varies regularly with exponent E, i.e. (2.2) holds and every eigenvalue of E has real part exceeding $\frac{1}{4}$. Then

$$A_n\left(\sum_{i=1}^{n} Z_i Z'_i - B_n\right)A_n^t \Rightarrow W \tag{4.2}$$

where A_n is taken from the definition (2.2) above, $B_n = EZ_1 Z'_1 I(\|A_n Z_1\| \leqslant 1)$, and W is a nonnormal operator stable random element of the vector space $\mathscr{M}_s^d$ of symmetric $d \times d$ matrices with real entries.

THEOREM 4.1. *Suppose that $\hat{\Gamma}_n(h)$ is the sample covariance matrix defined by (4.1) where X_t is the moving average (3.1) and $\{Z_n\}$ are i.i.d. with common distribution μ on $\mathbb{R}^d$. Suppose that μ varies regularly with exponent E, where every eigenvalue of E has real part exceeding $\frac{1}{4}$ and (2.2) holds. Assume that $Ex'Z_n = 0$ for all $x \in \mathbb{R}^d$ such that $E|x'Z_n| < \infty$. Then for all h we have*

$$nA_n\left[\hat{\Gamma}_n(h) - \sum_{j=-\infty}^{\infty} C_j B_n C_{j+h}^t\right]A_n^t \Rightarrow \sum_{j=-\infty}^{\infty} C_j W C_{j+h}^t \tag{4.3}$$

where A_n, B_n and W are as in (4.2).

The method of proof is similar to that of Theorem 3.2 above but much more involved. The assertion of Theorem 4.1 follows easily from the following two main propositions whose proofs are included in the appendix. The first key proposition asserts that the quadratic terms of $\hat{\Gamma}_n(h)$ dominate.

PROPOSITION 4.2. *Under the assumptions of Theorem 4.2 we have*

$$A_n\left[n\hat{\Gamma}_n(h) - \sum_{t=1}^{n}\sum_{j=-\infty}^{\infty} C_j Z_{t-j} Z'_{t-j} C_{j+h}^t\right]A_n^t \xrightarrow{P} 0 \tag{4.4}$$

© Blackwell Publishers Ltd 2000

as $n \to \infty$.

The next proposition establishes the convergence of the quadratic terms of the sample autocovariance matrix to the limit in (4.3).

PROPOSITION 4.3. *Under the assumptions of Theorem 4.1 we have*

$$A_n\left[\sum_{t=1}^{n}\sum_{j=-\infty}^{\infty} C_j(Z_{t-j}Z'_{t-j} - B_n)C^t_{j+h}\right]A^t_n \Rightarrow \sum_{j=-\infty}^{\infty} C_jWC^t_{j+h} \tag{4.5}$$

as $n \to \infty$.

PROOF OF THEOREM 4.1. Combining Propositions 4.2 and 4.3 we have

$$\begin{aligned} nA_n\left[\hat{\Gamma}_n(h) - \sum_{j=-\infty}^{\infty} C_jB_nC^t_{j+h}\right]A^t_n &= A_n\left[n\hat{\Gamma}_n(h) - \sum_{t=1}^{n}\sum_{j=-\infty}^{\infty} C_jB_nC^t_{j+h}\right]A^t_n \\ &= A_n\left[n\hat{\Gamma}_n(h) - \sum_{t=1}^{n}\sum_{j=-\infty}^{\infty} C_jZ_{t-j}Z'_{t-j}C^t_{j+h}\right]A^t_n \\ &\quad + A_n\left[\sum_{t=1}^{n}\sum_{j=-\infty}^{\infty} C_j(Z_{t-j}Z'_{t-j} - B_n)C^t_{j+h}\right]A^t_n \\ &\Rightarrow 0 + \sum_{j=-\infty}^{\infty} C_jWC^t_{j+h} \end{aligned}$$

which concludes the proof.

REMARK 4.4. If $a_p < \frac{1}{2}$ then $EZ_1Z'_1$ exists and consequently the autocovariance set as $\frac{1}{2}$ matrix $\Gamma(h) = EX_tX'_{t+h}$ is well-defined. In this case Meerschaert and Scheffler (1999) show that we can take $B_n = EZ_1Z'_1$ in (4.2) and then (4.3) becomes

$$nA_n[\hat{\Gamma}_n(h) - \Gamma(h)]A^t_n \Rightarrow \sum_{j=-\infty}^{\infty} C_jWC^t_{j+h}. \tag{4.6}$$

Since $A_n \to 0$ slower than $n^{-1/2}$ when $a_p < \frac{1}{2}$, we also have in this case that $\hat{\Gamma}_n(h) \to \Gamma(h)$ in probability, and (4.6) provides the rate of convergence. If $a_1 > \frac{1}{2}$ then Meerschaert and Scheffler (1999) show that we can take $B_n = 0$ in (4.2) and then (4.3) becomes

$$nA_n\hat{\Gamma}_n(h)A^t_n \Rightarrow \sum_{j=-\infty}^{\infty} C_jWC^t_{j+h}. \tag{4.7}$$

© Blackwell Publishers Ltd 2000

Since $A_n \to 0$ faster than $n^{-1/2}$ when $a_1 > \frac{1}{2}$, we also have in this case that $\hat{\Gamma}_n(h)$ is not bounded in probability, and (4.7) provides the rate at which it blows up.

An application of Proposition 3.1 in Resnick (1986) shows that the regular variation (2.2) implies weak convergence of the associated point processes. All of the results in this paper can also be established using point process methods. In the special case of scalar norming (where (2.3) holds with $A_n = a_n^{-1} I$ for all n) we say that $\{Z_n\}$ belongs to the domain of attraction of Y. In this special case a result equivalent to Theorem 4.1 was obtained by Davis, Marengo, and Resnick (1985) and Davis and Marengo (1990) using point process methods. In this case the random matrix W is multivariate stable. Since we are norming by constants in this case, one immediately obtains the asymptotic distribution of the sample autocorrelations using the continuous mapping theorem. It is not possible to extend this argument to the general case considered in this paper (with norming by linear operators).

In Theorem 4.1, the assumption that $Ex'Z_n = 0$ when the mean exists can be removed if we use the centered version of the sample covariance matrix defined by

$$\hat{G}_n(h) = \frac{1}{n}\sum_{t=1}^{n}(X_t - \overline{X})(X_{t+h} - \overline{X})'. \tag{4.8}$$

where $\overline{X} = 1/n\sum_{t=1}^{n} X_t$ is the sample mean.

THEOREM 4.5. *Suppose that $\hat{G}_n(h)$ is the sample autocovariance matrix defined by (4.8) where X_t is the moving average (3.1) and $\{Z_n\}$ are i.i.d. with common distribution μ on $\mathbb{R}^d$. Suppose that μ varies regularly with exponent E, where every eigenvalue of E has real part exceeding $\frac{1}{4}$. Then for all h we have*

$$nA_n\left[\hat{G}_n(h) - \sum_{j=-\infty}^{\infty} C_j B_n C_{j+h}^t\right]A_n^t \Rightarrow \sum_{j=-\infty}^{\infty} C_j W C_{j+h}^t, \tag{4.9}$$

where A_n, B_n and W are as in (4.2).

PROOF. Note that the difference between the two formulas (4.1) and (4.8) can be written in the form

© Blackwell Publishers Ltd 2000

$$\frac{1}{n}\sum_{t=1}^{n}(X_t - \overline{X})(X_{t+h} - \overline{X})' - \frac{1}{n}\sum_{t=1}^{n} X_t X'_{t+h}$$

$$= -\frac{1}{n}\sum_{t=1}^{n} X_t \overline{X}' - \frac{1}{n}\sum_{t=1}^{n} \overline{X} X'_{t+h} + \frac{1}{n}\sum_{t=1}^{n} \overline{X}\,\overline{X}'$$

$$= -\overline{X}\,\overline{X}' - \overline{X}\left(\overline{X} - \frac{1}{n}\sum_{t=1}^{h} X_t + \frac{1}{n}\sum_{t=n+1}^{n+h} X_t\right)' + \overline{X}\,\overline{X}'$$

$$= \overline{X}\,\overline{X}' + o_P(\overline{X}\,\overline{X}')$$

and so it suffices to show that $nA_n\overline{X}\,\overline{X}'A_n^t \to 0$ in probability. For this it suffices to show that for all unit vectors $x \in V_i$ and $y \in V_l$ we have

$$nx'A_n\overline{X}\,\overline{X}'A_n^t y = \sqrt{n}x'A_n\overline{X}\cdot\sqrt{n}y'A_n\overline{X} \xrightarrow{\mathrm{P}} 0$$

and so it suffices to show that $\sqrt{n}x'A_n\overline{X} \to 0$ in probability for any unit vector $x \in V_i$ for any $i = 1, \ldots, p$.

If $a_i < 1$ then EP_iZ_t exists. Note that (4.8) is not changed if we replace Z_t by $Z_t - EP_iZ_t$ so that without loss of generality $EP_iZ_t = 0$. Let $S_n = X_1 + \cdots + X_n$ so that $\overline{X} = S_n/n$. In the case $a_i < 1$, Meerschaert (1993), along with the spectral decomposition shows that we can take $P_ib_n = EP_iZ_t = 0$ in (2.3) and then we have from (3.3) that $A_nP_iS_n = nA_nP_i\overline{X} \Rightarrow U$ and so $n^{1/2}A_nP_i\overline{X} \to 0$ in probability. It follows that $n^{1/2}x'A_n\overline{X} \to 0$ in probability. If $a_i > 1$ then Meerschaert (1993) along with the spectral decomposition shows that we can also take $P_ib_n = 0$ in (2.3) and it follows as before that $n^{1/2}x'A_n\overline{X} \to 0$ in probability.

