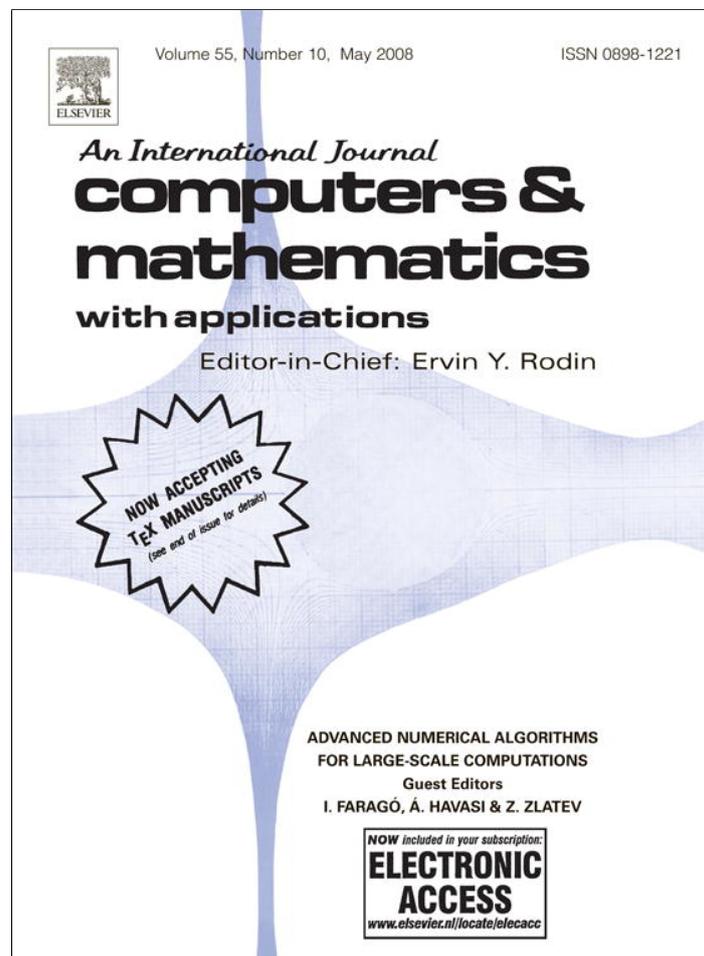


Provided for non-commercial research and education use.
Not for reproduction, distribution or commercial use.



This article appeared in a journal published by Elsevier. The attached copy is furnished to the author for internal non-commercial research and education use, including for instruction at the authors institution and sharing with colleagues.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to personal, institutional or third party websites are prohibited.

In most cases authors are permitted to post their version of the article (e.g. in Word or Tex form) to their personal website or institutional repository. Authors requiring further information regarding Elsevier's archiving and manuscript policies are encouraged to visit:

<http://www.elsevier.com/copyright>



Numerical solutions for fractional reaction–diffusion equations

Boris Baeumer^a, Mihály Kovács^{a,*}, Mark M. Meerschaert^b

^aDepartment of Mathematics, University of Otago, Dunedin, New Zealand

^bDepartment of Statistics and Probability, Michigan State University, A413 Wells Hall, East Lansing, MI 48824-1027, USA

Abstract

Fractional diffusion equations are useful for applications in which a cloud of particles spreads faster than predicted by the classical equation. In a fractional diffusion equation, the second derivative in the spatial variable is replaced by a fractional derivative of order less than two. The resulting solutions spread faster than the classical solutions and may exhibit asymmetry, depending on the fractional derivative used. Fractional reaction–diffusion equations combine the fractional diffusion with a classical reaction term. In this paper, we develop a practical method for numerical solution of fractional reaction–diffusion equations, based on operator splitting. Then we present results of numerical simulations to illustrate the method, and investigate properties of numerical solutions. We also discuss applications to biology, where the reaction term models species growth and the diffusion term accounts for movements.

© 2007 Elsevier Ltd. All rights reserved.

Keywords: Fractional reaction–diffusion; Operator semigroups; Operator splitting; Infinitely divisible distributions; Invasive species

1. Introduction

Reaction–diffusion equations are useful in many areas of science and engineering [1–5]. In applications to population biology, the reaction term models growth, and the diffusion term accounts for migration [6,7]. The classical diffusion term originates from a model in physics [8,9]. Recent research indicates that the classical diffusion equation is inadequate to model many real situations, where a particle plume spreads faster than that predicted by the classical model, and may exhibit significant asymmetry [10]. These situations are called anomalous diffusion [11,12]. One popular model for anomalous diffusion is the fractional diffusion equation, where the usual second derivative in space is replaced by a fractional derivative of order $0 < \alpha < 2$ [13,14]. Solutions to the fractional diffusion equation spread at a faster rate than the classical diffusion equation, and may exhibit asymmetry. However, the fundamental solutions of these equations still exhibit useful scaling properties that make them attractive for applications.

The classical diffusion equation $\partial u / \partial t = D \partial^2 u / \partial x^2$ is closely connected to the central limit theorem of statistics, which states that a normalised sum of independent and identically distributed random variables has a probability distribution that converges to a normal distribution as the number of summands tends to infinity [15,16]. Here

* Corresponding author.

E-mail addresses: bbaeumer@maths.otago.ac.nz (B. Baeumer), mkovacs@maths.otago.ac.nz (M. Kovács), mcubed@stt.msu.edu (M.M. Meerschaert).

“normalised” means we divide the sum by $n^{1/2}$, where n is the number of summands. Thinking of the random variables as particle jumps leads to a close connection with the diffusion equation, and forms the basis for the well-known result that the fundamental solution to the diffusion equation is a family of normal probability densities. A random walk formed using these particle jumps converges to a stochastic process called Brownian motion. The probability density of a Brownian motion at time $t > 0$ is normal with the standard deviation (spread) proportional to $t^{1/2}$. These probability densities also give the fundamental solution to the diffusion equation.

The fractional diffusion equation $\partial u/\partial t = D \partial^\alpha u/\partial x^\alpha$ relates to another central limit theorem. The usual result assumes that the individual random jump has a finite standard deviation. If instead we assume that the jumps X have power-law probability tails $P(|X| > r) \approx r^{-\alpha}$ for some $0 < \alpha < 2$, then the standard deviation is infinite, and under certain technical assumptions the distribution of the normalised sum converges to another distribution called a stable distribution [15,17]. In this case, we normalise by $n^{1/\alpha}$, and the limit Y also has power-law probability tails $P(|Y| > r) \approx r^{-\alpha}$ for the same α . A random walk formed using these particle jumps converges to a stochastic process called a stable Lévy motion, whose probability densities spread proportional to $t^{1/\alpha}$, and these densities are the fundamental solution to the fractional diffusion equation [18,19]. Particle traces are random fractals of dimension α , even in the classical case $\alpha = 2$ [20]. Since the parameter α codes the scaling, the order of the derivative, and the fractal dimension, there are several possibilities for model fitting.

In the multivariable case, classical diffusion equations spread at the same rate $t^{1/2}$ and represent the probability densities of multivariable normal random vectors. This normal limit comes from the multivariable central limit theorem, as the normalised sum of independent particle jumps, each jump represented by a random vector \mathbf{X} . If the vector particle jumps have power-law probability tails $P(\|\mathbf{X}\| > r) \approx r^{-\alpha}$ for some $0 < \alpha < 2$, then another central limit theorem says that, under certain technical conditions, the limiting distribution of particle jumps is a multivariable stable law \mathbf{Y} that retains the power-law tails $P(\|\mathbf{Y}\| > r) \approx r^{-\alpha}$ [16,17]. The associated vector random walk converges to a multivariable stable Lévy motion, whose probability densities spread proportional to $t^{1/\alpha}$ [16, Example 11.2.18.], and these densities are the fundamental solution to the vector fractional diffusion equation [19]. If the power-law probability tail index α varies with the spatial coordinate, then the associated diffusion equation involves fractional derivatives of different order in each coordinate [21,22].

Numerical solutions of fractional diffusion equations have recently been developed by several authors. Constant coefficient equations can be solved by the Fourier inversion, since these equations have analytic solutions in the Fourier space [19,21,22]. Variable coefficient equations admit finite difference solutions, based on a finite difference approximation to the fractional derivative [23–29]. Other approaches include variations on the method of lines [30–32], a finite element scheme [33–35], and particle tracking [36].

