Speculative option valuation and the fractional diffusion equation

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Abstract — *In financial markets not only returns, but also waiting times between consecutive trades are random variables and it is possible to apply continuous-time random walks (CTRWs) as phenomenological models of high-frequency prices. Based on these considerations, in this extended abstract, some results are outlined which can be useful for speculative option valuation.*

1 The basic mapping onto continuous-time random walks

High-frequency financial data can be phenomenologically mapped onto continuous-time random walks (CTRWs), also called point or renewal processes with reward [1].

Let $S(t)$ denote the price of an asset or the value of an index at time t. In finance, returns rather than prices are more convenient variables. Following Parkinson [2], let us introduce the variable $x(t) = \log S(t)$, that is the logarithm of the price. For small price variations, $\Delta S = S(t_{i+1}) - S(t_i)$, the return $r = \Delta S / S(t_i)$ and the logarithmic return $r_{log} = \log[S(t_{i+1})/S(t_i)]$ virtually coincide.

As also waiting times $\tau_i = t_{i+1} - t_i$ between two consecutive trades are stochastic variables, the time series $\{x(t_i)\}$ is characterised by $\varphi(\xi, \tau)$, the *joint probability density function* of log-returns $\xi_i = x(t_{i+1}) - x(t_i)$ and of waiting times $\tau_i = t_{i+1} - t_i$. The joint density satisfies the normalization condition $\int \int d\xi d\tau \varphi(\xi, \tau) = 1$. An important property of CTRWs is that log-returns and waiting times are independent and identically distributed random variables. However, there can be a dependence between the two random variables. One can define the two marginal densities in the usual way: $\lambda(\xi) = \int d\tau \varphi(\xi, \tau)$ and $\psi(\tau) = \int d\xi \varphi(\xi, \tau)$. Both $\lambda(\xi)$ and $\psi(\tau)$ are normalized to 1. If ξ and τ are independent, the joint probability density function $\varphi(\xi, \tau)$ is given by the product of the two marginal densities:

$$\varphi(\xi, \tau) = \lambda(\xi)\psi(\tau); \tag{1}$$

if they are not independent, then, according to the definition of conditional probability, one has:

$$\varphi(\xi, \tau) = \lambda(\xi)\psi(\tau|\xi) = \lambda(\xi|\tau)\psi(\tau), \tag{2}$$

where $\psi(\tau|\xi)$ and $\lambda(\xi|\tau)$ are conditional probability densities.

Let us now introduce the function $p(x, t)$, that is the probability density function of finding the value x of the price logarithm (which is the diffusing quantity in our case) at time t given that the log-price was 0 at time 0. Let us define the Fourier-Laplace transform of $p(x, t)$ as:

$$\widetilde{\widehat{p}}(\kappa, s) = \int_0^{+\infty} dt \int_{-\infty}^{+\infty} dx \, \exp(-st + i\kappa x) \, p(x, t) \, . \tag{3}$$

Montroll, Sher and coauthors [3, 4] have shown that the Fourier-Laplace transform of $p(x, t)$ is given by:

$$\widetilde{\widehat{p}}(\kappa, s) = \frac{1 - \widetilde{\psi}(s)}{s} \frac{1}{1 - \widetilde{\widehat{\varphi}}(\kappa, s)} \, , \tag{4}$$

where $\widetilde{\psi}(s)$ is the Laplace transform of $\psi(t)$ and:

$$\widetilde{\widehat{\varphi}}(\kappa, s) = \int_0^{+\infty} d\tau \int_{-\infty}^{+\infty} d\xi \, \exp(-s\tau + i\kappa\xi) \, \varphi(\xi, \tau) \, . \tag{5}$$

The space-time version of eq. (4) can be derived by purely probabilistic considerations [5]. The following integral equation gives the probability density, $p(x, t)$, for the walker being in position x at time t, conditioned by the fact that it was in position $x = 0$ at time $t = 0$:

$$p(x, t) = \delta(x) \, \Psi(t) + \int_0^t \int_{-\infty}^{+\infty} \varphi(x - x', t - t') \, p(x', t') \, dt' \, dx', \tag{6}$$

where $\Psi(\tau)$ is the so-called survival function. $\Psi(\tau)$ is related to the marginal waiting-time probability density $\psi(\tau)$ as follows:

$$\Psi(\tau) = 1 - \int_0^\tau \psi(\tau') \, d\tau' = \int_\tau^\infty \psi(\tau') \, d\tau'. \tag{7}$$