If $a_i = 1$ we can apply Lemma 2.1 part (ii) with $\zeta = 1 + \varepsilon > 1/a_i$ for any $\varepsilon > 0$. By (3.3) we have $A_n(S_n - na_n) \Rightarrow U$, and so we have $nA_n(\overline{X} - a_n) \Rightarrow U$, and then $nx'A_n(\overline{X} - a_n) \Rightarrow x'U$ by continuous mapping. Then $\sqrt{n}x'A_n(\overline{X} - a_n) \to 0$ in probability and so it suffices to show that $\sqrt{n}x'A_na_n \to 0$, where without loss of generality $a_n = \sum_j C_jb_n$ and $b_n = EZ_1I(\|A_nZ_1\| \leq 1)$. Argue as in the proof of lemma 5.1 that $U_1(r, \theta) < CU_{1+\varepsilon}(r, \theta)$ for all r large and all θ unit vectors in V_i. Then as in the proof of Lemma 5.1 we have (letting $A_n^t x = r_n\theta_n$ with $r_n > 0$ and $\|\theta_n\| = 1$) that

$$\begin{aligned}|nx'A_nb_n| &\leq nE|\langle A_nZ_1, x\rangle|I(\|A_nZ_1\| \leq 1)\\ &\leq C\|P_iA_n^{-1}\|^{\varepsilon}nE|\langle A_nZ_1, x\rangle|^{1+\varepsilon}I(|\langle A_nZ_1, x\rangle| \leq 1)\\ &< C\|P_iA_n^{-1}\|^{\varepsilon}K_1\end{aligned}$$

for all n large, and so for all large n we have

© Blackwell Publishers Ltd 2000

$$
\begin{aligned}
|n^{1/2}x'A_n a_n| &= n^{-1/2}\left|\sum_{j=-\infty}^{\infty} nx'A_n C_j b_n\right| \\
&\leqslant n^{-1/2}\sum_{j=-\infty}^{\infty}\|C_j\|\,|nx'A_n b_n| \\
&\leqslant n^{-1/2}C\|P_i A_n^{-1}\|^{\varepsilon}K_1\sum_{j=-\infty}^{\infty}\|C_j\|
\end{aligned}
$$

where the series converges. Since $\log(n^{-1/2}\|P_i A_n^{-1}\|^{\varepsilon})/\log n \to -\frac{1}{2}+a_i\varepsilon < 0$ for small $\varepsilon > 0$ we obtain $n^{1/2}x'A_n a_n \to 0$, which concludes the proof.

REMARK 4.6. Davis, Marengo, and Resnick (1985) and Davis and Marengo (1990) also establish the joint convergence of the sample autocovariance matrices over all lags $|h| \leqslant h_0$, in the special case of scalar norming. It is straightforward to extend the results in this paper to establish joint convergence as well. In Meerschart and Scheffler (1998) we apply this joint convergence to compute the asymptotic distribution of the sample cross-correlations for a bivariate moving average. This extends the results of Davis *et al.* to the case where the norming operator A_n and C_j are diagonal. The general case, where both A_n and C_j are arbitrary linear operators, remains open.

5. APPENDIX

In this section we prove Proposition 4.2 and Proposition 4.3 above and hence complete the proof of Theorem 4.1. Throughout this section we assume that the assumptions of Theorem 4.1 hold. That is, we assume that $\{Z_n\}$ are i.i.d. with common distribution μ on $\mathbb{R}^d$, and that μ varies regularly with index E, where every eigenvalue of E has real part exceeding $\frac{1}{4}$ and (2.2) holds. Assume further that $Ex'Z_n = 0$ for all $x \in \mathbb{R}^d$ such that $E|x'Z_n| < \infty$.

Recall from section 2 that in view of the spectral decomposition we can (and hence will) assume without loss of generality that A_n is block diagonal with respect to the direct sum decomposition $\mathbb{R}^d = V_1 \oplus \ldots \oplus V_p$, where the V_i-spaces are mutually orthogonal and form the spectral decomposition relative to E. Let $\frac{1}{4} < a_1 < \ldots < a_p$ denote the real parts of the eigenvalues of E.

Since the proofs of the two propositions are quite long we split them into several lemmas.

PROOF OF PROPOSITION 4.2. Let $V_1, \ldots, V_p$ be the spectral decomposition of E and $a_1 < \cdots < a_p$ the associated real spectrum (see section two above). Recall that $V_i \perp V_j$ and that P_i is the orthogonal projection onto V_i. We also assume as before that (3.2) holds for some $\delta < 1/a_p$ with $\delta \leqslant 1$. It will suffice to show that

$$
x'A_n\left[n\hat{\Gamma}_n(h) - \sum_{t=1}^{n}\sum_{j=-\infty}^{\infty} C_j Z_{t-j}Z'_{t-j}C^t_{j+h}\right]A^t_n y \xrightarrow{\text{P}} 0 \tag{5.1}
$$

for all unit vectors $x \in V_i$ and $y \in V_l$. Note that

© Blackwell Publishers Ltd 2000

$$n\hat{\Gamma}_n(h) = \sum_{t=1}^{n} X_t X'_{t+h} = \sum_{t=1}^{n}\left(\sum_{j=-\infty}^{\infty} C_j Z_{t-j}\right)\left(\sum_{q=-\infty}^{\infty} C_q Z_{t+h-q}\right)'$$

$$= \sum_{t=1}^{n}\left(\sum_{j=-\infty}^{\infty} C_j Z_{t-j}\right)\left(\sum_{k=-\infty}^{\infty} C_{k+h} Z_{t-k}\right)' \quad (\text{set } k = q-h)$$

$$= \sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} C_j Z_{t-j} Z'_{t-k} C^t_{k+h}$$

$$= \sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\left(C_j Z_{t-j} Z'_{t-j} C^t_{j+h} + \sum_{k\neq j} C_j Z_{t-j} Z'_{t-k} C^t_{k+h}\right)$$

$$= \sum_{t=1}^{n}\sum_{j=-\infty}^{\infty} C_j Z_{t-j} Z'_{t-j} C^t_{j+h} + \sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\sum_{k\neq j} C_j Z_{t-j} Z'_{t-k} C^t_{k+h}$$

so that (5.1) is equivalent to

$$x' A_n \left[\sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\sum_{k\neq j} C_j Z_{t-j} Z'_{t-k} C^t_{k+h}\right] A^t_n y \xrightarrow{\mathrm{P}} 0. \tag{5.2}$$

We now have to consider several cases separately.

CASE I: Suppose that $x \in V_i$ and $y \in V_l$ are unit vectors with $a_i + a_l > 1$. Choose $\delta > 0$ such that $\delta > 1/(a_i + a_l)$, $\delta < 1/a_i$, $\delta < 1/a_l$, and $\delta \leqslant 1$. Write $C^t_j A^t_n x = r\theta$ where $r > 0$ and $\|\theta\| = 1$. Then $r = \|C^t_j A^t_n x\| = \|C^t_j A^t_n P_i x\| = \|C^t_j P_i A^t_n x\| \leqslant \|C_j\|\,\|P_i A_n\|$ and so

$$E|x' A_n C_j Z_1|^\delta = E|(C^t_j A^t_n x)' Z_1|^\delta$$

$$= E|r\theta' Z_1|^\delta$$

$$\leqslant \|P_i A_n\|^\delta \|C_j\|^\delta E|\langle Z_1,\, \theta\rangle|^\delta$$

$$= \|P_i A_n\|^\delta \|C_j\|^\delta E|\langle Z_1,\, P_i\theta\rangle|^\delta$$

$$= \|P_i A_n\|^\delta \|C_j\|^\delta E|\langle P_i Z_1,\, \theta\rangle|^\delta$$

$$\leqslant \|P_i A_n\|^\delta \|C_j\|^\delta E\|P_i Z_1\|^\delta$$

and likewise

$$E|Z'_2 C^t_{k+h} A^t_n y|^\delta \leqslant \|P_l A_n\|^\delta \|C_{k+h}\|^\delta E\|P_l Z_2\|^\delta.$$