The classical one-dimensional reaction–diffusion equation

$$\frac{\partial u(x, t)}{\partial t} = D \frac{\partial^2 u(x, t)}{\partial x^2} + \tilde{f}(u(x, t)), \quad u(x, 0) = u_0(x) \tag{1.1}$$

is used in population biology to model the spread of invasive species [6,7]. Here $u(x, t)$ is the population density at location $x \in \mathbb{R}$ and time $t > 0$. The first term on the right-hand side is the diffusion term; it models migration. The second term is the reaction term that models population growth; a typical choice is the Kolmogorov–Fisher equation $\tilde{f}(u(x, t)) = ru(x, t)(1 - u(x, t)/K)$ where r is the intrinsic growth rate of a species and K is the environmental carrying capacity, representing the maximum sustainable population density. A more general fractional reaction–diffusion equation

$$\frac{\partial u(x, t)}{\partial t} = D \frac{\partial^\alpha u(x, t)}{\partial x^\alpha} + \tilde{f}(u(x, t)), \quad u(x, 0) = u_0(x) \tag{1.2}$$

with $0 < \alpha \leq 2$ appears in [37]. Solutions to (1.2) exhibit accelerating fronts with power-law leading edges [38], behaviour seen in many invasive species [39–44]. Eq. (1.2) is a special case of the reaction–diffusion equation

$$\frac{\partial u(x, t)}{\partial t} = [Au(\cdot, t)](x) + \tilde{f}(x, u(x, t)), \quad u(x, 0) = u_0(x) \quad x \in \mathbb{R}^d, \tag{1.3}$$

where A is a pseudo-differential operator of the form (3.5) (see also [45]) and $\tilde{f} : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$. Numerical solutions of (1.3) are obtained in this paper by the method of sequential operator splitting, focusing in particular on the case

where A represents a multivariable fractional derivative operator. Our approach is based on the theory of operator semigroups. We write (1.3) as an ordinary differential equation in t on a suitable Banach space X , and then use abstract functional analytic results to show the convergence of the operator splitting scheme. Operator splitting methods have been applied to classical reaction–diffusion equations in biology [46]. Several operator splitting techniques exist in the literature [47–51]. The sequential splitting used in this paper was chosen because it yields useful error bounds in some cases of practical interest, see Corollary 4.6. Also, this splitting method leads to discrete-time growth-dispersal models, which are widely used in applications to population biology, see Remark 4.7. Results of this paper should prove useful in population biology, and they may also find wide applications in geophysics and finance, where classical reaction–diffusion equations are commonly used, and where fractional diffusion is often observed.

2. Analytical framework

Let X be a Banach space with associated norm $\|v\|$, and consider the abstract reaction–diffusion equation

$$\dot{u}(t) = Au(t) + f(u(t)), \quad t > 0, \quad u(0) = u_0, \tag{2.1}$$

where $u : [0, \infty) \rightarrow X$ and $f : X \rightarrow X$. Here A is the generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on X , a one parameter family of linear operators on $T(t) : X \rightarrow X$ such that: $T(0) = I$, the identity operator ($Iu = u$); each $T(t)$ is bounded, meaning that there exists a real number $M > 0$ depending on $t > 0$, such that $\|T(t)u\| \leq M\|u\|$ for all $u \in X$; $T(t+s) = T(t)T(s)$ for $t, s \geq 0$; $t \mapsto T(t)u$ is continuous in the Banach space norm, for all $u \in X$; and the generator $Au = \lim_{h \rightarrow 0^+} h^{-1}(T(h)u - u)$ exists for at least some nonzero $u \in X$. We call the set $\mathcal{D}(A) \subset X$ for which this limit exists the domain of the linear operator A , and we say that the semigroup $\{T(t)\}_{t \geq 0}$ is generated by A . We say that $u : [0, \delta) \rightarrow X$ is a local classical/strong solution of (2.1) if u is continuous on $[0, \delta)$, continuously differentiable on $(0, \delta)$, $u(t) \in \mathcal{D}(A)$ for $t \in (0, \delta)$, and u satisfies (2.1) on $(0, \delta)$. If δ can be chosen arbitrarily large, then u is a global classical/strong solution of (2.1). A function $u : [0, \delta) \rightarrow X$ is a local mild solution of (2.1) if u is continuous and satisfies the corresponding integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(u(s)) ds \tag{2.2}$$

for $0 \leq t < \delta$. We note that the integral in (2.2) is a Bochner integral [52–55], an extension of the Lebesgue integral to the Banach space setting which coincides with a Riemann integral if the integrand is continuous in the Banach space norm. If δ can be chosen arbitrarily large, then u is a global mild solution of (2.1).

The reaction–diffusion equation (2.1) has two important special cases, the reaction equation

$$\dot{u}(t) = f(u(t)), \quad t > 0, \quad u(0) = u_0 \tag{2.3}$$

and the diffusion equation

$$\dot{u}(t) = Au(t), \quad t > 0, \quad u(0) = u_0. \tag{2.4}$$

It is well known that the abstract reaction–diffusion equation (2.1) can be solved via an operator splitting method called the Trotter Product Formula, as long as solutions to the two component equations (2.3) and (2.4) can be computed. We summarise these known results in Theorem 2.1 for the convenience of the reader.

We say that $f : X \rightarrow X$ is globally Lipschitz continuous, if for some $K > 0$, we have $\|f(u) - f(v)\| \leq K\|u - v\|$ for all $u, v \in X$, and is locally Lipschitz continuous, if the latter holds for $\|u\|, \|v\| \leq M$ with $K = K(M)$ for any $M > 0$. If the reaction equation (2.3) has a unique global mild solution $u(t) = S(t)u_0$ for any initial condition $u_0 \in X$, then the collection of nonlinear operators $\{S(t)\}_{t \geq 0}$ forms a semigroup called the flow of the abstract differential equation $\dot{u} = f(u)$. Then we say that the collection $\{S(t)\}_{t \geq 0}$ is generated by f . If the reaction–diffusion equation (2.1) has a unique global mild solution $u(t) = W(t)u_0$ for any $u_0 \in X$, then the collection of nonlinear operators $\{W(t)\}_{t \geq 0}$ forms a semigroup called the solution operators of the abstract differential equation $\dot{u} = Au + f(u)$. Theorem 2.1 asserts that the mild solution to the abstract reaction–diffusion equation (2.1) can be computed as an approximation using the solution operators $\{S(t)\}_{t \geq 0}$ of the abstract reaction equation $\dot{u} = f(u)$, and $\{T(t)\}_{t \geq 0}$ of the abstract diffusion equation $\dot{u} = Au$. Iterations of the form used here, with $u_{n+1} = [T(\tau)S(\tau)]u_n$ or $U_{n+1} = [S(\tau)T(\tau)]U_n$ are called sequential splitting, a type of operator splitting.

Theorem 2.1. *Suppose that X is a Banach space and $f : X \rightarrow X$ is globally Lipschitz continuous in the Banach space norm. Then the reaction equation (2.3) has a unique global strong solution $u(t) = S(t)u_0$ for any initial condition $u_0 \in X$, and this flow generated by f is given by*

$$u(t) = S(t)u_0 = u_0 + \int_0^t f(u(s)) \, ds. \tag{2.5}$$

If A is the generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on X , then the diffusion equation (2.4) has a unique global mild solution $u(t) = T(t)u_0$ for any initial condition $u_0 \in X$, and if $u_0 \in \mathcal{D}(A)$, the domain of the generator, then this is also the unique global strong solution. Then for any $u_0 \in X$ the abstract reaction–diffusion equation (2.1) has a unique global mild solution

$$u(t) = W(t)u_0 = T(t)u_0 + \int_0^t T(t-s)f(u(s)) \, ds \tag{2.6}$$

that can be computed by the Trotter Product Formula

$$W(t)u_0 = \lim_{n \rightarrow \infty} \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right]^n u_0 = \lim_{n \rightarrow \infty} \left[S\left(\frac{t}{n}\right) T\left(\frac{t}{n}\right) \right]^n u_0. \tag{2.7}$$

If $u_0 \in \mathcal{D}(A)$ and $f : X \rightarrow X$ is continuously differentiable, then (2.6) is the unique global strong solution of the abstract reaction–diffusion equation (2.1) and this strong solution can also be computed via (2.7).

Proof. If $f : X \rightarrow X$ is globally Lipschitz continuous, then for all $u_0 \in X$ there is a unique global mild solution $u(t) := W(t)u_0$ of (2.1) with $\|W(t)u_0 - W(t)v_0\| \leq M_T \|u_0 - v_0\|$, $t \in [0, T]$. See, for example, [55, Section 6.1]. Since the reaction equation (2.3) is a special case of (2.1) with $A = 0$, it follows that (2.3) has a unique mild solution given by (2.2) with $T(t)u_0 = u_0$, and hence (2.5) holds for all $t > 0$. This is also a strong solution, since if u and f are continuous, then $t \mapsto f(u(t))$ is continuous, and $t \mapsto \int_0^t f(u(s)) \, ds$ is differentiable with $\frac{d}{dt} \int_0^t f(u(s)) \, ds = f(u(t))$. See, for example, [54, p. 67]. Therefore $t \rightarrow u(t)$ is differentiable, too, in view of (2.5) and the fact that $\dot{u}(t) = f(u(t))$. Hence u is a strong solution. Since the diffusion equation (2.4) is a special case of (2.1) with $f(u) = 0$, it follows that (2.3) has a unique global mild solution $u(t) = T(t)u_0$, and this is a strong solution if $u_0 \in \mathcal{D}(A)$, see for example [52, Proposition 3.1.9]. Now, the solution operator $W(t)u_0$ to the abstract reaction–diffusion (2.1) can be computed by the Trotter Product Formula

$$W(t)u_0 = \lim_{n \rightarrow \infty} \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right]^n u_0 = \lim_{n \rightarrow \infty} \left[S\left(\frac{t}{n}\right) T\left(\frac{t}{n}\right) \right]^n u_0, \quad u_0 \in X, \tag{2.8}$$

see, for example, [56–58]. If $u_0 \in \mathcal{D}(A)$ and $f : X \rightarrow X$ is continuously differentiable, then u is also a strong solution by [55, Chapter 6, Theorem 1.5]. \square

