SEMISTABLE LÉVY MOTION

MARK M. MEERSCHAERT AND HANS-PETER SCHEFFLER

Abstract. Semistable Lévy motions have stationary independent increments with semistable distributions. They can be realized as scaling limits of simple random walks, extending the familiar Lévy motions. Generators of stable semigroups are fractional derivatives, and the semistable generators provide a new approximation to fractional derivatives. Semistable Lévy motions and semistable generators may be useful in physics to model anomalous diffusion.

1. Introduction

Stable Lévy motions are useful in physics and geology to describe anomalous diffusion, in which a cloud of particles spreads faster than the classical diffusion model predicts [3, 2, 5, 11, 23]. A fractional diffusion equation models this behavior using fractional derivatives [1, 6, 25]. Fractional derivatives are also generators of stable semigroups [7, 10]. In this paper, we develop the corresponding theory of semistable Lévy motions, semistable generators, and the corresponding diffusion equations. Semistable Lévy motions have stationary independent increments with semistable distributions, a natural generalization of stable distributions [12, 13, 17, 19, 22]. These motions are scaling limits of random walks, and in some cases they can be used to approximate Lévy motions. Since a semistable distribution is infinitely divisible, it defines a continuous convolution semigroup, whose generator provides a useful approximation to the fractional derivative. This new approximation is appealing because, unlike the fractional difference quotient, it has a nice scaling property. The resulting semistable diffusion equation may also have applications in physics, as a more flexible model for anomalous diffusion. At the end of this paper, we show that fractional difference approximations to fractional derivatives can also be realized as generators of continuous convolution semigroups associated with certain infinitely divisible laws.

2. Scaling limits of simple random walks

Let Y be a random variable whose probability distribution ν is strictly (b, c) semistable for some $c > 1$. This means that ν is infinitely divisible and $\nu^c =$

Key words and phrases. semistable law, fractional derivative, anomalous diffusion.
Meerschaert was partially supported by NSF–DES grant 9980484.

$c^{1/\alpha}\nu$ where $\alpha = \log c/\log b$, ν^t is the t–fold convolution power of the infinitely divisible law ν, and $a\nu(dx) = \nu(a^{-1}dx)$ is the probability distribution of the random variable aY. Then necessarily $0 < \alpha \leq 2$. If $\alpha = 2$ then ν is mean zero normal, otherwise the variance of Y is infinite, see for example [15] Corollary 7.4.4. Let $Z, Z_1, Z_2, Z_3, \ldots$ be independent random variables whose common distribution μ belongs to the strict domain of semistable attraction of ν. This means that for some $a_n > 0$, and some increasing sequence of positive integers (k_n) such that

$$k_{n+1}/k_n \to c \quad \text{as } n \to \infty \tag{2.1}$$

we have

$$a_n \sum_{j=1}^{k_n} Z_j \Rightarrow Y \tag{2.2}$$

where $\Rightarrow$ denotes weak convergence. In this case we write $\mu \in \mathrm{DOSA}(\nu, c)$. If (2.2) holds with $k_n = n$ we say that μ belongs to the strict domain of attraction of ν, we write $\mu \in \mathrm{DOA}(\nu)$, and the limit law ν is stable with index α, meaning that ν is infinitely divisible and $\nu^t = t^{1/\alpha}\nu$ for every $t > 0$.

A sequence of positive real numbers (b_n) is said to vary regularly with index ρ if $b_{[\lambda n]}/b_n \to \lambda^\rho$ as $n \to \infty$ for every $\lambda > 0$ [24]. A random variable X is nondegenerate if it is not almost surely constant. A sequence of random variables X_n is stochastically compact if for any increasing sequence of positive integers (n_i) there exists a subsequence (n_j) and a nondegenerate random variable X such that $X_{n_j} \Rightarrow X$. If $\mu \in \mathrm{DOSA}(\nu, c)$ and (Z_n) and Y are as above then for some regularly varying sequence of positive reals (b_n) with index $-1/\alpha$ the sequence of random sums

$$\left\{ b_n \sum_{j=1}^{n} Z_j : n \geq 1 \right\}$$

is stochastically compact with limit set contained in

$$\{Y_\lambda : 1 \leq \lambda \leq c\}$$

where Y_λ is a random variable with distribution $\lambda^{-1/\alpha}\nu^\lambda$. In fact, if we let $p_n = \max\{p : k_p \leq n\}$ and $\lambda_n = n/k_{p_n}$ then for any sequence of positive integers there exists a subsequence (n') such that $\lambda_{n'} \to \lambda \in [1, c]$ and $b_{n'} \sum_{j=1}^{n'} Z_j \Rightarrow Y_\lambda$, see [15] Theorem 8.3.18.