2 Limit theorems

The problem of the diffusive limit of the solutions to eq. (6) has been addressed by Gorenflo, Mainardi and the present author in various papers, dealing mainly with the uncoupled case, where the joint probability density $\varphi(\xi, \tau)$ can be factorized in terms of its marginals [6, 7, 8, 9]. The diffusive limit in the coupled case is discussed by Meerschaert *et al.* [10]. The coupled case is relevant as, in general, log-returns and waiting times are not independent [11]. Based on the results summarized in [9] and discussed in [7], it is possible to prove the following theorem for the coupled case:

Theorem

Let $\varphi(\xi,\tau)$ be the (coupled) joint probability density of a CTRW. If, under the scaling $\xi \to h\xi$ and $\tau \to r\tau$, the Fourier-Laplace transform of $\varphi(\xi,\tau)$ behaves as follows:

$$\widetilde{\widehat{\varphi}}_{h,r}(\kappa,s) = \widetilde{\widehat{\varphi}}(h\kappa,rs) \tag{8}$$

and if, for $h \to 0$ and $r \to 0$, the asymptotic relation holds:

$$\widetilde{\widehat{\varphi}}_{h,r}(\kappa,s) = \widetilde{\widehat{\varphi}}(h\kappa,rs) \sim 1 - \mu|h\kappa|^{\alpha} - \nu(rs)^{\beta}, \tag{9}$$

with $0 < \alpha \leq 2$ and $0 < \beta \leq 1$. Then, under the scaling relation $\mu h^{\alpha} = \nu r^{\beta}$, the solution of the (scaled) coupled CTRW master (integral) equation, eq. (6), $p_{h,r}(x,t)$, weakly converges to the Green function of the fractional diffusion equation, $u(x,t)$, for $h \to 0$ and $r \to 0$.

Proof
The Fourier-Laplace transform of the scaled conditional probability density $p_{h,r}(x,t)$ is given by:

$$\widetilde{\widehat{p}}_{h,r}(\kappa,s) = \frac{1-\widetilde{\psi}(rs)}{s}\frac{1}{1-\widetilde{\widehat{\varphi}}(h\kappa,rs)}. \tag{10}$$

Replacing eq. (9) in eq. (10) and observing that $\widetilde{\psi}(s) = \widetilde{\widehat{\varphi}}(0,s)$, one asymptotically gets for small h and r:

$$\widetilde{\widehat{p}}_{h,r}(\kappa,s) \sim \frac{\nu r^{\beta} s^{\beta-1}}{\nu r^{\beta} s^{\beta} + \mu h^{\alpha}|\kappa|^{\alpha}}, \tag{11}$$

which for vanishing h and r, under the hypotheses of the theorem, converges to:

$$\widetilde{\widehat{p}}_{0,0}(\kappa,s) = \widetilde{\widehat{u}}(\kappa,s) = \frac{s^{\beta-1}}{s^{\beta}+|\kappa|^{\alpha}}, \tag{12}$$

where $\widetilde{\widehat{u}}(\kappa,s)$ is the Fourier-Laplace transform of the Green function of the fractional diffusion equation (see the remarks below). The asymptotic equivalence in the space-time domain, between $p_{0,0}(x,t)$ and $u(x,t)$, the inverse Fourier-Laplace transform of $\widetilde{\widehat{u}}(\kappa,s)$, is ensured by the continuity theorem for sequences of characteristic functions, after the application of the analogous theorem for sequences of Laplace transforms [12]. There is convergence in law or weak convergence for the corresponding probability distributions and densities.

Remark 1

The fractional diffusion problem referred to in the previous theorem can be written as follows:

$$\begin{aligned} \frac{\partial^{\beta}}{\partial t^{\beta}}u(x,t) &= \frac{\partial^{\alpha}}{\partial|x|^{\alpha}}u(x,t), \quad 0<\alpha\leq 2, \quad 0<\beta\leq 1, \\ u(x,0^{+}) &= \delta(x), \quad x\in(-\infty,+\infty), \quad t>0, \end{aligned} \tag{13}$$

where the notation is inspired by the one used by Saichev and Zaslavsky [13], $\partial^\alpha/\partial|x|^\alpha$ is the Riesz derivative: a pseudo-differential operator with symbol $-|\kappa|^\alpha$, and $\partial^\beta/\partial t^\beta$ is the Caputo derivative, related to the Riemann–Liouville fractional derivative. For a sufficiently well-behaved function $f(t)$, the Caputo derivative is defined by the following equation, for $0 < \beta < 1$:

$$\frac{d^\beta}{dt^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\beta} \, d\tau - \frac{t^{-\beta}}{\Gamma(1-\beta)} f(0^+), \tag{14}$$

and reduces to the ordinary first derivative for $\beta = 1$. The Laplace transform of the Caputo derivative of a function $f(t)$ is:

$$\mathcal{L}\left(\frac{d^\beta}{dt^\beta} f(t);\, s\right) = s^\beta \widetilde{f}(s) - s^{\beta-1} f(0^+). \tag{15}$$