3. Fractional derivatives

Fractional derivatives are the generators of strongly continuous semigroups defined via convolution with infinitely divisible families of probability measures. Suppose that Y is a random variable on \mathbb{R}^d with probability distribution ω , so that $P(Y \in B) = \omega(B)$ for any Borel set $B \subseteq \mathbb{R}^d$, and define the Fourier transform $\hat{\omega}(\lambda) = \int e^{-i\langle \lambda, x \rangle} \omega(dx)$. Let $\omega^n = \omega * \dots * \omega$ denote the n -fold convolution of ω with itself. We say that Y (or ω) is *infinitely divisible*, if for each $n = 1, 2, 3, \dots$ there exist independent random variables Y_{n1}, \dots, Y_{nn} with the same distribution ω_n , such that $Y_{n1} + \dots + Y_{nn}$ is identically distributed with Y . Since the distribution of a sum of independent random variables is the convolution of their individual distributions, it follows that $\omega_n^n = \omega$. Hence, since the Fourier transform maps convolutions to products, we also have $\hat{\omega}_n(\lambda)^n = \hat{\omega}(\lambda)$. For $x := (x_1, \dots, x_d) \in \mathbb{R}^d$, we denote the standard Euclidean norm by $|x|$; that is, $|x|^2 = \sum_{i=1}^d x_i^2$. The Lévy representation (see, e.g., Theorem 3.1.11 in [16]) states that ω is infinitely divisible if and only if $\hat{\omega}(\lambda) = e^{\psi(\lambda)}$, where

$$\psi(\lambda) = -i\langle \lambda, a \rangle - \frac{1}{2}\langle \lambda, Q\lambda \rangle + \int_{x \neq 0} \left(e^{-i\langle \lambda, x \rangle} - 1 + \frac{i\langle \lambda, x \rangle}{1 + |x|^2} \right) \phi(dx), \tag{3.1}$$

where $a \in \mathbb{R}$, Q is a symmetric non-negative definite $d \times d$ matrix with real entries, and the Lévy measure ϕ is a σ -finite Borel measure on $\mathbb{R}^d \setminus \{0\}$, such that

$$\int_{x \neq 0} \min\{1, |x|^2\} \phi(dx) < \infty. \tag{3.2}$$

The triple $[a, Q, \phi]$ is unique, and we call this the Lévy representation of the infinitely divisible law ω . It follows that we can define the convolution power ω^t to be the infinitely divisible law with the Lévy representation $[ta, tQ, t\phi]$, so that ω^t has the Fourier transform $e^{t\psi(k)}$ for any $t \geq 0$. Then, we obtain from the uniqueness of the Lévy representation that

$$\omega^t * \omega^s = \omega^{t+s} \tag{3.3}$$

for any $s, t \geq 0$.

Let $C_0(\mathbb{R}^d)$ denote the Banach space of continuous functions $u : \mathbb{R}^d \rightarrow \mathbb{R}$, such that $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$, endowed with the supremum norm $\|u\| = \sup\{|u(x)| : x \in \mathbb{R}^d\}$. Let $C_0^2(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$ denote the set of functions $u \in C_0(\mathbb{R}^d)$ whose first- and second-order partial derivatives exists, are continuous, and vanish as $|x| \rightarrow \infty$. It is well known that every infinitely divisible distribution is associated with a strongly continuous semigroup on $C_0(\mathbb{R}^d)$ (see, for example [45], Example 4.1.3.) via

$$[T(t)u](x) := \int_{\mathbb{R}^d} u(x - y) \omega^t(dy), \quad u \in C_0(\mathbb{R}^d). \tag{3.4}$$

Generally, it is not possible to characterise the domain of the generator in terms of function spaces, however we can identify a reasonably large subset of the domain of the generator, together with a generator formula (see, for example, [45,59–61]). The following statement, using a slightly different version of the Lévy representation, can be found in [62, Theorem 31.5] and it also follows immediately from [63, Theorem 2.12].

Proposition 3.1. *Let $X := C_0(\mathbb{R}^d)$ and $(A, \mathcal{D}(A))$ denote the generator of the semigroup defined in (3.4), where ω is an infinitely divisible probability measure on \mathbb{R}^d . Then, $C_0^2(\mathbb{R}^d) \subset \mathcal{D}(A)$ and*

$$[Au](x) = -a \cdot \nabla u(x) + \frac{1}{2} \nabla \cdot Q \nabla u(x) + \int_{y \neq 0} \left(u(x - y) - u(x) + \frac{y \cdot \nabla u(x)}{1 + |y|^2} \right) \phi(dy) \tag{3.5}$$

for any $u \in C_0^2(\mathbb{R}^d)$.

Fractional derivatives were introduced by Leibnitz around the same time as their integer-order cousins [64]. The simplest mathematical description of the fractional derivative $d^\alpha u/dx^\alpha$ is the function whose Fourier transform is $(i\lambda)^\alpha \hat{u}(\lambda)$, where $\hat{u}(\lambda) = \int e^{-i\lambda x} u(x) dx$ is the usual Fourier transform. This extends the familiar formula for the Fourier transform of an integer-order derivative. For representations in a real space, and more information on fractional derivatives, see for example [64,65]. For $0 < \alpha < 1$ the Fourier transform $\hat{\omega}(\lambda)^t = e^{-t(i\lambda)^\alpha}$ yields an infinitely divisible probability measure called a *stable distribution*. Heuristically, this indicates a generator formula: since $T(t)u$ has Fourier transform $e^{-t(i\lambda)^\alpha} \hat{u}(\lambda)$ the difference quotient $h^{-1}(T(h)u - u)$ has the Fourier transform $h^{-1}(e^{-h(i\lambda)^\alpha} - 1)\hat{u}(\lambda) \rightarrow -(i\lambda)^\alpha \hat{u}(\lambda)$ as $h \rightarrow 0+$, suggesting the generator formula $A = -d^\alpha/dx^\alpha$ for this semigroup. For a rigorous proof, see [66]. For $1 < \alpha \leq 2$ a similar formula $\hat{\omega}(\lambda)^t = e^{t(i\lambda)^\alpha}$ also yields a stable distribution, with the special case $\alpha = 2$ corresponding to a normal or Gaussian distribution. Here the generator of the associated semigroup is $A = d^\alpha/dx^\alpha$ without the minus sign (see, also [59]). The negative fractional derivative $d^\alpha u/d(-x)^\alpha$ has the Fourier transform $(-i\lambda)^\alpha \hat{u}(\lambda)$. The simplest multivariable fractional derivative is the fractional Laplacian $\Delta^{\alpha/2}$ for $0 < \alpha \leq 2$, the inverse Fourier transform of $-|\lambda|^\alpha \hat{u}(\lambda)$. The fractional Laplacian can be considered as a fractional power of the Laplacian operator in the classical sense of Balakrishnan [52,54,59,67,68].

Since fractional derivatives are (negative) generators of infinitely divisible semigroups, they can be computed from the generator formula (3.5). The stable law ω with Fourier transform $\hat{\omega}(\lambda) = e^{-(i\lambda)^\alpha}$ for $0 < \alpha < 1$ has the Lévy representation $[a, 0, \phi]$, where $\phi(r, \infty) = Cr^{-\alpha}$ supported on the positive reals, and a, C are chosen to make $\psi(\lambda) = (i\lambda)^\alpha$ in (3.1), see [16, Section 7.3] for details. In this context, the exponent $-\psi(\lambda)$ is called the *Fourier symbol* of the operator A . The fractional Laplacian comes from the generator of a non-normal multivariable stable

law, an infinitely divisible law on \mathbb{R}^d with the Lévy representation $[a, 0, \phi]$, where $\phi\{x : \|x\| > r\} = Cr^{-\alpha}$ is radially symmetric. A more general fractional derivative of order α on \mathbb{R}^d comes from the family of stable laws with the Lévy representation $[a, 0, \phi]$, where

$$\phi \left\{ x : |x| > r, \frac{x}{|x|} \in B \right\} = Cr^{-\alpha} M(B)$$

for any Borel subset B of the unit sphere $S_{d-1} \subset \mathbb{R}^d$, and M is a probability distribution on S_{d-1} called the *mixing measure*. This fractional derivative has the Fourier symbol

$$\int_{|\theta|=1} (i(\lambda, \theta))^\alpha M(d\theta)$$

in the case $1 < \alpha \leq 2$, see [19] for more details.