Then

© Blackwell Publishers Ltd 2000

$$E\left|x'A_n\left[\sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\sum_{k\neq j}C_jZ_{t-j}Z'_{t-k}C^t_{k+h}\right]A^t_n y\right|^\delta$$

$$= E\left|\sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\sum_{k\neq j}x'A_nC_jZ_{t-j}Z'_{t-k}C^t_{k+h}A^t_n y\right|^\delta$$

$$\leqslant E\left[\left(\sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\sum_{k\neq j}|x'A_nC_jZ_{t-j}Z'_{t-k}C^t_{k+h}A^t_n y|\right)^\delta\right]$$

$$\leqslant E\left[\sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\sum_{k\neq j}|x'A_nC_jZ_{t-j}Z'_{t-k}C^t_{k+h}yA^t_n y|^\delta\right]$$

$$= E\left[\sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\sum_{k\neq j}|x'A_nC_jZ_{t-j}|^\delta|Z'_{t-k}C^t_{k+h}yA^t_n y|^\delta\right]$$

$$= \sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\sum_{k\neq j}E(|x'A_nC_jZ_{t-j}|^\delta|Z'_{t-k}C^t_{k+h}yA^t_n y|^\delta)$$

$$= \sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\sum_{k\neq j}E|x'A_nC_jZ_{t-j}|^\delta E|Z'_{t-k}C^t_{k+h}yA^t_n y|^\delta$$

by independence of Z_{t-j}, Z_{t-k}

$$\leqslant \sum_{t=1}^{n}\sum_{j=-\infty}^{\infty}\sum_{k\neq j}\|P_iA_n\|^\delta\|C_j\|^\delta E\|P_iZ_1\|^\delta\cdot\|P_lA_n\|^\delta\|C_{k+h}\|^\delta E\|P_lZ_2\|^\delta$$

$$= n\|P_iA_n\|^\delta\|P_lA_n\|^\delta E\|P_iZ_1\|^\delta E\|P_lZ_2\|^\delta\sum_{j=-\infty}^{\infty}\sum_{k\neq j}\|C_j\|^\delta\|C_{k+h}\|^\delta$$

where $E\|P_iZ_1\|^\delta<\infty$ because $\delta<1/a_i$, $E\|P_lZ_2\|^\delta<\infty$ because $\delta<1/a_l$, and

$$\sum_{j=-\infty}^{\infty}\sum_{k\neq j}\|C_j\|^\delta\|C_{k+h}\|^\delta \leqslant \left(\sum_{j=-\infty}^{\infty}\|C_j\|^\delta\right)\left(\sum_{j=-\infty}^{\infty}\|C_{k+h}\|^\delta\right)$$

$$= \left(\sum_{j=-\infty}^{\infty}\|C_j\|^\delta\right)^2<\infty$$

and furthermore since

$$\frac{\log\|P_iA_n\|}{\log n}\to -a_i \quad\text{and}\quad \frac{\log\|P_lA_n\|}{\log n}\to -a_l$$

we have

$$\frac{\log(n\|P_iA_n\|^\delta\|P_lA_n\|^\delta)}{\log n}\to 1-\delta a_i-\delta a_l<0$$

© Blackwell Publishers Ltd 2000

since $\delta > 1/(a_i + a_l)$. Then $n\|P_iA_n\|^\delta \|P_lA_n\|^\delta \to 0$ and it follows that (5.2) holds in this case.

CASE II: Suppose that $x \in V_i$ and $y \in V_l$ are unit vectors with $a_i + a_l \leqslant 1$. Then $a_i < 1$ and $a_l < 1$ so $E\|P_iZ_1\| < \infty$ and $E\|P_lZ_2\| < \infty$. Then by assumption $Ex'Z_1 = 0$ and $EZ_2'y = 0$. Define $Z_{in} = Z_iI(\|A_nZ_i\| \leqslant 1)$ and $\mu_n = EZ_{in}$ and let

$$A_{i,j,n} = (Z_{in} - \mu_n)(Z_{jn} - \mu_n)'$$

$$B_{i,j,n} = Z_{in}\mu_n' - \mu_n Z_{jn}'$$

$$C_{i,j,n} = Z_iZ_j'I(\|A_nZ_i\| > 1 \text{ or } \|A_nZ_j\| > 1)$$

$$D_{i,j,n} = -\mu_n\mu_n'$$

so that $EA_{i,j,n} = 0$ and $Z_iZ_j' = A_{i,j,n} + B_{i,j,n} + C_{i,j,n} + D_{i,j,n}$. Then let

$$A = \sum_{t=1}^{n} \sum_{j=-\infty}^{\infty} \sum_{k \neq j} x'A_nC_jA_{t-j,t-k,n}C_{k+h}^tA_n^ty$$

and similarly for B, C, D to obtain

$$x'A_n\left[\sum_{t=1}^{n} \sum_{j=-\infty}^{\infty} \sum_{k \neq j} C_jZ_{t-j}Z_{t-k}'C_{k+h}^t\right]A_n^ty = A + B + C + D$$

and so in order to establish (5.2) it suffices to show that each of A, B, C, D tends to zero in probability. Let $\overline{Z}_{in} = Z_{in} - \mu_n$ so that $A_{ijn} = \overline{Z}_{in}\overline{Z}_{jn}'$. Since $\overline{Z}_{in}$ and $\overline{Z}_{jn}$ are independent with mean zero for $i \neq j$ it is easy to check that $EA = 0$. Then

$$\begin{aligned}
\operatorname{var}(A) &= E\left(\sum_{t=1}^{n} \sum_{j=-\infty}^{\infty} \sum_{k \neq j} x'A_nC_j\overline{Z}_{t-j,n}\overline{Z}_{t-k,n}'C_{k+h}^tA_n^ty\right)^2 \\
&= E\left[\left(\sum_{t_1=1}^{n} \sum_{j_1=-\infty}^{\infty} \sum_{k_1 \neq j_1} x'A_nC_{j_1}\overline{Z}_{t_1-j_1,n}\overline{Z}_{t_1-k_1,n}'C_{k_1+h}^tA_n^ty\right)\right. \\
&\quad \left.\cdot\left(\sum_{t_2=1}^{n} \sum_{j_2=-\infty}^{\infty} \sum_{k_2 \neq j_2} x'A_nC_{j_2}\overline{Z}_{t_2-j_2,n}\overline{Z}_{t_2-k_2,n}'C_{k_2+h}^tA_n^ty\right)\right] \\
&= \sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\sum_{j_1=-\infty}^{\infty}\sum_{j_2=-\infty}^{\infty}\sum_{k_1 \neq j_1}\sum_{k_2 \neq j_2} E[\langle \overline{Z}_{t_1-j_1,n}, C_{j_1}^tA_n^tx\rangle \\
&\quad \cdot\langle \overline{Z}_{t_1-k_1,n}, C_{k_1+h}^tA_n^ty\rangle\langle \overline{Z}_{t_2-j_2,n}, C_{j_2}^tA_n^tx\rangle\langle \overline{Z}_{t_2-k_2,n}, C_{k_2+h}^tA_n^ty\rangle].
\end{aligned}$$

If any one of $t_1 - j_1$, $t_1 - k_1$, $t_2 - j_2$, or $t_2 - k_2$ is distinct then this term vanishes since $E\overline{Z}_{in} = 0$, using the independence to write the mean of the product as the product of the means. The only nonvanishing terms for a given t_1, j_1, k_1 occur when either

(a) $t_1 - j_1 = t_2 - j_2$ and $t_1 - k_1 = t_2 - k_2$
(b) $t_1 - j_1 = t_2 - k_2$ and $t_1 - k_1 = t_2 - j_2$.

Given t_1, j_1, k_1 for every $t_2 = 1, \ldots, n$ there exists a unique j_2, k_2 satisfying (a) or (b). Indeed we have

© Blackwell Publishers Ltd 2000

(a) $j_2 = (t_2 - t_1) + j_1$ and $k_2 = (t_2 - t_1) + k_1$
(b) $j_2 = (t_2 - t_1) + k_1$ and $k_2 = (t_2 - t_1) + j_1$

so that $j_2 \neq k_2$ is guaranteed. Now let $p = t_1 - j_1,\ q = t_1 - k_1,\ j = j_1,\ t = t_1,\ k = k_1$, and $\Delta = t_2 - t_1$ so that in case

(a) $t_2 - j_2 = p,\ t_2 - k_2 = q,\ j_2 = \Delta + j$, and $k_2 = \Delta + k$
(b) $t_2 - j_2 = q,\ t_2 - k_2 = p,\ j_2 = \Delta + k$, and $k_2 = \Delta + j$.

Then in case (a) the only nonvanishing terms for a given t, j, k are of the form

$$\begin{aligned}
&E[\langle \overline{Z}_{pn}, C_j^t A_n^t x\rangle\langle \overline{Z}_{qn}, C_{k+h}^t A_n^t y\rangle\langle \overline{Z}_{pn}, C_{\Delta+j}^t A_n^t x\rangle\langle \overline{Z}_{qn}, C_{\Delta+k+h}^t A_n^t y\rangle]\\
&\qquad = E[\langle \overline{Z}_{pn}, C_j^t A_n^t x\rangle\langle \overline{Z}_{pn}, C_{\Delta+j}^t A_n^t x\rangle\langle \overline{Z}_{qn}, C_{k+h}^t A_n^t y\rangle\langle \overline{Z}_{qn}, C_{\Delta+k+h}^t A_n^t y\rangle\\
&\qquad = E[x' A_n C_j \overline{Z}_{pn}\overline{Z}'_{pn} C_{\Delta+j}^t A_n^t x] E[y' A_n C_{k+h}\overline{Z}_{qn}\overline{Z}'_{qn} C_{\Delta+k+h}^t A_n^t y]\\
&\qquad = (x' A_n C_j M_n C_{\Delta+j}^t A_n^t x)(y' A_n C_{k+h} M_n C_{\Delta+k+h}^t A_n^t y)
\end{aligned}$$

where $M_n = E\overline{Z}_{in}\overline{Z}'_{in}$. In case (b) the only nonvanishing terms are of the form