Remark 2

The problem of Remark 1 can be solved by Fourier-Laplace transforming eq. (13) and then inverting back the Fourier-Laplace transform of the solution. The Fourier-Laplace transform of the solution is indeed:

$$\widetilde{\widehat{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^\beta + |\kappa|^\alpha}, \tag{16}$$

as stated in the proof above. The solution turns out to be:

$$u(x,t) = \frac{1}{t^{\beta/\alpha}} W_{\alpha,\beta}\left(\frac{x}{t^{\beta/\alpha}}\right), \tag{17}$$

where $W_{\alpha,\beta}(u)$ is given by:

$$W_{\alpha,\beta}(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\kappa \; \mathrm{e}^{-i\kappa u} E_\beta(-|\kappa|^\alpha), \tag{18}$$

the inverse Fourier transform of a Mittag-Leffler function: $E_\beta(\cdot)$.

Remark 3

An important consequence of the above theorem is the following corollary showing that in the case of marginal densities with finite first moment of waiting times and finite second moment of log-returns, the limiting density $u(x,t)$ is the solution of the ordinary diffusion equation (and thus the limiting process is the Wiener process). The corollary can be used to justify the popular Geometric Brownian Motion model of stock prices, here with expected return set to zero. However, in order to derive this result, no reference is necessary to the Efficient Market Hypothesis [14, 15].

Corollary

If the Fourier-Laplace transform of $\varphi(\xi,\tau)$ is regular for $\kappa = 0$ and $s = 0$, and, moreover, the marginal waiting-time density, $\psi(\tau)$, has finite first moment τ_0 and the marginal jump density, $\lambda(\xi)$, is symmetric with finite second moment σ^2, then the limiting solution of the master (integral) equation for the coupled CTRW is the Green function of the ordinary diffusion equation.

Proof
Due to the hypothesis of regularity in the origin and to the properties of Fourier and Laplace transforms, we have that:

$$\widetilde{\widehat{\varphi}}_{h,r}(\kappa,s) = \widetilde{\widehat{\varphi}}(h\kappa, rs) = \widetilde{\widehat{\varphi}}(0,0) + \frac{1}{2}\left(\frac{\partial^2 \widetilde{\widehat{\varphi}}}{\partial \kappa^2}\right)_{(0,0)} h^2\kappa^2 + \left(\frac{\partial \widetilde{\widehat{\varphi}}}{\partial s}\right)_{(0,0)} rs + \ldots$$
$$\sim 1 - \frac{\sigma^2}{2}h^2\kappa^2 - \tau_0 rs, \tag{19}$$

and, as a consequence of the theorem, under the scaling $h^2\sigma^2/2 = \tau_0 r$, one gets, for vanishing h and r:

$$\widetilde{\widehat{p}}_{0,0}(k,s) = \widetilde{\widehat{u}}(k,s) = \frac{1}{s+k^2}, \tag{20}$$

corresponding to the Green function of eq. (13) for $\alpha = 2$ and $\beta = 1$, that is of the ordinary diffusion equation.

Remark 4

A discussion of the meaning of the limit taken in the theorem can be useful. Replacing the jumps ξ by $h\xi$ and the waiting times τ by $r\tau$ and letting h and r vanish, means that the jump size becomes smaller and smaller and also the time between two consecutive jumps decreases. The scaling requirement that $\mu h^\alpha = \nu r^\beta$ can be written as $h = Cr^{\beta/\alpha}$. If $\beta = 1$ and $\alpha = 2$, one recognizes the behaviour of the Wiener process: the limiting process, in this case, for vanishing h and r. As the variable x is a log-price, a consequence of the corollary is that, in the diffusive limit, one gets normal log-prices and log-normal prices. In other words, one recovers the stochastic part of the usual Geometric Brownian Motion model for the dynamic of prices in a financial market. Under the conditions of the theorem discussed above, this behaviour is generalized to a larger class of limiting stochastic processes.