Lemma 2.1. *For any $t > 0$ we have:*

(a) $\{b_{[s]} \sum_{j=1}^{[st]} Z_j : s \geq 1\}$ *is stochastically compact with limit set contained in* $\{\lambda^{-1/\alpha}\nu^{\lambda t} : 1 \leq \lambda \leq c\}$*;*

(b) $b_{k_n} \sum_{j=1}^{[k_n t]} Z_j \Rightarrow \nu^t$ *as* $n \to \infty$.

Proof. Write $[st] = [[s]t] + r_{s,t}$ so that $0 \leq r_{s,t} \leq t$ for all $s \geq 1$. Since the index of the regularly varying sequence (b_n) is negative, $b_n \to 0$ and hence

$$b_{[s]} \sum_{j=[[s]t]+1}^{[st]} Z_j \to 0 \quad \text{in probability}$$

as $s \to \infty$. Since

$$b_{[s]} \sum_{j=1}^{[st]} Z_j = b_{[s]} \sum_{j=1}^{[[s]t]} Z_j + b_{[s]} \sum_{j=[[s]t]+1}^{[st]} Z_j$$

part (a) will follow if we can show that the sequence of random variables

$$\left(b_n \sum_{j=1}^{[nt]} Z_j \right)_{n \geq 1}$$

is stochastically compact. Since $\{b_n \sum_{j=1}^{n} Z_j\}$ is stochastically compact, any sequence of positive integers contains a further sequence (n') along which $b_n \sum_{j=1}^{n} Z_j \Rightarrow Y_\lambda$ where Y_λ has distribution $\lambda^{-1/\alpha}\nu^\lambda$ for some $\lambda \in [1, c]$. An application of Proposition 3.3.7 of [15] yields (a). Part (b) follows directly from Proposition 3.3.7 of [15]. □

For any infinitely divisible law ν, there is a corresponding Lévy process $\{Y(t) : t \geq 0\}$, a stationary independent increment process which is continuous in law such that $Y(t)$ has distribution ν^t [4, 13, 20]. If ν is stable then $\{Y(t)\}$ is also called a Lévy motion. In the special case where ν is normal, $\{Y(t)\}$ is a Brownian motion. If ν is (b, c) semistable for some $c > 1$ then we call the corresponding Lévy process $\{Y(t)\}$ a semistable Lévy motion. The index $\alpha = \log c / \log b$ has more or less the same interpretation as the index of a stable Lévy motion. In particular, for nonnormal semistable laws with index $\alpha < 2$ the probability tail $P(|Y(t)| > y)$ diminishes like $y^{-\alpha}$, so that the moments $E|Y(t)|^\rho$ exist for $0 < \rho < \alpha$ and diverge for $\rho \geq \alpha$.

Lévy motions are useful because they are the scaling limits of simple random walks. For $t \geq 0$ let

$$X(t) = \sum_{j=1}^{[t]} Z_j \tag{2.3}$$

where as before $Z, Z_1, Z_2, Z_3, \ldots$ are independent random variables with common distribution μ. If $EZ = 0$ and $\sigma^2 = EZ^2 < \infty$ then

$$\{c^{-1/2} X(ct)\} \stackrel{f.d.}{\Longrightarrow} \{Y(t)\} \quad \text{as } c \to \infty$$

where $\{Y(t)\}$ is a Brownian motion and the notation $\stackrel{f.d.}{\Longrightarrow}$ means that for any $0 < t_1 < \cdots < t_n$ we have

$$\left(c^{-1/2}X(ct_1), \ldots, c^{-1/2}X(ct_n)\right) \Rightarrow (Y(t_1), \ldots, Y(t_n)) \quad \text{as } c \to \infty. \tag{2.4}$$

More generally, if $\mu \in \mathrm{DOA}(\nu)$ and ν is stable with index $0 < \alpha \leq 2$ then

$$\{b_{[c]}X(ct)\} \stackrel{f.d.}{\Longrightarrow} \{Y(t)\} \quad \text{as } c \to \infty$$

where $\{Y(t)\}$ is a stable Lévy motion with index α. In physics, Lévy motions are used to model anomalous diffusion, which results from the power law probability tails of the particle jumps (Z_n), see for example [23].