$$\begin{aligned}
&E[\langle \overline{Z}_{pn}, C_j^t A_n^t x\rangle\langle \overline{Z}_{qn}, C_{k+h}^t A_n^t y\rangle\langle \overline{Z}_{qn}, C_{\Delta+k}^t A_n^t x\rangle\langle \overline{Z}_{pn}, C_{\Delta+j+h}^t A_n^t y\rangle]\\
&\qquad = E[\langle \overline{Z}_{pn}, C_j^t A_n^t x\rangle\langle \overline{Z}_{pn}, C_{\Delta+j+h}^t A_n^t y\rangle\langle \overline{Z}_{qn}, C_{k+h}^t A_n^t y\rangle\langle \overline{Z}_{qn}, C_{\Delta+k}^t A_n^t x\rangle]\\
&\qquad = (x' A_n C_j M_n C_{\Delta+j+h}^t A_n^t y)(y' A_n C_{k+h} M_n C_{\Delta+k}^t A_n^t x)
\end{aligned}$$

so that

$$\begin{aligned}
\operatorname{var}(A) = \sum_{t=1}^{n}\sum_{t+\Delta=1}^{n}\sum_{j=-\infty}^{\infty}\sum_{k\neq j}&[(x' A_n C_j M_n C_{\Delta+j}^t A_n^t x)(y' A_n C_{k+h} M_n C_{\Delta+k+h}^t A_n^t y)\\
&+ (x' A_n C_j M_n C_{\Delta+j+h}^t A_n^t y)(y' A_n C_{k+h} M_n C_{\Delta+k}^t A_n^t x)].
\end{aligned}$$

Now we need to bound the terms in this sum. Write $C_j^t x = r_j x_j$, $A_n^t x_j = a_{nj} x_{nj}$, $C_j^t y = \rho_j y_j$, and $A_n^t y_j = b_{nj} y_{nj}$ where x_j, x_{nj}, y_j, y_{nj} are unit vectors and r_j, a_{nj}, ρ_j, b_{nj} are positive reals. Then for example we have

$$\begin{aligned}
x' A_n C_j M_n C_{\Delta+j}^t A_n^t x &= x' C_j A_n M_n A_n^t C_{\Delta+j}^t x\\
&= (C_j^t x)' A_n M_n A_n^t C_{\Delta+j}^t x\\
&= r_j r_{\Delta+j} x'_j A_n M_n A_n^t x_{\Delta+j}
\end{aligned}$$

where

$$\begin{aligned}
x'_j A_n M_n A_n^t x_{\Delta+j} &= x'_j A_n E\overline{Z}_{1n}\overline{Z}'_{1n} A_n^t x_{\Delta+j}\\
&= E x'_j A_n \overline{Z}_{1n}\overline{Z}'_{1n} A_n^t x_{\Delta+j}\\
&= E\langle \overline{Z}_{1n}, A_n^t x_j\rangle\langle \overline{Z}_{1n}, A_n^t x_{\Delta+j}\rangle\\
&\leqslant \sqrt{E\langle \overline{Z}_{1n}, A_n^t x_j\rangle^2 E\langle \overline{Z}_{1n}, A_n^t x_{\Delta+j}\rangle^2}.
\end{aligned}$$

Since $\overline{Z}_{1n} = Z_{1n} - \mu_n$ where $\mu_n = EZ_{1n}$ we have for example that

© Blackwell Publishers Ltd 2000

$$\begin{aligned} E\langle \overline{Z}_{1n}, A_n^t x_j\rangle^2 &= E\langle Z_{1n} - \mu_n, A_n^t x_j\rangle^2 \\ &= E(\langle Z_{1n}, A_n^t x_j\rangle - E\langle Z_{1n}, A_n^t x_j\rangle)^2 \\ &= \mathrm{var}(\langle Z_{1n}, A_n^t x_j\rangle) \\ &\leqslant E[\langle Z_{1n}, A_n^t x_j\rangle^2] \end{aligned}$$

so that, calculating in a similar manner for the remaining terms, we have for example

$$x' A_n C_j M_n C_{\Delta+j}^t A_n^t x \leqslant r_j r_{j+\Delta} \sqrt{E\langle Z_{1n}, A_n^t x_j\rangle^2 E\langle Z_{1n}, A_n^t x_{\Delta+j}\rangle^2}$$

and then

$$\begin{aligned} \mathrm{var}(A) \leqslant \sum_{t=1}^{n} \sum_{t+\Delta=1}^{n} \sum_{j=-\infty}^{\infty} \sum_{k\neq j} &(r_j r_{j+\Delta}[E\langle Z_{1n}, A_n^t x_j\rangle^2 E\langle Z_{1n}, A_n^t x_{j+\Delta}\rangle^2]^{1/2} \\ &\cdot \rho_{k+h}\rho_{k+h+\Delta}[E\langle Z_{1n}, A_n^t y_{k+h}\rangle^2 E\langle Z_{1n}, A_n^t y_{k+h+\Delta}\rangle^2]^{1/2} \\ &+ r_j \rho_{j+h+\Delta}[E\langle Z_{1n}, A_n^t x_y\rangle^2 E\langle Z_{1n}, A_n^t y_{j+h+\Delta}\rangle^2]^{1/2} \\ &\cdot \rho_{k+h} r_{k+\Delta}[E\langle Z_{1n}, A_n^t y_{k+h}\rangle^2 E\langle Z_{1n}, A_n^t x_{k+\Delta}\rangle^2]^{1/2}). \end{aligned}$$

To bound the expectations on the right hand side of the inequality above we need

LEMMA 5.1. *For all unit vectors $x \in V_i$, $y \in V_l$, for some n_0, for all $n \geqslant n_0$ and all j, k, h, Δ, for some $K_n > 0$ satisfying $nK_n \to 0$ we have*

$$\begin{aligned} &[E\langle Z_{1n}, A_n^t x_j\rangle^2 E\langle Z_{1n}, A_n^t x_{j+\Delta}\rangle^2 \cdot E\langle Z_{1n}, A_n^t y_{k+h}\rangle^2 E\langle Z_{1n}, A_n^t y_{k+h+\Delta}\rangle^2]^{1/2} \\ &\quad + [E\langle Z_{1n}, A_n^t x_j\rangle^2 E\langle Z_{1n}, A_n^t y_{j+h+\Delta}\rangle^2 \cdot E\langle Z_{1n}, A_n^t y_{k+h}\rangle^2 E\langle Z_{1n}, A_n^t x_{k+\Delta}\rangle^2]^{1/2} \\ &\quad \leqslant K_n. \end{aligned} \tag{5.3}$$

PROOF OF LEMMA 5.1. Suppose that $a_i > \frac{1}{2}$. Then $a_l < \frac{1}{2}$ so $E\|P_i Z_1\|^2 = \infty$ while $E\|P_l Z_1\|^2 < \infty$. Recall that $P_l = P_l^2 = P_l^t$ for this orthogonal projection operator, $A_n P_l = P_l A_n$ and that $|\langle x, \theta\rangle \leqslant \|x\|$ when θ is a unit vector. Then for all unit vectors $\theta \in V_l$ we have

$$\begin{aligned} E\langle Z_{1n}, A_n^t \theta\rangle^2 &= E\langle A_n Z_1, \theta\rangle^2 I(\|A_n Z_1\| \leqslant 1) \\ &\leqslant E\langle A_n Z_1, \theta\rangle^2 \\ &= E\langle A_n Z_1, P_l\theta\rangle^2 \\ &= E\langle P_l A_n P_l Z_1, \theta\rangle^2 \\ &\leqslant E\|P_l A_n P_l Z_1\|^2 \\ &\leqslant \|P_l A_n\|^2 E\|P_l Z_1\|^2. \end{aligned}$$

Since $1/a_i < 2$ we can apply part (ii) of Lemma 2.1 with $\zeta = 2$ to get

$$nE\langle Z_{1n}, A_n^t \theta\rangle^2 = nE\langle A_n Z_1, \theta\rangle^2 I(\|A_n Z_1\| \leqslant 1) < K_2$$

for all $n \geqslant n_0$ and all $\|\theta\| = 1$ in V_i. Then (5.3) holds in this case with