Stable laws are distributional limits of sums of independent and identically distributed (i.i.d.) random variables, and in this context the Lévy measure codes the probability tail for the individual jumps, see [16, Section 3.1]. For example, the symmetric stable law on \mathbb{R}^d is the distributional limit of the normalised sums $n^{-1/\alpha}(X_1 + \dots + X_n)$, where X_i are i.i.d. symmetric random vectors on \mathbb{R}^d with $P\{|X| > r\} = Cr^{-\alpha}$, for $r > 0$ sufficiently large. The mixing measure codes the directions of large jumps, so that in the symmetric case, $M(d\theta)$ is a uniform distribution on S_{d-1} . Fractional derivatives of different order in each coordinate are (negative) generators of operator stable laws. Operator stable laws are distributional limits of i.i.d. random vectors normalised by linear operators rather than constants like $n^{-1/\alpha}$. For example, in \mathbb{R}^2 the operator stable law with Fourier symbol $(i\lambda_1)^{\alpha_1} + (i\lambda_2)^{\alpha_2}$ for $1 < \alpha_i \leq 2$ has independent stable components with index α_i and the Lévy measure concentrated on the coordinate axes. The generator of the associated semigroup is $d^{\alpha_1}/dx_1^{\alpha_1} + d^{\alpha_2}/dx_2^{\alpha_2}$.

4. Reaction–diffusion equations

Abstract reaction–diffusion equations of the form (2.1) can be solved approximately by the Trotter Product Formula as stated in Theorem 2.1, as long as the component equations (2.3) and (2.4) can be solved first, and assuming that the reaction function f is globally Lipschitz. To this end, our goal is to rewrite the partial differential equation (1.3) in the form of an abstract ordinary differential equation (2.1) on $X := C_0(\mathbb{R}^d)$, where $f : X \rightarrow X$ is defined via the function $\tilde{f} : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ as

$$[f(u)](x) = \tilde{f}(x, u(x)). \tag{4.1}$$

Then, it will suffice to consider operator splitting solutions for the abstract reaction–diffusion equation (2.1). For many problems of practical interest, the function f is not globally Lipschitz on X . For example, in applications to population biology the most common model is the Kolmogorov–Fisher equation with $f(u) = ru(1 - u/K)$ (discussed in Section 1), and here f is not globally Lipschitz. In this section, we show how to solve reaction–diffusion equations of type (1.3) by an operator splitting method, when the abstract function f defined via (4.1) is only locally Lipschitz. We use the basic idea from [57], essentially we truncate f to make it globally Lipschitz in such a way that the solutions to the modified reaction–diffusion equation will be equal to the solutions to the original equation, for initial functions u_0 , with $\|u_0\| \leq N$ for some integer N . Our approach requires that solutions remain uniformly bounded in time and space, hence we avoid the complications in [57] caused by solutions that blow-up in finite time. The results presented here are more general than the illustrative examples in [57], because we allow variable coefficients in the reaction term, i.e., $\tilde{f}(u(x), x)$ rather than just $\tilde{f}(u(x))$. While our results do follow from the general procedure in [57], we provide a self-contained proof here, since the arguments can be greatly simplified in the present case.

We call a Banach space X an *ordered Banach space* if it is a real Banach space endowed with a partial ordering \leq such that

- (1) $u \leq v$ implies $u + w \leq v + w$ for all $u, v, w \in X$.
- (2) $u \geq 0$ implies $\lambda u \geq 0$ for all $u \in X$ and $\lambda \geq 0$.
- (3) $0 \leq u \leq v$ implies $\|u\| \leq \|v\|$ for all $u, v \in X$.
- (4) The positive cone $X_+ := \{x \in X : x \geq 0\}$ is closed.

A typical example of an ordered Banach space is $C_0(\mathbb{R}^d)$ endowed with the partial ordering $u \leq v$ whenever $u(x) \leq v(x)$ for all $x \in \mathbb{R}^d$. Another example is $L_p(\mathbb{R}^d)$ ($1 \leq p \leq \infty$) endowed with the partial ordering $u \leq v$ whenever $u(x) \leq v(x)$ for $x \in \mathbb{R}^d$ almost everywhere. An operator A on an ordered Banach space is called *positive* if $0 \leq u \leq v$ implies $0 \leq Au \leq Av$. We also write $B \leq A$ if $0 \leq Bu \leq Au$ for any $u \geq 0$; see, for example, [69].

In what follows we discuss the reaction–diffusion (1.3), where the dispersion term is given in terms of the pseudo-differential operator (3.5), assuming non-negative initial data. First we show that, under suitable conditions on $\tilde{f}(x, y)$, equation (1.3) can be written in the form (2.1) on $X := C_0(\mathbb{R}^d)$. Assume that the following conditions hold:

- (H1) The function $\tilde{f} : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous;
- (H2) $\lim_{(|x|, y) \rightarrow (\infty, 0)} \tilde{f}(x, y) = 0$; that is, for any $\varepsilon > 0$, there is $\delta > 0$ and $C > 0$, such that $|\tilde{f}(x, y)| < \varepsilon$ whenever $|y| < \delta$ and $|x| > C$.
- (H3) The function $y \rightarrow \tilde{f}(x, y)$ is locally Lipschitz uniformly in x ; that is, for any $M > 0$ there is $K(M) > 0$ such that

$$|\tilde{f}(x, y) - \tilde{f}(x, z)| \leq K(M)|y - z|$$

for $|y|, |z| \leq M$ and $x \in \mathbb{R}^d$;

- (H4) $\tilde{f}(x, y) < 0$ for $x \in \mathbb{R}^d$ and $y \geq y_0$; and
- (H5) $\tilde{f}(x, 0) = 0$ for all $x \in \mathbb{R}^d$.

Note that in view of (H1) and (H5) assumption (H2) is a uniform continuity assumption on \tilde{f} at $(x, 0)$ for $|x|$ large. Let us introduce the cut-off function

$$[f_N(u)](x) := \tilde{f}_N(x, u(x)) := \begin{cases} 0 & \text{if } u(x) < 0, \\ \tilde{f}(x, u(x)) & \text{if } 0 \leq u(x) \leq y_0N, \\ \tilde{f}(x, y_0N) & \text{if } u(x) > y_0N, \end{cases} \quad (4.2)$$

where $N \in \mathbb{N}$. We have the following lemma.

Lemma 4.1. *Under assumptions (H1)–(H2), the abstract function f defined as $[f(u)](x) := \tilde{f}(x, u(x))$ maps $X := C_0(\mathbb{R}^d)$ to itself and so does f_N provided that (H5) holds, too. If, in addition, (H3) holds, then $f : X \rightarrow X$ is locally Lipschitz and $f_N : X \rightarrow X$ is globally Lipschitz.*

Proof. If $u \in X = C_0(\mathbb{R}^d)$ then $x \rightarrow \tilde{f}(x, u(x))$ is continuous on \mathbb{R}^d by (H1) and $\lim_{|x| \rightarrow \infty} \tilde{f}(x, u(x)) = 0$ by (H2). This shows that $f : X \rightarrow X$. If (H3) holds, then

$$|\tilde{f}(x, u(x)) - \tilde{f}(x, v(x))| \leq K(M) |u(x) - v(x)|, \quad \text{if } |u(x)|, |v(x)| \leq M$$

and hence, taking supremums on both sides,

$$\|f(u) - f(v)\| \leq K(M) \|u - v\|, \quad \text{if } \|u\|, \|v\| \leq M;$$

that is, $f : X \rightarrow X$ is locally Lipschitz. The statements about f_N follow by similar arguments. \square

For example, Lemma 4.1 includes the Kolmogorov–Fisher equation where

$$\tilde{f}(x, u(x)) = r(x)u(x)(1 - u(x)/K(x))$$

with variable coefficients $r = r(x) > 0$ and $K = K(x) > 0$ as long as r, K and $1/K$ are continuous and uniformly bounded. Or, in general, it includes functions

$$\tilde{f}(x, u(x)) := \sum_{i=1}^n K_i(x)[u(x)]^i \quad (4.3)$$

provided that

$$K_i \text{ are bounded continuous functions on } \mathbb{R}^d \text{ and (H4) holds.} \quad (4.4)$$

In population biology, Eq. (4.3) can be used to model the Allee-effect, i.e., the fact that for many species there is a minimum viable population m below which the species dies out. The minimum viable population can depend on environmental factors, and hence can be space-dependent. One commonly used model for this phenomenon is

$$\tilde{f}(x, u(x)) := r(x)u(x) \left(\frac{u(x)}{m(x)} - 1 \right) \left(1 - \frac{u(x)}{K(x)} \right),$$

see for example [70, p. 185].

Proposition 4.2. *Let $X := C_0(\mathbb{R}^d)$ and let f be given by (4.1) and assume that conditions (H1)–(H5) hold. Then the abstract differential equation*

$$\dot{u}(t) = f(u(t)), \quad u(0) = u_0 \geq 0 \tag{4.5}$$

has a unique strong global solution given by $u(t) = S(t)u_0$ for each non-negative $u_0 \in X$. For any positive integer $N \geq 2$, the abstract differential equation $\dot{u} = f_N(u)$, $u(0) = u_0 \geq 0$, where f_N is the Lipschitz continuous function defined in (4.2), also has a unique strong global solution given by $u(t) = S_N(t)u_0$ for each $u_0 \geq 0$ in X . Furthermore, if $N \in \mathbb{N}$ is such that $0 \leq u_0(x) \leq y_0N$ for all $x \in \mathbb{R}^d$, then $0 \leq [S(t)u_0](x) = [S_N(t)u_0](x) \leq y_0N$ for all $x \in \mathbb{R}^d$ and $t \geq 0$.