Domains of semistable attraction are the most general setting in which the behavior of sums of independent, identically distributed random variables can be usefully approximated by a limit distribution, see [15] p. 286. Our next result shows that a convergence like (2.4) also pertains in this case, but in the weaker sense of stochastic compactness.

Theorem 2.2. *Suppose that $Z, Z_1, Z_2, Z_3, \ldots$ are independent random variables with common distribution $\mu \in \mathrm{DOSA}(\nu, c)$ for some $c > 1$ and that* (2.1), (2.2) *and* (2.3) *hold. Then:*

(a) *Let $\{Y_\lambda(t)\}$ be a Lévy process such that $Y_\lambda(t)$ has distribution $\lambda^{-1/\alpha}\nu^{\lambda t}$. Then for any $0 < t_1 < \cdots < t_m$*

$$\left\{b_{[s]}\left(X(st_1), \ldots, X(st_m)\right) : s > 0\right\}$$

is stochastically compact with limit set contained in

$$\{(Y_\lambda(t_1), \ldots, Y_\lambda(t_m)) : \lambda \in [1, c]\}.$$

In other words, for any sequence $s_n \to \infty$ there exists a subsequence $s_{n'}$ such that

$$\left\{b_{s_{n'}}X(s_{n'}t)\right\} \stackrel{f.d.}{\Longrightarrow} \{Y_\lambda(t)\}$$

for some $\lambda \in [1, c]$;

(b) *Let $\{Y(t)\}$ be a semistable Lévy motion such that $Y(t)$ has distribution ν^t. Then for any $0 < t_1 < \cdots < t_m$*

$$b_{k_n}\left(X(k_nt_1), \ldots, X(k_nt_m)\right) \Rightarrow (Y(t_1), \ldots, Y(t_m))$$

or in other words

$$\{b_{k_n}X(k_nt)\} \stackrel{f.d.}{\Longrightarrow} \{Y(t)\}.$$

Remark 2.3. It is easy to check that if ν is strictly (b, c) semistable then so is $\lambda^{-1/\alpha}\nu^\lambda$, hence $\{Y_\lambda(t)\}$ is also a semistable Lévy motion with the same index α. If $\lambda = 1$ or $\lambda = c$ then since ν is (b, c) semistable it follows that

$$\{Y_\lambda(t)\} \stackrel{f.d.}{=} \{Y(t)\},$$

meaning that for any $0 < t_1 < \cdots < t_m$ the random vectors $(Y_\lambda(t_1), \ldots, Y_\lambda(t_m))$ and $(Y(t_1), \ldots, Y(t_m))$ are identically distributed.

Proof. We only consider the case $m = 2$, the other cases being similar. Given $0 < t_1 < t_2$ and $s > 0$ note that $[st_2] - [st_1] = [s(t_2 - t_1)] + r$ for some $r \in \{0, 1, 2\}$. It follows from Lemma 2.1 (a) that for any sequence $s_n \to \infty$ for some subsequence $s_{n'}$ and some $\lambda \in [1, c]$ we have

$$b_{[s_{n'}]}\mu^{[s_{n'}t]} \Rightarrow \lambda^{-1/\alpha}\nu^{\lambda t}$$

for all $t > 0$. It follows easily that

$$b_{[s_{n'}]}X(s_{n'}t_1) \Rightarrow Y_\lambda(t_1) \sim \lambda^{-1/\alpha}\nu^{\lambda t_1} \tag{2.5}$$

where $X \sim \mu$ means that the random variable X has distribution μ, and

$$\begin{aligned} b_{[s_{n'}]}\left(X(s_{n'}t_2) - X(s_{n'}t_1)\right) &= b_{[s_{n'}]} \sum_{j=[s_{n'}t_1]+1}^{[s_{n'}t_2]} Z_j \\ &\sim b_{[s_{n'}]}\mu^{[s_{n'}t_2]-[s_{n'}t_1]} \\ &= b_{[s_{n'}]}\mu^{[s_{n'}(t_2-t_1)]} * b_{[s_{n'}]}\mu^r \\ &\Rightarrow \lambda^{-1/\alpha}\nu^{\lambda(t_2-t_1)} \sim Y_\lambda(t_2) - Y_\lambda(t_1). \end{aligned} \tag{2.6}$$