Remark 5

In the coupled CTRW case, the relationship between the analytic behaviour of the Fourier-Laplace transform of $\varphi(\xi,\tau)$ and the governing equation for the limiting probability densities gives rise to a rich set of possibilities. The corollary of remark 3 shows that, if the marginal waiting-time density has a finite first moment and the marginal jump density has a finite second moment, then the limiting density coincides with the Green function of the ordinary diffusion equation. The reader is referred to refs. [10, 16] for further details.

3 Speculative option valuation

The knowledge of $p(x,t)$ is sufficient for speculative option valuation. Given a maturity T, the log-price distribution at maturity is given by $p(x,T)$. Let the function $y = \Pi(x,T)$ denote the pay-off of a European option at maturity. For instance, in the case of a plain-vanilla call European option, the payoff is:

$$y = \Pi(x,T) = \max\{\exp(x(T)) - E, 0\}, \tag{21}$$

where E is the exercise price. The knowledge of the density $p(x,T)$ immediately yields the knowledge of the transformed density, $q(y,T)$, by change of variable:

$$q(y,T) = p[\Pi^{-1}(y,T),T] \left| \frac{dx}{dy} \right| \tag{22}$$

In the diffusive limit, for a large class of CTRWs, the solution of eq. (6) weakly converges to the Green function in eq. (17) which can be used for speculative option pricing predictions. However, it is also possible to estimate $p(x,T)$ by means of Monte Carlo simulations [17].

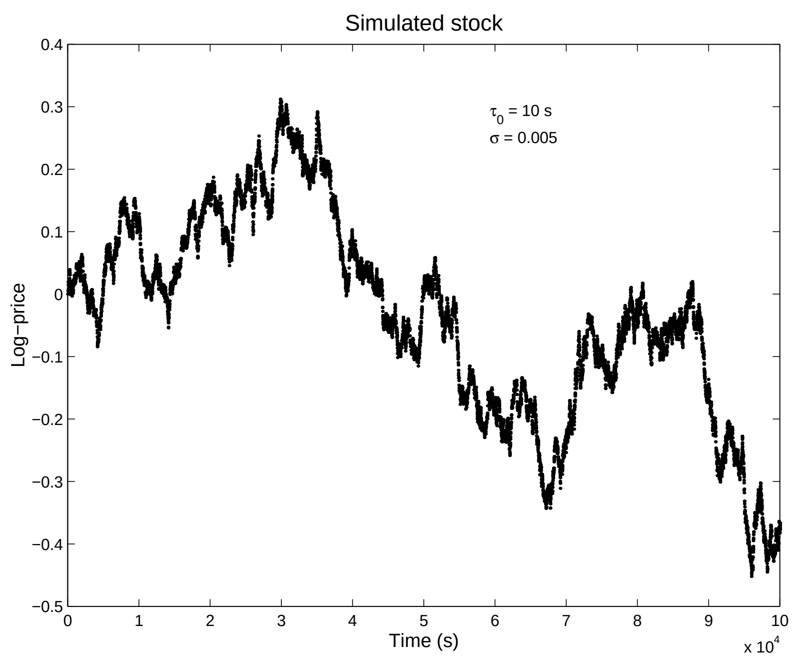

Figure 1: Simulated NCPP log-price as a function of time. This simulation includes 10000 log-prices. It takes a few minutes to run on an old Pentium II processor at 349 MHz.

For illustrative purposes, we shall consider a particular instance of continuous-time random walk, the Normal Compound Poisson Process (NCPP), following and simplifying the discussion in ref. [18]. The NCPP is characterized by an exponential marginal waiting-time density:

$$\psi(\tau) = \mu \mathrm{e}^{-\mu\tau}, \tag{23}$$

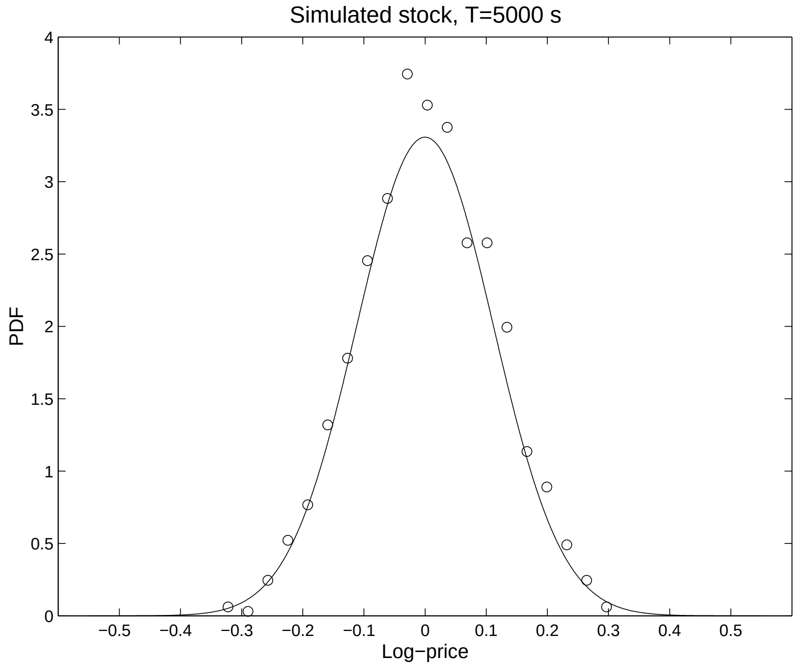

Figure 2: Theoretical probability density function (solid line) and simulated probability density function (circles) for a NCPP. $p(x,T)$ is computed for T=5000s, τ_0=10s, σ=0.005. The simulated probability density function is computed from the histogram of 1000 realisations.

and a normal marginal jump density:

$$\lambda(\xi) = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\xi^2/2\sigma}. \tag{24}$$

Both jumps and waiting times are independent and identically distributed variables. Moreover they are not correlated. For a NCPP, the density $p(x,t)$ can be computed exactly and is:

$$p(x,t) = \mathrm{e}^{-\mu t}\sum_{n=0}^{\infty}\frac{(\mu t)^n}{n!}\frac{1}{\sqrt{2\pi n}\sigma}\mathrm{e}^{-x^2/2n\sigma^2}. \tag{25}$$

In Figure 1, a Monte Carlo simulation of a NCPP is presented for $\sigma = 0.005$ and $\tau_0 = 10$s. In the Monte Carlo simulation, the initial log-price was set to zero, then a series of jump sizes has been generated from normal random digits with zero mean and the appropriate variance. Finally, and independently from jump-size generation, a series of waiting times has been extracted from exponentially distributed deviates. A MATLAB Monte Carlo routine can be obtained from the corresponding author. In Figure 2 a comparison is shown between the probability density function $p(x,T)$ computed by means of eq. (25) and estimated from the histogram of 1000 independent Monte Carlo realizations of the NCPP. Finally, in Figure 3, we plot the payoff histogram for a call European option with payoff given by eq. (21), where the initial price is $S(0) = 100$ and the strike price after T=5000s is $E = 100$. The histogram has been obtained from the 1000 Monte Carlo realizations described in Figure 2.

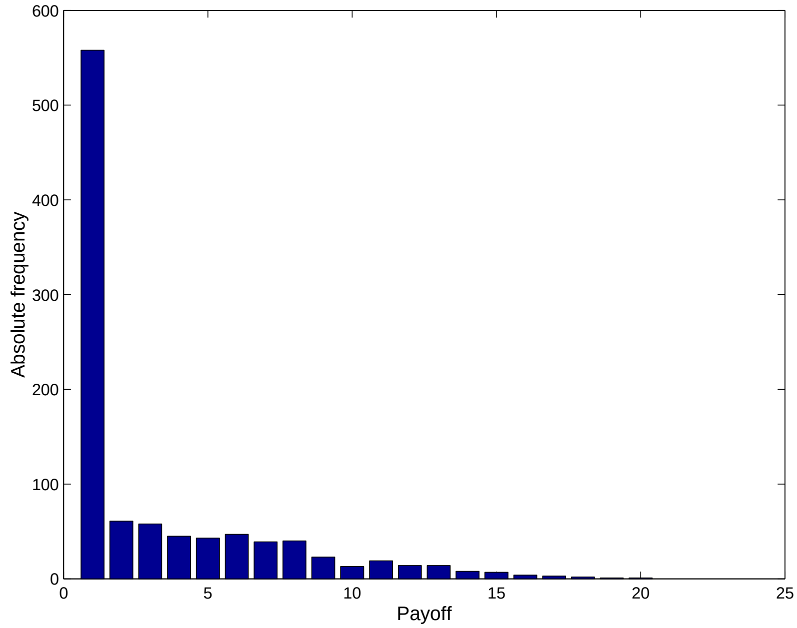

Figure 3: Payoff histogram for a very short-term plain vanilla call European option with initial price $S(0) = 100$ and strike price $E = 100$. The evolution of the underlying has been simulated 1000 times by means of a NCPP, with parameters given in Figure 2.

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