Proof. Consider the abstract initial value problem

$$\dot{u}(t) = f_N(u(t)), \quad u(0) = u_0 \geq 0 \in X$$

which has, by the Lipschitz continuity of f_N (see Lemma 4.1), a unique global strong solution $u(t) = S_N(t)u_0$ (see Theorem 2.1), where $S_N(\cdot)$ is the nonlinear semigroup generated by f_N . Hence, since the operator norm in this space is the supremum norm, it follows that the function $u_x(t) := [S_N(t)u_0](x)$ is, for each fixed $x \in \mathbb{R}^d$, the unique solution of the ordinary differential equation

$$\frac{d}{dt}u_x(t) = \tilde{f}_N(x, u_x(t)), \quad u_x(0) = u_0(x) \geq 0 \in \mathbb{R}.$$

Since $\tilde{f}_N(x, 0) = 0$, it follows easily using the uniqueness of solutions that $S_N(t)$ is positive for all $t \geq 0$; i.e., if $u_0(x) \geq 0$ for all $x \in \mathbb{R}^d$, then $u_x(t) = [u(t)](x) = [S_N(t)u_0](x) \geq 0$ for all $x \in \mathbb{R}^d$, and also if $u_0(x) \geq v_0(x)$ for all $x \in \mathbb{R}^d$, then $u_x(t) = [u(t)](x) = [S_N(t)u_0](x) \geq v_x(t) = [v(t)](x) = [S_N(t)v_0](x)$ for all $x \in \mathbb{R}^d$. Since assumption (H4) ensures that $\tilde{f}_N(x, u(x)) < 0$ for all $u(x) > y_0$, it also follows from uniqueness of solutions that, if $u_0(x) \leq y_0N$ for all $x \in \mathbb{R}^d$, then $[S_N(t)u_0](x) \leq y_0N$ for all $x \in \mathbb{R}^d$. Moreover, since $\tilde{f}_N(x, u(x)) = \tilde{f}(x, u(x))$ for $0 \leq u(x) \leq y_0N$, we see that $S_N(t)u_0$ also solves

$$\dot{u}(t) = f(u(t)), \quad u(0) = u_0. \tag{4.6}$$

Since f is locally Lipschitz by Lemma 4.1, [54, Chapter 3, Theorem 3.4.1] also implies that (4.6) has a unique local strong solution. The function $t \mapsto S_N(t)u_0$ is defined for all $t \geq 0$, and hence $S_N(t)u_0$ is the unique strong global solution $S(t)u_0$ of (4.6). \square

Now we come to the main result of this paper. It shows that the sequential splitting approximation of (2.1) is indeed convergent.

Theorem 4.3. *Let ω be infinitely divisible, let A denote the generator of the strongly continuous semigroup defined in (3.4) on $X := C_0(\mathbb{R}^d)$, and let f be given by (4.1). Assume that conditions (H1)–(H5) hold. Then (2.1) has a unique mild solution $u(t) = W(t)u_0$ for all $u_0 \geq 0$ in X given by the Trotter Product Formula*

$$W(t)u_0 = \lim_{n \rightarrow \infty} \left[T \left(\frac{t}{n} \right) S \left(\frac{t}{n} \right) \right]^n u_0 = \lim_{n \rightarrow \infty} \left[S \left(\frac{t}{n} \right) T \left(\frac{t}{n} \right) \right]^n u_0. \tag{4.7}$$

Proof. Let $N \in \mathbb{N}$ be such that $u_0(x) \leq y_0N$ for all $x \in \mathbb{R}^d$, and consider the abstract reaction–diffusion equation

$$\dot{u}(t) = Au(t) + f_N(u(t)), \quad u(0) = u_0 \geq 0. \tag{4.8}$$

Since $f_N : X \rightarrow X$ is globally Lipschitz continuous by Lemma 4.1 and A is a generator, there is a unique mild solution $u_N(t) = W_N(t)u_0$ of (4.8) given by the Trotter Product Formula

$$\begin{aligned} u_N(t) = W_N(t)u_0 &= \lim_{n \rightarrow \infty} \left[T\left(\frac{t}{n}\right) S_N\left(\frac{t}{n}\right) \right]^n u_0 \\ &= \lim_{n \rightarrow \infty} \left[S_N\left(\frac{t}{n}\right) T\left(\frac{t}{n}\right) \right]^n u_0, \end{aligned} \tag{4.9}$$

as stated in Theorem 2.1. The semigroup $\{T(t)\}_{t \geq 0}$ satisfies $0 \leq T(t)u_0 \leq T(t)v_0$ for $0 \leq u_0 \leq v_0$ and $t \geq 0$ since ω^t is a positive measure. If $0 \leq u_0(x) \leq y_0N$ for all $x \in \mathbb{R}^d$, then

$$0 \leq [T(t)u_0](x) \leq y_0N \int_{\mathbb{R}^d} \omega^t(ds) = y_0N \tag{4.10}$$

Therefore, by (4.10) and Proposition 4.2,

$$0 \leq \left[\left(T\left(\frac{t}{n}\right) S_N\left(\frac{t}{n}\right) \right)^n u_0 \right](x) = \left[\left(T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right)^n u_0 \right](x) \leq y_0N \tag{4.11}$$

and

$$0 \leq \left[\left(S_N\left(\frac{t}{n}\right) T\left(\frac{t}{n}\right) \right)^n u_0 \right](x) = \left[\left(S\left(\frac{t}{n}\right) T\left(\frac{t}{n}\right) \right)^n u_0 \right](x) \leq y_0N. \tag{4.12}$$

This also shows that $0 \leq [u_N(t)](x) \leq y_0N$ for all $x \in \mathbb{R}^d$ in view of (4.9). Therefore $u_N(t)$ is also a mild solution of (2.1), since $f_N(u) = f(u)$ if $0 \leq u(x) \leq y_0N$ for all $x \in \mathbb{R}^d$. Since f is locally Lipschitz continuous by Lemma 4.1, [55, Chapter 6, Theorem 1.4] implies that (2.1) has a unique local mild solution and since $u_N(t)$ is defined for all $t > 0$, it follows that $u_N(t)$ is the unique global mild solution of (2.1) given by the Trotter Product Formula (4.7) in view of (4.9), (4.11) and (4.12). \square

Under certain conditions, we also obtain a unique classical solution to the reaction–diffusion equation (2.1).

Corollary 4.4. *Let A be given by (3.5) and \tilde{f} by (4.3), and assume that (4.4) holds. If $u_0 \in C_0^2(\mathbb{R}^d) \geq 0$, then (2.1) has a unique classical solution u on $X = C_0(\mathbb{R}^d)$ given by the Trotter Product formula (4.7).*

Proof. It follows from (4.4) that \tilde{f} satisfies conditions (H1)–(H5). Then, it follows from Theorem 4.3 that (2.1) has a unique mild solution u given by the Trotter Product Formula (4.7). Since $u_0 \in \mathcal{D}(A)$ by Proposition 3.1, and since $f : X \rightarrow X$ is continuously differentiable, u is also the unique strong solution of (2.1) by [55, Chapter 6, Theorem 1.5].

Remark 4.5. Corollary 4.4 yields the existence and uniqueness of strong solutions $u(t) := W(t)u_0$ to the abstract differential equation (2.1) on the function space $X = C_0(\mathbb{R}^d)$ with the supremum norm. Then, it follows easily that $u(x, t) := [W(t)u_0](x)$ solves the original partial differential equation (1.3) pointwise. Furthermore, the sequential splitting (4.7) converges pointwise to $u(x, t)$, uniformly for $x \in \mathbb{R}^d$.

Next we present a useful result on operator splitting in the special case where the function $f(u) = ru(1 - u/K)$, the Kolmogorov–Fisher equation with constant coefficients. In this case, it can easily be shown that the flow of the abstract differential equation $\dot{u} = f(u)$ is given by $[S(t)u_0](x) = [\tilde{S}(t)](u_0(x))$, where

$$[\tilde{S}(t)](y) = K \left(1 - \frac{K - y}{K + y(e^{rt} - 1)} \right) \tag{4.13}$$

using integration by parts. The following result is similar to [57, Lemma 16].

Corollary 4.6. *Under the assumptions of Theorem 4.3, if $f(u) = ru(1 - u/K)$, where r and K are constants for $x \in \mathbb{R}^d$, then for all $n \in \mathbb{N}$,*

$$\left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right]^n u_0 \leq \left[T\left(\frac{t}{2n}\right) S\left(\frac{t}{2n}\right) \right]^{2n} u_0 \leq W(t)u_0$$

$$\leq \left[S\left(\frac{t}{2n}\right) T\left(\frac{t}{2n}\right) \right]^{2n} u_0 \leq \left[S\left(\frac{t}{n}\right) T\left(\frac{t}{n}\right) \right]^n u_0, \tag{4.14}$$

where $\{T(t)\}_{t \geq 0}$ is given by (3.4) and $\{S(t)\}_{t \geq 0}$ is defined in (4.13).