Since the left hand sides in (2.5) and (2.6) are independent, and likewise for the right hand sides, it follows that

$$b_{[s_{n'}]}\left(X(s_{n'}t_1), X(s_{n'}t_2) - X(s_{n'}t_1)\right) \Rightarrow \left(Y_\lambda(t_1), Y_\lambda(t_2) - Y_\lambda(t_1)\right)$$

and then the continuous mapping theorem yields

$$b_{[s_{n'}]}\left(X(s_{n'}t_1), X(s_{n'}t_2)\right) \Rightarrow \left(Y_\lambda(t_1), Y_\lambda(t_2)\right)$$

which proves (a). The proof of (b) is similar, using Lemma 2.1 (b) instead of Lemma 2.1 (a). □

3. Stable and semistable Lévy motions

In this section we elucidate the connection between stable and semistable Lévy motions. A strictly (b, c) semistable law satisfies $\nu^c = c^{1/\alpha}\nu$ where $\alpha = \log c / \log b$, and hence $\nu^t = t^{1/\alpha}\nu$ whenever $t = c^n$ for some integer n. A stable law satisfies $\nu^t = t^{1/\alpha}\nu$ for any $t > 0$. As the scale $c > 1$ of the semistable law tends to one, it is reasonable to expect that the behavior of the semistable law becomes more like that of a stable law.

Theorem 3.1. *Fix $0 < \alpha < 2$, let ν_c be a strictly $(c^{1/\alpha}, c)$ semistable law for each $c > 1$, and let $\{Y_c(t)\}$ be the associated semistable Lévy motion, so that*

$Y_c(t) \sim \nu_c^t$. If $\nu_c \Rightarrow \nu$ nondegenerate as $c \to 1$ then ν is stable with index α and

$$\{Y_c(t)\} \overset{f.d.}{\Longrightarrow} \{Y(t)\} \quad \text{as } c \to 1 \tag{3.1}$$

where $\{Y(t)\}$ is a Lévy motion with $Y(t) \sim \nu^t$.

Proof. Let $[a_c, 0, \phi_c]$ denote the Lévy representation of ν_c, see for example [15] Theorem 3.1.11. Then $\nu_c \Rightarrow \nu$ implies that ν is infinitely divisible and that $\phi_c \to \phi$ the Lévy measure of ν, see for example [15] Theorem 3.1.15. Corollary 7.4.4 of [15] implies that $\phi_c(t,\infty) = t^{-\alpha}\theta_1^{(c)}(\log t)$ and $\phi_c(-\infty,-t) = t^{-\alpha}\theta_2^{(c)}(\log t)$ where $\theta_i^{(c)}$ are nonnegative periodic functions with period $\log b$ where $b = c^{1/\alpha}$.

Lemma 3.2. *For some $\bar\theta_1, \bar\theta_2 \geq 0$ with $\bar\theta_1 + \bar\theta_2 > 0$ we have $\theta_i^{(c)}(x) \to \bar\theta_i$ for all $x > 0$ and $i = 1, 2$ as $c \to 1$.*

Proof. Let Suppose $0 < s < t$ and write $t = sb^{k(c)}\lambda(c)$ where $k(c)$ is an integer and $1 \leq \lambda(c) < b$. Then for $i = 1, 2$

$$\begin{aligned} t^{-\alpha}\theta_i^{(c)}(\log t) &\leq (sb^{k(c)})^{-\alpha}\theta_i^{(c)}(\log sb^{k(c)}) \\ &= (sb^{k(c)})^{-\alpha}\theta_i^{(c)}(\log s) \end{aligned}$$

so that

$$\begin{aligned} \theta_i^{(c)}(\log t) &\leq t^{\alpha}(sb^{k(c)})^{-\alpha}\theta_i^{(c)}(\log s) \\ &= \lambda(c)^{\alpha}\theta_i^{(c)}(\log s) \\ &< b^{\alpha}\theta_i^{(c)}(\log s) \\ &= c\theta_i^{(c)}(\log s) \end{aligned}$$

and similarly $\theta_i^{(c)}(\log t) \geq c^{-1}\theta_i^{(c)}(\log s)$ so that we have

$$c^{-1}\theta_i^{(c)}(\log s) \leq \theta_i^{(c)}(\log t) \leq c\theta_i^{(c)}(\log s) \tag{3.2}$$