© Blackwell Publishers Ltd 2000

$$K_n = 2[(K_2/n)^2(\|P_lA_n\|^2E\|P_lZ_1\|^2)^2]^{1/2}$$

$$= 2(K_2/n)\|P_lA_n\|^2E\|P_lZ_1\|^2$$

so that $nK_n \to 0$ since $\|P_lA_n\| \to 0$. If $a_l > \frac{1}{2}$ so that $a_i < \frac{1}{2}$ then a similar argument establishes (5.3) with $K_n = 2(K_2/n)\|P_iA_n\|^2E\|P_iZ_1\|^2$ and again $nK_n \to 0$. Otherwise we have $a_i \leqslant \frac{1}{2}$ and $a_l \leqslant \frac{1}{2}$. If both $E\|P_iZ_1\|^2$ and $E\|P_lZ_1\|^2$ exist then (5.3) holds with $K_n = 2\|P_iA_n\|^2\|P_lA_n\|^2E\|P_iZ_1\|^2E\|P_lZ_1\|^2$ and then

$$\frac{\log nK_n}{\log n} \sim \frac{\log(n\|P_iA_n\|^2\|P_lA_n\|^2)}{\log n} \to 1 - 2(a_i + a_l) < 0$$

since $a_i > \frac{1}{4}$ and $a_l > \frac{1}{4}$ by assumption, so again $nK_n \to 0$. Otherwise suppose that $a_i \leqslant \frac{1}{2}$ and $a_l \leqslant \frac{1}{2}$ with $E\|P_iZ_1\|^2 = \infty$ and $E\|P_lZ_1\|^2 < \infty$. Then $a_i = \frac{1}{2}$ and so we can apply part (ii) of Lemma 2.1 with $\zeta = 2 + \varepsilon$ for any $\varepsilon > 0$. Observe that for all t and all θ we have

$$\begin{aligned}
U_2(t, \theta) &= \int_{|\langle z,\theta\rangle| \leqslant t} |\langle z, \theta\rangle|^2\mu(dz)\\
&= \int_{0 \leqslant |\langle z,\theta\rangle| < 1} |\langle z, \theta\rangle|^2\mu(dz) + \int_{1 \leqslant |\langle z,\theta\rangle| \leqslant t} |\langle z, \theta\rangle|^2\mu(dz)\\
&\leqslant \int_{0 \leqslant |\langle z,\theta\rangle| < 1} |\langle z, \theta\rangle|^2\mu(dz) + \int_{1 \leqslant |\langle z,\theta\rangle| \leqslant t} |\langle z, \theta\rangle|^{2+\varepsilon}\mu(dz)\\
&\leqslant 1 + U_{2+\varepsilon}(t, \theta)
\end{aligned}$$

and since $U_{2+\varepsilon}(t, \theta) \to \infty$ uniformly on $\|\theta\| = 1$ in V_i, for some t_0 and $C > 0$ we have for all $t \geqslant t_0$ and all $\|\theta\| = 1$ in V_i that $U_2(t, \theta) \leqslant CU_{2+\varepsilon}(t, \theta)$. As in the proof of Lemma 2.1 we can choose n_0 such that $\|A_n\theta\| \leqslant t_0^{-1}$ for all $\|\theta\| = 1$ and all $n \geqslant n_0$. Enlarge n_0 if necessary so that Lemma 2.1 also applies. Then

$$\begin{aligned}
nE\langle Z_{1n}, A_n^tx_j\rangle^2 &= nE\langle Z_1, A_n^tx_j\rangle^2 I(\|A_nZ_1\| \leqslant 1)\\
&\leqslant na_{nj}^2U_2(a_{nj}^{-1}, x_{nj})\\
&\leqslant Ca_{nj}^{-\varepsilon}\cdot na_{nj}^{2+\varepsilon}U_{2+\varepsilon}(a_{nj}^{-1}, x_{nj})\\
&= Ca_{nj}^{-\varepsilon}\cdot nE|\langle A_nZ_1, x_j\rangle|^{2+\varepsilon}I(|\langle A_nZ_1, x_j\rangle| \leqslant 1)\\
&\leqslant Ca_{nj}^{-\varepsilon}\cdot K_1
\end{aligned}$$

for all $n \geqslant n_0 = n_0(t_0)$ and all j. Note also that for all j and all n we have $a_{nj} = \|A_nx_j\| \geqslant \min\{\|A_n\theta\|: \|\theta\| = 1 \text{ in } V_i\} = 1/\|P_iA_n^{-1}\|$ so that $a_{nj}^{-\varepsilon} \leqslant \|P_iA_n^{-1}\|^{\varepsilon}$ for all n and all j. Then (5.3) holds in this case with

$$\begin{aligned}
K_n &= [(n^{-1}K_1\|P_iA_n^{-1}\|^{\varepsilon})^2(\|P_lA_n\|^2E\|P_iZ_1\|^2)^2]^{1/2}\\
&= n^{-1}K_1\|P_iA_n^{-1}\|^{\varepsilon}\|P_lA_n\|^2E\|P_iZ_1\|^2
\end{aligned}$$

so that

$$\frac{\log nK_n}{\log n} \sim \frac{\log(\|P_iA_n^{-1}\|^{\varepsilon}\|P_lA_n\|^2)}{\log n} \to \varepsilon a_i - 2a_l < 0$$

for all $\varepsilon > 0$ sufficiently small. Note that $\varepsilon > 0$ can be chosen independently of x, y, j and so $nK_n \to 0$ here too. Finally if $a_i \leqslant \frac{1}{2}$ and $a_l \leqslant \frac{1}{2}$ with $E\|P_iZ_1\|^2 < \infty$ and

© Blackwell Publishers Ltd 2000

$E\|P_l Z_1\|^2 = \infty$ then (5.3) holds with $K_n = n^{-1} K_1 \|P_l A_n^{-1}\|^{\varepsilon} \|P_i A_n\|^2 E\|P_i Z_1\|^2$ and $\log(nK_n)/\log n \to \varepsilon a_l - 2a_i < 0$ for $\varepsilon > 0$ sufficiently small, and again $nK_n \to 0$. This concludes the proof of Lemma 5.1.

Returning to the proof of Proposition 4.2, we get from Lemma 5.1

$$\begin{aligned}
\operatorname{var}(A) &\leqslant K_n \sum_{t=1}^{n} \sum_{t+\Delta=1}^{n} \sum_{j=-\infty}^{\infty} \sum_{k \neq j} (r_j r_{j+\Delta} \rho_{k+h} \rho_{k+h+\Delta} + r_j \rho_{j+h+\Delta} \rho_{k+h} r_{k+\Delta}) \\
&\leqslant K_n \sum_{t=1}^{n} \sum_{\Delta=1-t}^{n-t} \sum_{j=-\infty}^{\infty} \sum_{k \neq j} (\|C_j\| \, \|C_{j+\Delta}\| \, \|C_{k+h}\| \, \|C_{k+h+\Delta}\| \\
&\qquad + \|C_j\| \, \|C_{j+h+\Delta}\| \, \|C_{k+h}\| \, \|C_{k+\Delta}\|) \\
&\leqslant nK_n 2 \left(\sum_{j=-\infty}^{\infty} \|C_j\| \right)^4 \to 0
\end{aligned}$$

and so $A \to 0$ in probability.

Next we have

$$B = \sum_{t=1}^{n} \sum_{j=-\infty}^{\infty} \sum_{k \neq j} x' A_n C_j [Z_{t-j,n} \mu_n' + \mu_n Z_{t-k,n}'] C_{k+h}^t A_n^t y$$

and we will let

$$\begin{aligned}
B_{rs}^{jk} &= x' A_n C_j [Z_{rn} \mu_n' + \mu_n Z_{sn}'] C_k^t A_n^t y \\
&= (C_j x)' A_n [Z_{rn} \mu_n' + \mu_n Z_{sn}'] A_n^t (C_k^t y) \\
&= r_j r_k (A_n^t x_j)' [Z_{rn} \mu_n' + \mu_n Z_{sn}'] (A_n^t y_k) \\
&= r_j r_k [B_r^{(1)} + B_s^{(2)}]
\end{aligned}$$

where

$$\begin{aligned}
B_r^{(1)} &= (A_n^t x_j)' Z_{rn} \mu_n' A_n^t y_k \\
B_s^{(2)} &= (A_n^t x_j)' \mu_n Z_{sn}' A_n^t y_k
\end{aligned}$$

In the next lemma we derive bounds on the expectation of B_{rs}^{lk}.

LEMMA 5.2. *For all unit vectors $x \in V_i$* and *$y \in V_l$, for some n_0, for all $n \geqslant n_0$, for all j, k, r, s we have*

$$E|B_{rs}^{jk}| \leqslant \|C_j\| \, \|C_k\| K_n \tag{5.4}$$

where $nK_n \to 0$.

PROOF OF LEMMA 5.2. Write

$$\begin{aligned}
E|B_{rs}^{jk}| &= r_j r_k E|B_r^{(1)} + B_s^{(2)}| \\
&\leqslant \|C_j\| \, \|C_k\| (E|B_r^{(1)}| + E|B_s^{(2)}|)
\end{aligned}$$

where

© Blackwell Publishers Ltd 2000

$$E|B_r^{(1)}| = E|\langle Z_{1n}, A_n^t x_j\rangle\langle \mu_n, A_n^t y_k\rangle|$$
$$= |\langle \mu_n, A_n^t y_k\rangle| E|\langle Z_{1n}, A_n^t x_j\rangle|.$$

Since $a_i < 1$ we have $E\|P_i Z_1\| < \infty$. Then $E|\langle Z_{1n}, A_n^t x_j\rangle| \leqslant \|P_i A_n\| E\|P_i Z_1\|$ as in the proof of Lemma 5.1. Apply part (iv) of Lemma 2.1 with $\eta = 1$ (so that $\eta < 1/a_l$). Using the fact that $E\langle Z_1, y_k\rangle = 0$ we have for some $n_1 > 0$ that

$$n|\langle \mu_n, A_n^t y_k\rangle| = n|\langle EZ_{1n}, A_n^t y_k\rangle|$$
$$= n|E\langle Z_{1n}, A_n^t y_k\rangle|$$
$$= n|E\langle Z_1, A_n^t y_k\rangle I(\|A_n Z_1\| \leqslant 1)|$$
$$= n|E\langle Z_1, A_n^t y_k\rangle I(\|A_n Z_1\| > 1)|$$
$$\leqslant nE|\langle A_n Z_1, y_k\rangle| I(\|A_n Z_1\| > 1) < K_4$$

for all $n \geqslant n_1$ and all k. It follows that

$$nE|B_r^{(1)}| \leqslant K_4 \|P_i A_n\| E\|P_i Z_1\|$$

for all $n \geqslant n_1$ and all k. A similar argument (note that $a_l < 1$ so that $E\|P_l Z_1\| < \infty$ and apply part (iv) of Lemma 2.1 again with $\eta = 1$, so that $\eta < 1/a_i$) shows that for some $n_2 > 0$ we have

$$nE|B_r^{(2)}| \leqslant K_4 \|P_l A_n\| E\|P_l Z_1\|$$

for all $n \geqslant n_2$ and all j. Letting $n_0 = \max\{n_1, n_2\}$ we have that (5.4) holds with

$$nK_n = K_4(\|P_i A_n\| E\|P_i Z_1\| + \|P_l A_n\| E\|P_l Z_1\|)$$

for all j, k, r, s and all $n \geqslant n_0$. Then $nK_n \to 0$ since $\|A_n\| \to 0$, which concludes the proof of Lemma 5.2.