Proof. The proof is essentially identical to (33) in [37]. \square

Remark 4.7. The sequential splitting scheme (4.7) used in Theorem 4.3 for solving the reaction–diffusion equation (1.3) is motivated by applications to population biology. There the operator $S(\tau)$ represents the growth process, the operator $T(\tau)$ is the migration process, and the sequential operator splitting has the interpretation of first growing and then migrating, or vice versa. Suppose that $T(t)$ is the convolution semigroup (3.4) associated with an infinitely divisible probability distribution ω^t , and that $f_t(x)$ is the probability density of this infinitely divisible law. Then the iteration formula

$$u_{n+1}(x) = [T(\tau)S(\tau)u_n](x) = \int_{\mathbb{R}^d} [S(\tau)(u_n)](x - y) f_\tau(y) dy \tag{4.15}$$

corresponds to a commonly used discrete-time model in population biology [6,7,37]. In this context, the probability density $f_\tau(x)$ is called a *dispersal kernel*, since it represents the distance travelled by a randomly selected member of the species during one time step of duration τ [7,71]. Formula (4.15) expresses that the population increases via an application of the growth operator $S(\tau)$, and then spreads out via an application of the dispersal operator $T(\tau)$. In some species, population growth is a seasonal event, so that the discrete-time formula (4.15) is the fundamental model, and the reaction–diffusion equation (1.3) is merely a useful continuum approximation. In those applications, the connection with the continuous time model (1.3) can be useful to identify appropriate dispersal kernels. The infinitely divisible kernels are often convenient, since they can be adjusted to any time scale, and have a clear probabilistic interpretation.

5. Numerical experiments

In this section, we implement the sequential operator splitting procedure to solve fractional reaction–diffusion equations. We consider the partial differential equation

$$\frac{\partial u}{\partial t} = C \frac{\partial^\alpha u}{\partial x^\alpha} + D \frac{\partial^\beta u}{\partial y^\beta} + ru \left(1 - \frac{u}{K}\right), \tag{5.1}$$

where $1 < \alpha, \beta \leq 2$, $u = u(x, y, t)$ is defined over the set $(x, y) \in \mathbb{R}^2$ and $t \geq 0$, and we compute numerical solutions of the initial value problem assuming the radially symmetric initial condition

$$u(x, y, t = 0) = \min \left\{ 0.8, 10e^{-x^2 - y^2} \right\}. \tag{5.2}$$

Note that the initial function is continuous and tends to zero as the norm of the vector (x, y) tends to infinity, and hence this initial function is an element of the space $C_0(\mathbb{R}^2)$. Then the conditions of Theorem 4.3 and Corollary 4.4 are satisfied, which guarantees the convergence of the splitting formula (4.7) in view of Remark 4.5. The splitting algorithm requires us to evaluate the solution operators $S(t)$ and $T(t)$. The solution operator $S(t)$ will be evaluated via the explicit solution formula (4.13), and note that this procedure is equally valid if the coefficients r, K vary with the spatial variables (x, y) . The solution operator $T(t)$ will be computed via the convolution formula

$$T(t)u(x, y) = \int_{(x,y) \in \mathbb{R}^2} u(x - r, y - s) f_t(r, s) dr ds \tag{5.3}$$

where $f_t(x, y)$ is the probability density function of the operator stable law μ^t , whose Fourier transform is given by

$$\hat{\omega}(\lambda_1, \lambda_2)^t = e^{tC(i\lambda_1)^\alpha + tD(i\lambda_2)^\beta}.$$

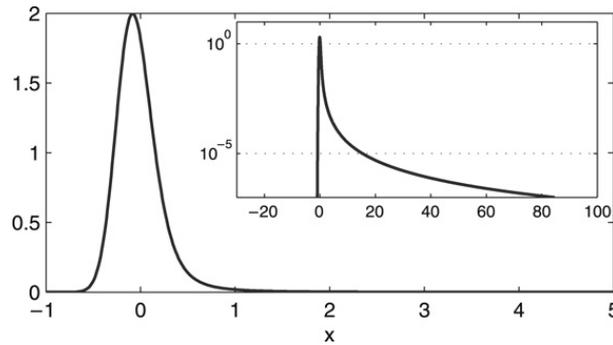


Fig. 1. A typical one-dimensional kernel $g_\tau(x)$ with $\alpha = 1.7$, $C = 0.4$, and $\tau = 0.1$. Note the long right tail and marked asymmetry highlighted in the semi-log plot inset.

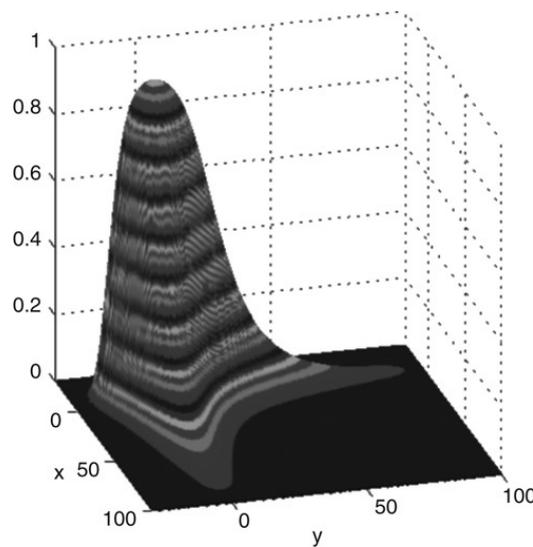


Fig. 2. Solution to the fractional reaction–diffusion equation (5.1) at time $t = 40$ with initial condition (5.2) and parameter values $\alpha = \beta = 1.7$, $C = D = 0.4$, $r = 0.2$, and $K = 1$.

In this case, the Fourier transform decomposes into a product of two terms, and since products in Fourier space correspond to convolutions in real space, the double integral (5.3) decomposes into a pair of convolutions

$$T(t)u(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x - r, y - s)g_t(r) dr h_t(s) ds,$$

where g_t is the probability density function of the stable law with Fourier transform $e^{tC(i\lambda)^\alpha}$, and h_t is the probability density function of the stable law with Fourier transform $e^{tD(i\lambda)^\beta}$. Fast and accurate numerical computation of the stable densities is accomplished using the method of Nolan [73], based on an integral representation in Zolotarev [72, Thm 2.2.3]. Then, the operator $T(t)$ is computed via two numerical convolutions, one for each variable x and y . This is, of course, another application of operator splitting. Fig. 1 illustrates the shape of the one-dimensional kernel, i.e., the stable probability density. Note that the tail falls off rather slowly, indicating a strong non-local effect. This is typical of fractional diffusion models, and accounts for their super-diffusive character. Finally, once the solution operators $S(t)$ and $T(t)$ are computed, the Trotter Product Formula (4.7) can be used to obtain a faithful approximation to the solution of the fractional reaction–diffusion equation (5.1).

As a first illustration of the method, we solve the fractional reaction–diffusion equation (5.1) with initial condition (5.2), assuming that $\alpha = \beta = 1.7$, $C = D = 0.4$, $r = 0.2$, and $K = 1$. Fig. 2 illustrates the solution at time $t = 40$. This solution was computed using a time step of $\tau = 0.1$ and a spatial grid of $\Delta x = \Delta y = 0.5$. Note the elongated tails in the x and y directions, which are characteristic of the anomalous diffusion component. Also note that, in the fractional case, the solution is strongly asymmetric and clusters along the axes.

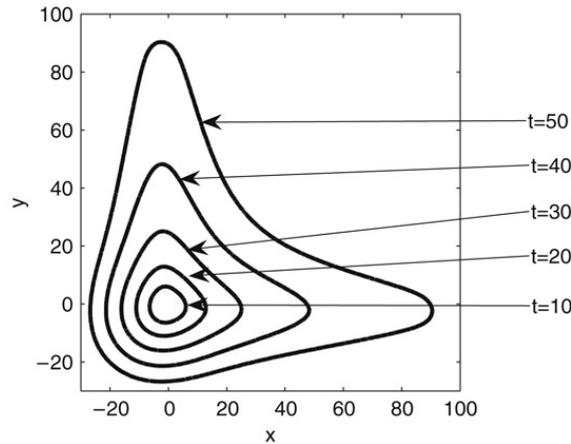


Fig. 3. Level sets $u = 0.1$ at different times of the solution to the fractional reaction–diffusion equation (5.1) with initial condition (5.2) and parameter values $\alpha = \beta = 1.7$, $C = D = 0.4$, $r = 0.2$, and $K = 1$. The level sets illustrate the accelerating front.

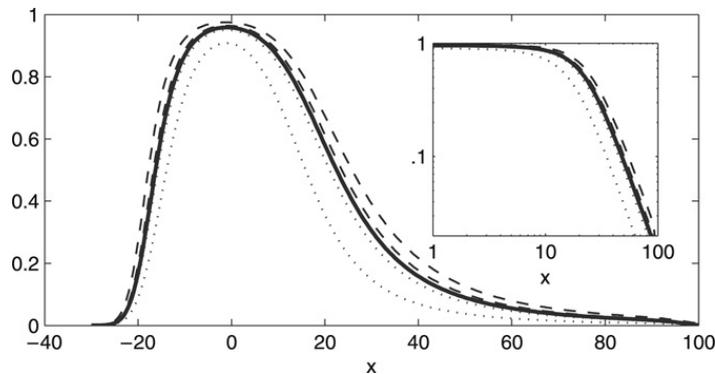


Fig. 4. A slice ($y = 0$) of the solution from Fig. 2, with dotted and dashed lines indicating approximate outer and inner solutions computed by using time steps of $\tau = 8$ and $\tau = 2$. The inset shows the power-law tail of the solution curve.