for all $c > 1$. Choose $s > 0$ so that $\phi\{-s, s\} = 0$ and let $\bar\theta_1 = s^{\alpha}\phi(s,\infty)$ and $\bar\theta_2 = s^{\alpha}\phi(-\infty,-s)$. Since ν is nondegenerate with no normal component, we must have $\bar\theta_1 + \bar\theta_2 = s^{\alpha}\phi\{x : |x| > s\} > 0$. Since $\phi_c \to \phi$ as $c \to 1$ we also have $\theta_1^{(c)}(\log s) = s^{\alpha}\phi_c(s,\infty) \to s^{\alpha}\phi(s,\infty) = \bar\theta_1$ and similarly $\theta_2^{(c)}(\log s) \to \bar\theta_2$. Now the lemma follows immediately from (3.2). □

Since $\phi_c \to \phi$ it follows from Lemma 3.2 that for any $t > 0$ such that $\phi\{-t, t\} = 0$ we have

$$\phi_c(t,\infty) = t^{-\alpha}\theta_1^{(c)}(\log t) \to t^{-\alpha}\bar\theta_1 = \phi(t,\infty)$$

and similarly $\phi(-\infty,-t) = t^{-\alpha}\bar\theta_2$. Since $\phi\{-t, t\} = 0$ for all but countably many $t > 0$ it follows that $\phi(t,\infty) = t^{-\alpha}\bar\theta_1$ and $\phi(-\infty,-t) = t^{-\alpha}\bar\theta_2$ for all

$t > 0$. Then it follows from [15] Corollary 7.3.4 and the uniqueness of the Lévy representation that ν is a stable law with index α. Hence $\{Y(t)\}$ is an α–stable Lévy motion.

Since $\nu_c \Rightarrow \nu$ as $c \to 1$ we also have $\hat{\nu}_c(x) \to \hat{\nu}(x)$ for all real x. Then $\hat{\nu}_c(x)^t \to \hat{\nu}(x)^t$ as well, and so $\nu_c^t \Rightarrow \nu^t$ as $c \to 1$ for every $t > 0$. In other words, $Y_c(t) \Rightarrow Y(t)$ as $c \to 1$ for each $t > 0$. Since all of these processes have independent increments, it follows that for any $0 < t_1 < \cdots < t_m$ we have

$$(Y_c(t_1), \ldots, Y_c(t_m) - Y_c(t_{m-1})) \Rightarrow (Y(t_1), \ldots, Y(t_m) - Y(t_{m-1}))$$

and then the continuous mapping theorem implies that

$$(Y_c(t_1), \ldots, Y_c(t_m)) \Rightarrow (Y(t_1), \ldots, Y(t_m))$$

which finishes the proof. □

4. Generators of semistable semigroups

Suppose again that ν is a $(c^{1/\alpha}, c)$ strictly semistable law. Since ν is infinitely divisible, we can define a continuous semigroup of linear operators

$$T_\nu(t)f(x) = f * \nu^t(x) = \int f(x-y)\nu^t(dy)$$

where $t > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is any bounded continuous function, see for example Feller [7] p.294. The infinitesimal generator of this semigroup is a linear operator defined by

$$A_\nu f(x) = \lim_{t\downarrow 0} t^{-1}\left(T_\nu(t)f(x) - f(x)\right)$$

for all $f \in D(A_\nu)$, the domain of the operator A_ν which is the space of all functions f for which this limit exists. It follows from Theorem 4.1.14 of [9] that all C^2-functions vanishing at infinity belong to $D(A_\nu)$. If ν is infinitely divisible with Lévy representation $[a, b, \phi]$ then the characteristic function $\hat{\nu}(k) = \int e^{ikx}\nu(dx) = e^{\psi(k)}$ where

$$\psi(k) = iak - \frac{b}{2}k^2 + \int_{x\neq 0}\Big(e^{ikx} - 1 - \frac{ikx}{1+x^2}\Big)\phi(dx), \tag{4.1}$$

and the generator is given by

$$A_\nu f(x) = -af'(x) + \frac{b}{2}f''(x) + \int\left(f(x-y) - f(x) + \frac{yf'(x)}{1+y^2}\right)\phi(dy), \tag{4.2}$$

see for example Hille and Phillips [10] Section 23.14. If $\alpha < 2$ (so that ν has no normal component) then $b = 0$.