Returning to the proof of Proposition 4.2, Lemma 5.2 yields

$$E|B| \leqslant \sum_{t=1}^{n} \sum_{j=-\infty}^{\infty} \sum_{k \neq j} \|C_j\| \, \|C_{k+h}\| K_n$$
$$\leqslant nK_n \left(\sum_{j=-\infty}^{\infty} \|C_j\| \right)^2 \to 0$$

and so $B \to 0$ in probability.

Next we write

$$C_{rs}^{jk} = x' A_n C_j Z_r Z_s' I(\|A_n Z_r\| > 1 \text{ or } \|A_n Z_s\| > 1) C_k^t A_n^t y$$
$$= r_j r_k \langle Z_r, A_n^t x_j\rangle\langle Z_s, A_n^t y_k\rangle I(\|A_n Z_r\| > 1 \text{ or } \|A_n Z_s\| > 1)$$

so that

$$E|C_{rs}^{jk}| \leqslant \|C_j\| \, \|C_k\| E(|\langle Z_r, A_n^t x_j\rangle\langle Z_s, A_n^t y_k\rangle| I(\|A_n Z_r\| > 1 \text{ or } \|A_n Z_s\| > 1))$$
$$\leqslant \|C_j\| \, \|C_k\| (E|x_j' A_n Z_r Z_s' A_n^t y_k| I(\|A_n Z_r\| > 1) + E|x_j' A_n Z_r Z_s' A_n^t y_k| I(\|A_n Z_s\| > 1)).$$

The next lemma gives bounds on $E|C_{jk}^{rs}|$.

LEMMA 5.3. *For all unit vectors $x \in V_i$ and $y \in V_l$, for some n_0, for all $n \geqslant n_0$, and all j, k, r, s we have*

© Blackwell Publishers Ltd 2000

$$E|C_{jk}^{rs}| \leqslant \|C_j\| \, \|C_k\| K_n \tag{5.5}$$

where $nK_n \to 0$.

PROOF OF LEMMA 5.3. Write

$$E|x_j' A_n Z_r Z_s' A_n^t y_k| I(\|A_n Z_r\| > 1) = E|\langle Z_s, A_n^t y_k\rangle| E|\langle Z_r, A_n^t x_j\rangle| I(\|A_n Z_r\| > 1)$$

where

$$\begin{aligned} E|\langle Z_s, A_n^t y_k\rangle| &= b_{nk} E|\langle Z_s, y_{nk}\rangle| \\ &\leqslant b_{nk} E\|P_l Z_2\| \\ &\leqslant \|P_l A_n\| E\|P_l Z_2\| \end{aligned}$$

and

$$nE|\langle Z_r, A_n^t x_j\rangle| I(\|A_n Z_r\| > 1) \leqslant K_4$$

and similarly

$$nE|x_j' A_n Z_r Z_s' A_n^t y_k| I(\|A_n Z_s\| > 1) \leqslant \|P_i A_n\| E\|P_i A_n\| K_4$$

so that (5.5) holds with

$$nK_n = K_4(\|P_i A_n\| E\|P_i Z_1\| + \|P_l A_n\| E\|P_l Z_1\|)$$

which is the same K_n as for Lemma 5.2, so $nK_n \to 0$. This concludes the proof of Lemma 5.3.

In the proof of Proposition 4.2 we get using Lemma 5.3

$$\begin{aligned} E|C| &\leqslant \sum_{t=1}^{n} \sum_{j=-\infty}^{\infty} \sum_{k \neq j} \|C_j\| \, \|C_{k+h}\| K_n \\ &\leqslant nK_n \left(\sum_{j=-\infty}^{\infty} \|C_j\| \right)^2 \to 0 \end{aligned}$$

and so $C \to 0$ in probability.

Finally we can write

$$\begin{aligned} D_{jk} &= x' A_n C_j \mu_n \mu_n' C_k^t A_n^t y \\ &= r_j r_k \langle \mu_n, A_n^t x_j\rangle \langle \mu_n, A_n^t y_k\rangle. \end{aligned}$$

and use the bound proved in

LEMMA 5.4. *For all unit vectors* $x \in V_i$ *and* $y \in V_l$, *for some* n_0, *for all* $n \geqslant n_0$, *and all* j, k, *we have*

$$|D_{jk}| \leqslant \|C_j\| \, \|C_k\| K_n$$

where $nK_n \to 0$.

PROOF OF LEMMA 5.4. Recall that for all large n we have $|\langle \mu_n, A_n^t x_j\rangle| < K_4/n$ from the proof of Lemma 5.2 above. Likewise $|\langle \mu_n, A_n^t y_k\rangle| \leqslant K_4/n$ so we can take $K_n = (K_4/n)^2$, which concludes the proof of Lemma 5.4.

Now we have from Lemma 5.4

© Blackwell Publishers Ltd 2000

$$|D| \leqslant \sum_{t=1}^{n} \sum_{j=-\infty}^{\infty} \sum_{k\neq j} \|C_j\| \, \|C_{k+h}\| K_n$$

$$\leqslant nK_n \left(\sum_{j=-\infty}^{\infty} \|C_j\| \right)^2 \to 0$$

which completes the proof of Proposition 4.2.

We now prove Proposition 4.3.

PROOF OF PROPOSITION 4.3. Define

$$S_n^{(m)} = \sum_{|j|\leqslant m} C_j \left(\sum_{t=1}^{n} Z_{t-j} Z'_{t-j} \right) C_{j+h}^t$$

$$d_n^{(m)} = n \sum_{|j|\leqslant m} C_j B_n C_{j+h}^t$$

$$W^{(m)} = \sum_{|j|\leqslant m} C_j W C_{j+h}^t$$

for $m \geqslant 1$. We will now apply (4.2) to show that

$$\left(A_n \left(\sum_{t=1}^{n} Z_{t-j} Z'_{t-j} - B_n \right) A_n^t \colon |j| \leqslant m \right) \Rightarrow (W, \ldots, W). \tag{5.6}$$

Since we know that the convergence holds for the $j = 0$ component, it suffices to show that the difference between this component and any other component tends to zero in probability. This difference consists of at most $2m$ identically distributed terms, any one of which must tend to zero in probability in view of the convergence (4.2) above, and so (5.6) holds. Now by the continuous mapping theorem we obtain from (5.6) that

$$A_n(S_n^{(m)} - d_n^{(m)}) A_n^t \Rightarrow W^{(m)}$$

and since

$$\sum_{|j|>m} \|C_j W C_{j+h}^t\| \leqslant \|W\| \sum_{|j|>m} \|C_j\| \, \|C_{j+h}\|$$

$$\leqslant \|W\| \left(\sum_{|j|>m} \|C_j\|^2 \right)^{1/2} \left(\sum_{|j|>m} \|C_{j+h}\|^2 \right)^{1/2} \to 0$$

as $m \to \infty$ we also have

$$W^{(m)} \to \sum_{j=-\infty}^{\infty} C_j W C_{j+h}^t$$

almost surely. Then in order to establish (4.5) it suffices to show (e.g. by Billingsley (1968) Theorem 4.2) that

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(\|T\| > \varepsilon) = 0 \tag{5.7}$$

where

© Blackwell Publishers Ltd 2000

$$T = \sum_{|j|>m} \sum_{t=1}^{n} A_n C_j (Z_{t-j} Z'_{t-j} - B_n) C^t_{j+h} A^t_n.$$

Note that $\|T\|^2 = \sum_{i,j} |e'_i T e_j|^2$ where $e_1, \ldots, e_d$ is the standard basis for $\mathbb{R}^d$. Then in order to show that (5.7) holds it suffices to show that