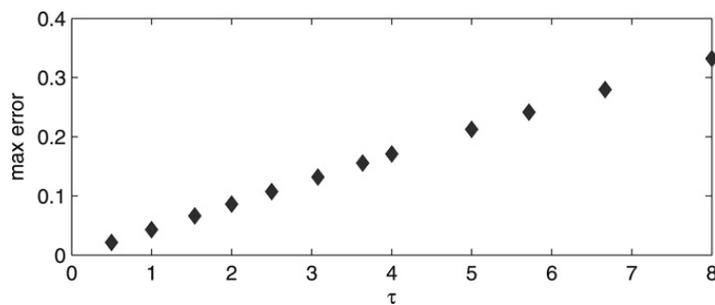


Fig. 5. Maximal distance between the upper and lower solutions at time $t = 40$ for various time steps τ , indicating $O(\tau)$ convergence.

Another interesting feature of the solutions to the fractional reaction–diffusion equation is their accelerating fronts. Fig. 3 shows the level sets $u = 0.1$ at times $t = 10, 20, \dots, 50$. The accelerating fronts are apparent, particularly along the coordinate axes. In applications to biology, where dispersion kernels similar to that in Fig. 1 are often observed, this accelerating front could represent the advance of an invasive species.

A closer examination of the expanding tail is shown in Fig. 4, which represents the slice $y = 0$ from Fig. 2. Note the power-law tail indicated by the straight line asymptotics on the inset log–log plot. The power-law behaviour is inherited from the stable convolution kernel. The dotted and dashed lines in Fig. 4 illustrate the monotone convergence guaranteed by Corollary 4.6 in this constant coefficient case. Fig. 5 indicates that the order of convergence is $O(\tau)$.

Next we consider the solution to the fractional reaction–diffusion equation (5.1) in the case where the coefficients of the reaction term vary in space. We set $C = 0.15$, $D = 0.4$, $r = 0.2$, and let K vary in space. In particular, we

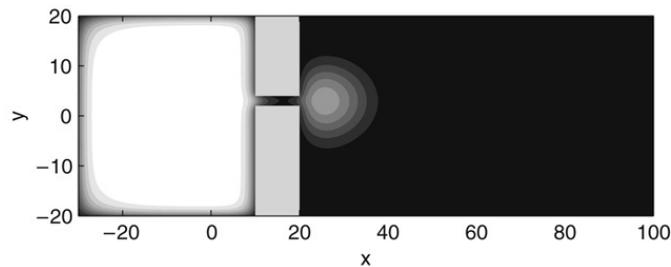


Fig. 6. Solution to the classical reaction–diffusion equation (5.1) at time $t = 90$ with initial condition (5.2) and parameter values $\alpha = \beta = 2$, $C = 0.15$, $D = 0.4$, $r = 0.2$, and $K = 1$ outside the region $10 < x < 20$ and $y < 2$ or $y > 4$. We set $K = 10^{-6}$ inside this region to create a slitted barrier, through which the solution propagates slowly.

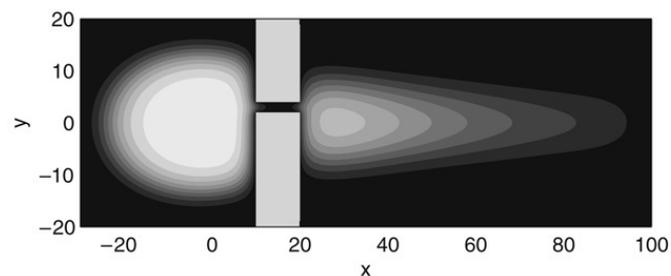


Fig. 7. Solution to the fractional reaction–diffusion equation (5.1) at time $t = 50$ with the same parameter values as in Fig. 6 except that now $\alpha = 1.7$. As the plume is more or less symmetric about $y = 0$, this illustrates how in the fractional case the solution mainly propagates across the barrier rather than through the slit.

set $K(x, y) = 10^{-6}$ if $10 < x < 20$ and $y < 2$ or $y > 4$, $K = 1$ outside this region, and smoothly interpolate in between. In applications to biology, this might represent a region where populations cannot grow, due to unfavourable environmental conditions. The geometry is a slitted barrier, through which the solution will eventually penetrate. First we consider the case where $\alpha = \beta = 2$. Fig. 6 shows the solution in this case, in plan view, at time $t = 90$. Because of the classical diffusion term in the x coordinate, the solution is very slow to penetrate the barrier.

Next we change $\alpha = 1.7$ to represent anomalous diffusion, and repeat the experiment. Fig. 7 shows that by time $t = 50$, even earlier than the snapshot $t = 90$ illustrated in Fig. 6, the solution has penetrated significantly, and is spreading in the y direction as well. Due to the strongly non-local character of the stable convolution kernel shown in Fig. 1, it is much easier for members of the population to cross over the barrier via long “jumps.” This striking characteristic of fractional reaction–diffusion equations may be significant for predicting the likely effects of population control efforts for nuisance species.

Acknowledgements

M. Kovács is partially supported by the postdoctoral grant No. 623-2005-5078 of the Swedish Research Council and the research grant (CZN-14/2005) of the Science and Technology Foundation. M.M. Meerschaert is partially supported by the USA National Science Foundation grants DMS-0417869 and DMS-0706440.

References

- [1] N.F. Britton, Reaction–Diffusion Equations and Their Applications to Biology, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1986.
- [2] R.S. Cantrell, C. Cosner, Spatial ecology via reaction–diffusion equations, in: Wiley Series in Mathematical and Computational Biology, John Wiley & Sons Ltd., Chichester, 2003.
- [3] P. Grindrod, The Theory and Applications of Reaction–Diffusion Equations, 2nd edn, in: Oxford Applied Mathematics and Computing Science Series, The Clarendon Press Oxford University Press, New York, 1996.
- [4] F. Rothe, Global solutions of reaction–diffusion systems, in: Lecture Notes in Mathematics, vol. 1072, Springer-Verlag, Berlin, 1984.
- [5] J. Smoller, Shock waves and reaction–diffusion equations, 2nd edn, in: Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 258, Springer-Verlag, New York, 1994.
- [6] J.D. Murray, Mathematical Biology. I,II, 3rd edn, in: Interdisciplinary Applied Mathematics, vols. 17, 18, Springer-Verlag, New York, 2002.