For $a > 0$ define the dilation operator $\delta_a f(x) = f(ax)$. The next result shows that the generator of a semistable continuous convolution semigroup has a pleasant scaling property.

Proposition 4.1. *If ν is $(c^{1/\alpha}, c)$ strictly semistable then $cA_\nu f(x) = A_\nu(\delta_{c^{1/\alpha}} f)(c^{-1/\alpha}x)$.*

Proof. From the definition of the generator A_ν we have that

$$\begin{aligned} cA_\nu f(x) &= \lim_{t\downarrow 0} \frac{c}{t}\big(f * \nu^t(x) - f(x)\big) \\ &= \lim_{s\downarrow 0} \frac{1}{s}\left(f * \nu^{cs}(x) - f(x)\right). \end{aligned}$$

Since $\nu^c = c^{1/\alpha}\nu$ we obtain

$$\begin{aligned} f * \nu^{cs}(x) &= f * (c^{1/\alpha}\nu^s)(x) \\ &= \int f(x-y) c^{1/\alpha}\nu^s(dy) \\ &= \int f(x - c^{1/\alpha}y)\nu^s(dy) \\ &= \int (\delta_{c^{1/\alpha}} f)(c^{-1/\alpha}x - y)\nu^s(dy) \\ &= (\delta_{c^{1/\alpha}} f) * \nu^s(c^{-1/\alpha}x) \end{aligned}$$

and hence

$$\begin{aligned} cA_\nu f(x) &= \lim_{s\downarrow 0} \frac{1}{s}\big((\delta_{c^{1/\alpha}} f) * \nu^s(c^{-1/\alpha}x) - (\delta_{c^{1/\alpha}} f)(c^{-1/\alpha}x)\big) \\ &= A_\nu(\delta_{c^{1/\alpha}} f)(c^{-1/\alpha}x) \end{aligned}$$

which finishes the proof. □

Recall from Section 3 that

$$\phi(t,\infty) = t^{-\alpha}\theta_1(\log t) \quad \text{and} \quad \phi(-\infty,-t) = t^{-\alpha}\theta_2(\log t) \tag{4.3}$$

where θ_i are nonnegative periodic functions with period $\log b$ where $b = c^{1/\alpha}$. Then since ν has no normal component, the generator A_ν is given by

$$\begin{aligned} A_\nu f(x) &= -af'(x) \\ &\quad - \int_0^\infty \left(f(x-y) - f(x) + \frac{yf'(x)}{1+y^2}\right) d\left(y^{-\alpha}\theta_1(\log y)\right) \\ &\quad - \int_0^\infty \left(f(x+y) - f(x) - \frac{yf'(x)}{1+y^2}\right) d\left(y^{-\alpha}\theta_2(\log y)\right). \end{aligned} \tag{4.4}$$

Proposition 4.2. *If ν is $(c^{1/\alpha}, c)$ semistable with index $0 < \alpha < 1$ and Lévy measure* (4.3) *then for some $\tilde{a} \in \mathbb{R}$*

$$
\begin{aligned}
A_\nu f(x) = \tilde{a} f'(x) &- \int_0^\infty f'(x-y) y^{-\alpha} \theta_1(\log y) dy \\
&+ \int_0^\infty f'(x+y) y^{-\alpha} \theta_2(\log y) dy.
\end{aligned} \tag{4.5}
$$

Proof. Let $[a, 0, \phi]$ be the Lévy representation of ν, and define

$$
\tilde{a} = -a - \int_0^\infty \frac{y}{1+y^2} d\left(y^{-\alpha}\theta_1(\log y)\right) + \int_0^\infty \frac{y}{1+y^2} d\left(y^{-\alpha}\theta_2(\log y)\right).
$$