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(|x' T y| > \varepsilon) = 0 \tag{5.8}$$

for $x \in V_i$ and $y \in V_l$ where $1 \leqslant i,\ l \leqslant p$. Write $T = A + B - C$ where

$$A = \sum_{|j|>m} \sum_{t=1}^{n} A_n C_j Z_{t-j} Z'_{t-j} C^t_{j+h} A^t_n I(\|A_n Z_{t-j}\| \leqslant 1)$$

$$B = \sum_{|j|>m} \sum_{t=1}^{n} A_n C_j Z_{t-j} Z'_{t-j} C^t_{j+h} A^t_n I(\|A_n Z_{t-j}\| > 1)$$

$$C = \sum_{|j|>m} \sum_{t=1}^{n} A_n C_j B_n C^t_{j+h} A^t_n.$$

Since $T = A + B - C$ it suffices to show that (5.8) holds with T replaced by A, B, or C. We begin with B. Choose $\delta > 0$ with $\delta < 1/a_p$ and $\delta \leqslant 1$ and let $\delta_1 = \delta/2 < 1$. Use the Markov inequality along with the fact that $|x + y|^{\delta_1} \leqslant |x|^{\delta_1} + |y|^{\delta_1}$ to obtain

$$\begin{aligned}
P(|x' B y| > \varepsilon) &\leqslant \varepsilon^{-\delta_1} E|x' B y|^{\delta_1} \\
&\leqslant \varepsilon^{-\delta_1} \sum_{|j|>m} nE|x' C_j A_n Z_1 Z'_1 A^t_n C^t_{j+h} y|^{\delta_1} I(\|A_n Z_1\| > 1) \\
&\leqslant \varepsilon^{-\delta_1} \sum_{|j|>m} \|C_j\|^{\delta_1} \|C_{j+h}\|^{\delta_1} nE|x'_j A_n Z_1 Z'_1 A^t_n y_{j+h}|^{\delta_1} I(\|A_n Z_1\| > 1) \\
&\leqslant \varepsilon^{-\delta_1} \left[\sum_{|j|>m} \|C_{j+h}\|^{\delta} \right]^{1/2} \\
&\quad \cdot \left[\sum_{|j|>m} \|C_j\|^{\delta} (nE|x'_j A_n Z_1 Z'_1 A^t_n y_{j+h}|^{\delta_1} I(\|A_n Z_1\| > 1))^2 \right]^{1/2}
\end{aligned}$$

by the Schwartz inequality. Then in order to show that (5.8) holds with T replaced by B it will suffice to show that for some $K_4 > 0$ we have

$$nE|x'_j A_n Z_1 Z'_1 A^t_n y_{j+h}|^{\delta_1} I(\|A_n Z_1\| > 1) \leqslant K_4 \tag{5.9}$$

for all j, h and all n large. Apply the Schwartz inequality again to see that

$$\begin{aligned}
&nE|x'_j A_n Z_1 Z'_1 A^t_n y_{j+h}|^{\delta_1} I(\|A_n Z_1\| > 1) \\
&\quad = nE|\langle A_n Z_1, x_j \rangle \langle A_n Z_1, y_{j+h} \rangle|^{\delta_1} I(\|A_n Z_1\| > 1) \\
&\quad \leqslant \sqrt{nE|\langle A_n Z_1, x_j \rangle|^{\delta} I(\|A_n Z_1\| > 1) \cdot nE|\langle A_n Z_1, y_{j+h} \rangle|^{\delta} I(\|A_n Z_1\| > 1)}
\end{aligned}$$

where $nE|\langle A_n Z_1, x_j \rangle|^{\delta} I(\|A_n Z_1\| > 1) < K_4$ for all n large and all j by an application of

© Blackwell Publishers Ltd 2000

Lemma 2.1 part (iv) with $\eta = \delta < 1/a_i$. A similar argument holds for the remaining term, which establishes (5.9), and so (5.8) holds with T replaced by B.

Next we look at A and C which requires us to consider different cases. First suppose that $a_i > \frac{1}{2}$ and $a_l > \frac{1}{2}$. As in the proof of Proposition 4.2 we write $C_j^t x = r_j x_j$, $A_n^t x_j = a_{nj} x_{nj}$, $C_j^t y = \rho_j y_j$ and $A_n^t y_j = b_{nj} y_{nj}$ where x_j, x_{nj}, y_j, y_{nj} are unit vectors and r_j, a_{nj}, ρ_j, b_{nj} are positive reals. By the Markov inequality we get

$$\begin{aligned} P(|x'Ay| > \varepsilon) &\leqslant \varepsilon^{-1} E|x'Ay| \\ &\leqslant \varepsilon^{-1} \sum_{|j|>m} nE|x'C_j A_n Z_1 Z_1' A_n^t C_{j+h}^t y| I(\|A_n Z_1\| \leqslant 1) \\ &\leqslant \varepsilon^{-1} \sum_{|j|>m} \|C_j\| \, \|C_{j+h}\| nE|\langle Z_1, A_n^t x_j\rangle \langle Z_1, A_n^t y_{j+h}\rangle| I(\|A_n Z_1\| \leqslant 1). \end{aligned}$$

Next we will show that for some $K_2 > 0$ we have

$$nE|\langle Z_1, A_n^t x_j\rangle \langle Z_1, A_n^t y_{j+h}\rangle| I(\|A_n Z_1\| \leqslant 1) \leqslant K_2 \tag{5.10}$$

for all j, h and all n large. By the Schwartz inequality we have that the left hand side of (5.10) is bounded above by

$$\sqrt{nE\langle Z_1, A_n^t x_j\rangle^2 I(\|A_n Z_1\| \leqslant 1) nE|\langle Z_1, A_n^t y_{j+h}\rangle^2 I(\|A_n Z_1\| \leqslant 1)}$$

where we have $nE\langle Z_1, A_n^t x_j\rangle^2 I(\|A_n Z_1\| \leqslant 1) < K_2$ independent of j and n large as in the proof of Lemma 5.1 above, since $a_i > \frac{1}{2}$ (so that $\zeta = 2 > 1/a_i$ in Lemma 2.1 part (ii)). Similarly $nE\langle Z_1, A_n^t y_{j+h}\rangle^2 I(\|A_n Z_1\| \leqslant 1) < K_2$ independent of j and n large by another application of Lemma 2.1 part (ii) (note that $a_l > \frac{1}{2}$ so that $\zeta = 2 > 1/a_l$). Hence (5.10) holds for all j, h and all n large, and so for n large we have

$$\begin{aligned} P(|x'Ay| > \varepsilon) &\leqslant \varepsilon^{-1} K_2 \sum_{|j|>m} \|C_j\| \, \|C_{j+h}\| \\ &\leqslant \varepsilon^{-1} K_2 \left(\sum_{|j|>m} \|C_j\|^2 \right)^{1/2} \left(\sum_{|j|>m} \|C_{j+h}\|^2 \right)^{1/2} \end{aligned}$$

which tends to zero as $m \to \infty$, and so (5.8) holds with T replaced by A in this case. Since $|x'Cy| = |Ex'Ay|$ we also have that (5.8) holds with T replaced by C in this case. Finally suppose that either $a_i \leqslant \frac{1}{2}$ or $a_l \leqslant \frac{1}{2}$ (or both). As in the proof of Proposition 4.2 define $Z_{in} = Z_i I(\|A_n Z_i\| \leqslant 1)$ and let $Q_{in} = Z_{in} Z_{in}' - EZ_{in} Z_{in}' = Z_{in} Z_{in}' - B_n$. By the Markov inequality we get

$$\begin{aligned} P(|x'(A-C)y| > \varepsilon) &\leqslant \varepsilon^{-2} E[(x'(A-C)y)^2] \\ &= \varepsilon^{-2} E\left(\sum_{|j|>m} \sum_{t=1}^{n} x' A_n C_j Q_{t-j,n} C_{j+h}^t A_n^t y \right)^2 \\ &= \varepsilon^{-2} \sum_{|j|>m} \sum_{t=1}^{n} \sum_{|j'|>m} \sum_{t'=1}^{n} E(x' A_n C_j Q_{t-j,n} C_{j+h}^t A_n^t y \cdot x' A_n C_{j'} Q_{t'-j',n} C_{j'+h}^t A_n^t y). \end{aligned}$$

Since Q_{in} are independent with mean zero the only terms that contribute to the sum above are those for which $t - j = t' - j'$. Then the sum above equals

© Blackwell Publishers Ltd 2000

$$\sum_{|j|>m}\sum_{t=1}^{n}\sum_{j'=1-t+j}^{n-t+j} r_j\rho_{j+h}r_{j'}\rho_{j'+h}E(x_j'A_nQ_{t-j,n}A_n^t y_{j+h}x_{j'}'A_nQ_{t-j,n}A_n^t y_{j'+h}).$$

Next we will show that for some $K_2 > 0$ we have

$$nE(x_j'A_nQ_{1n}A_n^t y_{j+h}x_{j'}'A_nQ_{1n}A_n^t y_{j'+h}) \leqslant K_2 \tag{5.11}$$

for all j, j', h and all n large. Write

$$\begin{aligned} x_j'A_nQ_{1n}A_n^t y_k &= x_j'A_nZ_{1n}Z_{1n}'A_n^t y_k - Ex_j'A_nZ_{1n}Z_{1n}'A_n^t y_k \\ &= \langle A_nZ_{1n}, x_j\rangle\langle A_nZ_{1n}, y_k\rangle - E\langle A_nZ_{1n}, x_j\rangle\langle A_nZ_{1n}, y_k\rangle, \\ &= \xi_{nj}\beta_{nk} - E\xi_{nj}\beta_{nk} \end{aligned}$$

where

$$\xi_{nj} = \langle A_nZ_{1n}, x_j\rangle.$$
$$\beta_{nk} = \langle A_nZ_{1n}, y_k\rangle$$