- [7] M. Neubert, H. Caswell, Demography and dispersal: Calculation and sensitivity analysis of invasion speed for structured populations, *Ecology* 81 (6) (2000) 1613–1628.
- [8] L.J.B. Bachelier, *Théorie de la Spéculation*, Gauthier-Villars, Paris, 1900.
- [9] A. Einstein, *Investigations on the Theory of the Brownian Movement*, Dover Publications Inc., New York, 1956, (Edited with notes by R. Fürth, Translated by A. D. Cowper).
- [10] I.M. Sokolov, J. Klafter, From diffusion to anomalous diffusion: A century after Einstein's Brownian motion, *Chaos* 15 (2) (2005) 26–103.
- [11] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, *Phys. Rep.* 339 (1) (2000).
- [12] R. Metzler, J. Klafter, The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics, *J. Phys. A* 37 (31) (2004) R161–R208.
- [13] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, The fractional-order governing equation of Lévy motion, *Water Resources Res.* 36 (2000) 1413–1424.
- [14] A. Chaves, A fractional diffusion equation to describe Lévy flights, *Phys. Lett. A* 239 (1998) 13–16.
- [15] W. Feller, *An Introduction to Probability Theory and Applications*, vols. I, II, John Wiley and Sons, 1966.
- [16] M.M. Meerschaert, H.P. Scheffler, *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*, Wiley Interscience, New York, 2001.
- [17] G. Samorodnitsky, M.S. Taqqu, *Stable Non-Gaussian Random Processes*, Chapman & Hall/CRC, 1994.
- [18] M.M. Meerschaert, H.P. Scheffler, Limit theorems for continuous-time random walks with infinite mean waiting times, *J. Appl. Probab.* 41 (3) (2004) 623–638.
- [19] M.M. Meerschaert, D.A. Benson, B. Baeumer, Multidimensional advection and fractional dispersion, *Phys. Rev. E* 59 (1999) 5026–5028.
- [20] S.J. Taylor, The measure theory of random fractals, *Math. Proc. Cambridge Philos. Soc.* 100 (1986) 383–406.
- [21] M.M. Meerschaert, D.A. Benson, B. Baeumer, Operator Lévy motion and multiscaling anomalous diffusion, *Phys. Rev. E* 63 (2001) 1112–1117.
- [22] R. Schumer, D.A. Benson, M.M. Meerschaert, B. Baeumer, Multiscaling fractional advection–dispersion equations and their solutions, *Water Resources Res.* 39 (2003) 1022–1032.
- [23] Z. Deng, V.P. Singh, L. Bengtsson, Numerical solution of fractional advection-dispersion equation, *J. Hydraulic Eng.* 130 (2004) 422–431.
- [24] V.E. Lynch, B.A. Carreras, D. del-Castillo-Negrete, K.M. Ferreira-Mejias, H.R. Hicks, Numerical methods for the solution of partial differential equations of fractional order, *J. Comput. Phys.* 192 (2003) 406–421.
- [25] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for two-sided space-fractional partial differential equations, *Appl. Numer. Math.* 56 (1) (2006) 80–90.
- [26] M.M. Meerschaert, H.P. Scheffler, C. Tadjeran, Finite difference methods for two-dimensional fractional dispersion equation, *J. Comput. Phys.* 211 (2006) 249–261.
- [27] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection-dispersion flow equations, *J. Comput. Appl. Math.* 172 (1) (2004) 65–77.
- [28] C. Tadjeran, M.M. Meerschaert, H.P. Scheffler, A second order accurate numerical approximation for the fractional diffusion equation, *J. Comput. Phys.* 213 (2006) 205–213.
- [29] C. Tadjeran, M.M. Meerschaert, A second order accurate numerical method for the two-dimensional fractional diffusion equation, *J. Comput. Phys.* 220 (2007) 813–823.
- [30] F. Liu, V. Ahn, I. Turner, Numerical solution of the fractional advection–dispersion equation, 2002, Preprint.
- [31] F. Liu, V. Ahn, I. Turner, P. Zhuang, Numerical simulation for solute transport in fractal porous media, *ANZIAM J.* 45 (E) (2004) C461–C473.
- [32] F. Liu, V. Ahn, I. Turner, Numerical solution of the space fractional Fokker–Planck equation, *J. Comput. Appl. Math.* 166 (2004) 209–219.
- [33] V.J. Ervin, J.P. Roop, Variational solution of fractional advection dispersion equations on bounded domains in R^d , *Numer. Methods Partial Differential Equations* 23 (2007) 256–281.
- [34] G.J. Fix, J.P. Roop, Least squares finite element solution of a fractional order two-point boundary value problem, *Comput. Math. Appl.* 48 (2004) 1017–1033.
- [35] J.P. Roop, Computational aspects of FEM approximation of fractional advection–dispersion equations on bounded domains in R^2 , *J. Comput. Appl. Math.* 193 (2005) 243–268.
- [36] Y. Zhang, D.A. Benson, M.M. Meerschaert, H.P. Scheffler, On using random walks to solve the space-fractional advection-dispersion equations, *J. Statist. Phys.* 123 (2006) 89–110.
- [37] B. Baeumer, M. Kovács, M.M. Meerschaert, Fractional reproduction-dispersal equations and heavy tail dispersal kernels, *Bull. Math. Biol.* 69 (2007) 2281–2297.
- [38] D. del Castillo-Negrete, B.A. Carreras, V.E. Lynch, Front dynamics in reaction–diffusion systems with levy flights: A fractional diffusion approach, *Phys. Rev. Lett.* 91 (1) (2003) 018302.
- [39] J.M. Bullock, R.T. Clarke, Long distance seed dispersal by wind: Measuring and modelling the tail of the curve, *Oecologia* 124 (4) (2000) 506–521.
- [40] J.S. Clark, M. Silman, R. Kern, E. Macklin, J. HilleRisLambers, Seed dispersal near and far: Patterns across temperate and tropical forests, *Ecology* 80 (5) (1999) 1475–1494.
- [41] J.S. Clark, M. Lewis, L. Horvath, Invasion by extremes: Population spread with variation in dispersal and reproduction, *The Amer. Naturalist* 157 (5) (2001) 537–554.
- [42] G.G. Katul, A. Porporato, R. Nathan, M. Siqueira, M.B. Soons, D. Poggi, H.S. Horn, S.A. Levin, Mechanistic analytical models for long-distance seed dispersal by wind, *The Amer. Naturalist* 166 (2005) 368–381.
- [43] E.K. Klein, C. Lavigne, H. Picault, M. Renard, P.H. Gouyon, Pollen dispersal of oilseed rape: Estimation of the dispersal function and effects of field dimension, *J. Appl. Ecology* 43 (10) (2006) 141–151.

- [44] E. Paradis, S.R. Baillie, W.J. Sutherland, Modeling large-scale dispersal distances, *Ecological Modelling* 151 (2–3) (2002) 279–292.
- [45] N. Jacob, Pseudo-differential operators and Markov processes, in: *Mathematical Research*, vol. 94, Akademie Verlag, Berlin, 1996.
- [46] A. Gerisch, J.G. Verwer, Operator splitting and approximate factorization for taxis-diffusion–reaction models, *Appl. Numer. Math.* 42 (2002) 159–176.
- [47] P. Csomós, I. Faragó, Á. Havasi, Weighted sequential splittings and their analysis, *Comput. Math. Appl.* 50 (2005) 1017–1031.
- [48] I. Faragó, Á. Havasi, Consistency analysis of operator splitting methods for C_0 -semigroups expression, *Semigroup Forum* 74 (1) (2007) 125–139.
- [49] G.I. Marchuk, Some application of splitting-up methods to the solution of mathematical physics problems, *Appl. Mat.* 13 (1968) 103–132.
- [50] G. Strang, Accurate partial difference methods I: Linear Cauchy problems, *Arch. Ration. Mech. Anal.* 12 (1963) 392–402.
- [51] G. Strang, On the construction and comparison of difference schemes, *SIAM. J. Numer. Anal.* 5 (1968) 506–517.
- [52] W. Arendt, C. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, in: *Monographs in Mathematics*, Birkhauser-Verlag, Berlin, 2001.
- [53] K.J. Engel, R. Nagel, One-parameter semigroups for linear evolution equations, in: *Graduate Texts in Mathematics*, vol. 194, Springer-Verlag, New York, 2000.
- [54] E. Hille, R.S. Phillips, *Functional Analysis and Semi-Groups*, American Mathematical Society, Providence, RI, 1974, Third printing of the revised edition of 1957, vol. XXXI, American Mathematical Society Colloquium Publications.
- [55] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential equations, in: *Applied Mathematical Sciences*, vol. 44, Springer-Verlag, New York, 1983.
- [56] H. Brézis, A. Pazy, Convergence and approximation of semigroups of nonlinear operators in Banach spaces, *J. Funct. Anal.* 9 (1972) 63–74.
- [57] M. Cliff, J.A. Goldstein, M. Wacker, Positivity, Trotter products, and blow-up, *Positivity* 8 (2) (2004) 187–208.
- [58] I. Miyadera, S. Ôharu, Approximation of semi-groups of nonlinear operators, *Tôhoku Math. J.* 22 (2) (1970) 24–47.
- [59] B. Baeumer, M. Kovács, Subordinated groups of linear operators: properties via the transference principle and the related unbounded operational calculus, (2006) (submitted for publication).
- [60] R.S. Phillips, On the generation of semigroups of linear operators, *Pacific J. Math.* 2 (1952) 343–369.
- [61] R.L. Schilling, Growth and Hölder conditions for sample paths of Feller processes, *Probab. Theory Related Fields* 112 (1998) 565–611.
- [62] K.-I. Sato, Lévy Processes and Infinitely Divisible Distributions, in: *Cambridge Studies in Advanced Mathematics*, vol. 68, Cambridge University Press, 1999.
- [63] B. Baeumer, M. Kovács, M.M. Meerschaert, Subordinated multiparameter groups of linear operators: Properties via the transference principle, *Functional Analysis and Evolution Equations: Dedicated to Günter Lumer*, Birkhäuser Basel, (2008) (in press).
- [64] K. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley and Sons, New York, 1993.
- [65] S. Samko, A. Kilbas, O. Marichev, *Fractional Integrals and derivatives: Theory and Applications*, Gordon and Breach, London, 1993.
- [66] M.M. Meerschaert, H.P. Scheffler, Semistable Lévy Motion, *Fract. Calc. Appl. Anal.* 5 (2002) 27–54.
- [67] A.V. Balakrishnan, An operational calculus for infinitesimal generators of semigroups, *Trans. Amer. Math. Soc.* 91 (1959) 330–353.
- [68] A.V. Balakrishnan, Fractional powers of closed operators and the semigroups generated by them, *Pacific J. Math.* 10 (1960) 419–437.
- [69] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H.P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, U. Schlotterbeck, One-parameter semigroups of positive operators, in: *Lecture Notes in Mathematics*, vol. 1184, Springer-Verlag, Berlin, 1986.
- [70] G.C. van Kooten, E.H. Blute, *The Economics of Nature. Managing Biological Assets*, Blackwell Publishers Inc., 2000.
- [71] D.R. Lockwood, A. Hastings, The effects of dispersal patterns on marine reserves: Does the tail wag the dog?, *Theoretical Population Biology* 61 (2002) 297–309.
- [72] V.M. Zolotarev, One-dimensional stable distributions, in: *Translations of Mathematical Monographs*, vol. 65, American Mathematical Society, Providence, RI, 1986.
- [73] J.P. Nolan, Numerical calculation of stable densities and distribution functions. Heavy tails and highly volatile phenomena, *Comm. Statist. Stochastic Models* 13 (4) (1997) 759–774.