Then

$$\begin{aligned} &E(x_j'A_nQ_{1n}A_n^t y_{j+h}x_{j'}'A_nQ_{1n}A_n^t y_{j'+h}) \\ &\quad= E(\xi_{nj}\beta_{n,j+h} - E\xi_{nj}\beta_{n,j+h})(\xi_{nj'}\beta_{n,j'+h} - E\xi_{nj'}\beta_{n,j'+h}) \\ &\quad= E\xi_{nj}\beta_{n,j+h}\xi_{nj'}\beta_{n,j'+h} - E\xi_{nj}\beta_{n,j+h}E\xi_{nj'}\beta_{n,j'+h} \\ &\qquad - E\xi_{nj}\beta_{n,j+h}E\xi_{nj'}\beta_{n,j'+h} + E\xi_{nj}\beta_{n,j+h}E\xi_{nj'}\beta_{n,j'+h} \\ &\quad= E\xi_{nj}\beta_{n,j+h}\xi_{nj'}\beta_{n,j'+h} - E\xi_{nj}\beta_{n,j+h}E\xi_{nj'}\beta_{n,j'+h} \end{aligned}$$

where

$$\begin{aligned} E|\xi_{nj}\beta_{n,j+h}\xi_{nj'}\beta_{n,j'+h}| &\leqslant (E(\xi_{nj}\beta_{n,j+h})^2)^{1/2}(E(\xi_{nj'}\beta_{n,j'+h})^2)^{1/2} \\ &\leqslant (E\xi_{nj}^4)^{1/4}(E\beta_{n,j+h}^4)^{1/4}(E\xi_{nj'}^4)^{1/4}(E\beta_{n,j'+h}^4)^{1/4} \\ &= (E\xi_{nj}^4E\beta_{n,j+h}^4E\xi_{nj'}^4E\beta_{n,j'+h}^4)^{1/4} \end{aligned}$$

by the Schwartz inequality, and similarly

$$E|\xi_{nj}\beta_{n,j+h}| \leqslant (E\xi_{nj}^4E\beta_{n,j+h}^4)^{1/4}$$
$$E|\xi_{nj'}\beta_{n,j'+h}| \leqslant (E\xi_{nj'}^4E\beta_{n,j'+h}^4)^{1/4}$$

so that together we have

$$E(x_j'A_nQ_{t-j,n}A_n^t y_{j+h}x_{j'}'A_nQ_{t-j,n}A_n^t y_{j'+h}) \leqslant 2(E\xi_{nj}^4E\beta_{n,j+h}^4E\xi_{nj'}^4E\beta_{n,j'+h}^4)^{1/4}$$

where

$$nE\xi_{nj}^4 = nE\langle A_nZ_{1n}, x_j\rangle^4 = nE\langle A_nZ_1, x_j\rangle^4 I(\|A_nZ_1\| \leqslant 1)$$

which is bounded above by some $K_2 < \infty$, for all $n \geqslant n_0$ and all j by an application of part (ii) of Lemma 2.1 with $4 = \zeta > 1/a_i$. A similar argument shows that $nE\beta_{nk}^4 < K_2$ for all k and all n large (apply part (ii) of Lemma 2.1 again with $4 = \zeta > 1/a_l$). Then (5.11) holds for j, j', h and all n large, and so for all large n we have

© Blackwell Publishers Ltd 2000

$$P(|x'(A-C)y| > \varepsilon) \leqslant \varepsilon^{-2} 2K_2 n^{-1} \sum_{|j|>m} \sum_{t=1}^{n} \sum_{j'=1-t+j}^{n-t+j} \|C_j\| \, \|C_{j+h}\| \, \|C_{j'}\| \, \|C_{j'+h}\|$$

$$= \varepsilon^{-2} 2K_2 \sum_{|j|>m} \|C_j\| \, \|C_{j+h}\| n^{-1} \sum_{t=1}^{n} \sum_{j'=1-t+j}^{n-t+j} \|C_{j'}\| \, \|C_{j'+h}\|$$

$$\leqslant \varepsilon^{-2} 2K_2 \left(\sum_{|j|>m} \|C_j\|^2 \right)^{1/2} \left(\sum_{|j|>m} \|C_{j+h}\|^2 \right)^{1/2}$$

$$\cdot \left(\sum_{j'=-\infty}^{\infty} \|C_{j'}\|^2 \right)^{1/2} \left(\sum_{j'=-\infty}^{\infty} \|C_{j'+h}\|^2 \right)^{1/2}$$

which tends to zero as $m \to \infty$, and so (5.8) holds with T replaced by $A - C$ in this case, which concludes the proof of Proposition 4.3.

REFERENCES

ANDERSON, P. and MEERSCHAERT, M. (1997) Periodic moving averages or random variables with regularly varying tails, *Ann. Statist.*, 25, 771–785.

ARAUJO, A. and GINÉ, E. (1980) *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, New York.

BHANSALI, R. (1993) Estimation of the impulse response coefficients of a linear process with infinite variance, *J. Multivariate Anal.*, 45, 274–290.

BILLINGSLEY, P. (1968) *Convergence of Probability Measures*, Wiley, New York.

BROCKWELL, P. and DAVIS, R. (1991) *Time Series: Theory and Methods*, 2nd Ed., Springer-Verlag, New York.

DAVIS, R. and RESNICK, S. (1985a) Limit theory for moving averages of random variables with regularly varying tail probabilities, *Ann. Probab.*, 13, 179–195.

DAVIS, R. and RESNICK, S. (1985b) More limit theory for the sample correlation function of moving averages, *Stoch. Proc. Appl.*, 20, 257–279.

DAVIS, R. and RESNICK, S. (1986) More limit theory for the sample covariance and correlation functions of moving averages, *Ann. Statist.*, 14, 533-558.

DAVIS, R., MARENGO, J. and RESNICK, S. (1985) Extremal properties of a class of multivariate moving averages, *Proceedings of the 45th International Statistical Institute*, Vol. 4, Amsterdam.

DAVIS, R. and MARENGO, J. (1990) Limit theory for the sample covariance and correlation matrix functions of a class of multivariate linear processes, *Commun. Statist. Stochastic Models*, 6, 483–497.

Feller, W. (1971) *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd Ed., Wiley, New York.

JANICKI, A. and WERON, A. (1994) *Simulation and Chaotic Behavior of α-Stable Stochastic Processes*, Marcel Dekker, New York.

JANSEN, D. and DE VRIES, C. (1991) On the frequency of large stock market returns: Putting booms and busts into perspective, *Review of Econ. and Statist.*, 23, 18–24.

KOKOSZKA, P. and TAQQU, M. (1994) Infinite variance stable ARMA processes, *J. Time Series Anal.*, 115, 203–220.

KOKOSZKA, P. and TAQQU, M. (1996) Parameter estimation for infinite variance fractional ARIMA, *Ann. Statist.*, 24, 1880–1913.

LORETAN, M. and PHILLIPS, P. (1994) Testing the covariance stationarity of heavy tailed time series, *J. Empirical Finance* 1, 211–248.

MANDELBROT, B. (1963) The variation of certain speculative prices, *J. Business*, 36, 394–419.

MEERSCHAERT, M. (1993) Regular variation and generalized domains of attraction in $\mathbb{R}^k$, *Stat. Probab. Lett.* 18, 233–239.

© Blackwell Publishers Ltd 2000

Meerschaert, M. (1994) Norming operators for generalized domains of attraction, *J. Theoretical Probab.*, 7, 793–798.
Meerschaert, M. and Scheffler, H. P. (1999) Sample covariance matrix for random vectors with regularly varying tails, *J. Theoretical Probab.*, 12, 821–838.
Meerschaert, M. and Scheffler, H. P. (1999) Multivariable regular variation of functions and measures, *J. Applied Analysis*., 5, 125–146.
Meerschaert, M. and Scheffler, H. P. (1998) Sample cross-correlations for moving averages with regularly varying tails, preprint.
Mikosch, T., Gadrich, T., Klüppenberg, C. and Adler, R. (1995) Parameter estimation for ARMA models with infinite variance innovations, *Ann. Statist.*, 23, 305–326.
Mikknik, S. and Rachev, S. (1998) *Asset and Option Pricing with Alternative Stable Models*, Wiley, New York.
Nikias, C. and Shao, M. (1995) *Signal Processing with Alpha Stable Distributions and Applications*, Wiley, New York.
Resnick, S. (1986) Point processes, regular variation, and weak convergence, *Adv. Appl. Probab.*, 18, 66–138.
Resnick, S. and Stărică, C. (1995) Consistency of Hill's estimator for dependent data, *J. Appl. Probab.*, 32, 139–167.
Samorodnitsky, G. and Taqqu, M. (1994) *Stable non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman and Hall, London.
Scheffler, H. P. (1998) Multivariate R–O varying measures, preprint.
Seneta, E. (1976) *Regularly Varying Functions*, Lecture Notes Mathematics, Springer, Berlin-Heidelberg, New York.
Sharpe, M. (1969) Operator-stable probability distributions on vector groups, *Trans. Amer. Math. Soc.* 136, 51–65.

© Blackwell Publishers Ltd 2000