Note that these two integrals converge since $0 < \alpha < 1$ and $\theta_i(x)$ is bounded. Then $A_\nu f(x) = \tilde{a} f'(x) + I_1 + I_2$ where

$$
\begin{aligned}
I_1 &= -\int_0^\infty \left(f(x-y) - f(x)\right) d\left(y^{-\alpha}\theta_1(\log y)\right) \\
&= -\int_0^\infty f'(x-y) y^{-\alpha} \theta_1(\log y) dy
\end{aligned}
$$

via integration by parts using the fact that by Taylor's formula we have $f(x-y) - f(x) = O(|y|)$ as $|y| \to 0$. Since $0 < \alpha < 1$ and θ_1 is bounded, this integral converges if f' is bounded and $f'(x) = O(x^{-1})$ as $|x| \to \infty$. Similarly

$$
\begin{aligned}
I_2 &= -\int_0^\infty \left(f(x+y) - f(x)\right) d\left(y^{-\alpha}\theta_2(\log y)\right) \\
&= \int_0^\infty f'(x+y) y^{-\alpha} \theta_2(\log y) dy
\end{aligned}
$$

which completes the proof. □

Proposition 4.3. *Suppose ν is $(c^{1/\alpha}, c)$ semistable with index $1 < \alpha < 2$ and Lévy measure* (4.3), *and let*

$$
h_i(y) = -\int_y^\infty t^{-\alpha} \theta_i(\log t) dt
$$

for $i = 1, 2$. Then for some $\tilde{a} \in \mathbb{R}$

$$
A_\nu f(x) = \tilde{a} f'(x) - \int_0^\infty f''(x-y) h_1(y) dy - \int_0^\infty f''(x+y) h_2(y) dy \tag{4.6}
$$

for any $f \in C_b^2(\mathbb{R})$.

Proof. Let $[a, 0, \phi]$ be the Lévy representation of ν, and define

$$\tilde{a} = -a + \int_0^\infty \left(y - \frac{y}{1+y^2} \right) \, d\left(y^{-\alpha}\theta_1(\log y)\right) - \int_0^\infty \left(y - \frac{y}{1+y^2} \right) d\left(y^{-\alpha}\theta_2(\log y)\right)$$

which converges since $1 < \alpha < 2$ and $\theta_i(x)$ is bounded. Then $A_\nu f(x) = \tilde{a} f'(x) + I_1 + I_2$ where

$$\begin{aligned} I_1 &= -\int_0^\infty \left(f(x-y) - f(x) + y f'(x)\right) d\left(y^{-\alpha}\theta_1(\log y)\right) \\ &= \int_0^\infty \left(-f'(x-y) + f'(x)\right) y^{-\alpha}\theta_1(\log y) dy \\ &= -\int_0^\infty f''(x-y) h_1(y) dy \end{aligned}$$

since $h_i'(y) = y^{-\alpha}\theta_i(\log y)$. Since θ_i is bounded, $y^{\alpha-1}h_1(y)$ is bounded, so the integral converges if f'' is bounded and $f''(x) = O(x^{-1})$ as $|x| \to \infty$. Similarly

$$\begin{aligned} I_2 &= -\int_0^\infty \left(f(x+y) - f(x) - y f'(x)\right) d\left(y^{-\alpha}\theta_2(\log y)\right) \\ &= \int_0^\infty \left(f'(x+y) - f'(x)\right) y^{-\alpha}\theta_2(\log y) dy \\ &= -\int_0^\infty f''(x+y) h_2(y) dy \end{aligned}$$

which completes the proof. □

5. Fractional derivatives and semistable approximations

Generators of stable semigroups can also be written in terms of fractional derivatives. Fractional derivatives are almost as old as ordinary derivatives, see Miller and Ross [18] for a brief history. Fractional derivatives are most easily understood in terms of the Fourier transform

$$\hat{f}(k) = \int_{-\infty}^\infty e^{ikx} f(x) dx.$$

The positive fractional derivative $d^\alpha f(x)/dx^\alpha$ has Fourier transform $(-ik)^\alpha \hat{f}(k)$, extending the well known formula where α is a positive integer, see for example Samko, Kilbas and Marichev [21]. It is also useful to define the negative fractional derivative $d^\alpha f(x)/d(-x)^\alpha$ which has Fourier transform $(ik)^\alpha \hat{f}(k)$.

Fractional derivatives are intimately related to stable Lévy motions [1, 6]. If ν is stable with index $\alpha < 2$ then its Lévy measure is of the form (4.3) with $\theta_i(t) = \bar\theta_i$ constant for $i = 1, 2$. Since $y^{-\alpha} I(y > 0)$ has Fourier transform $\Gamma(1-\alpha)(-ik)^{\alpha-1}$, the convolution

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty f'(x-y) y^{-\alpha} dy$$

has Fourier transform $(-ik)^\alpha \hat f(k)$, and similarly for $1 < \alpha < 2$ we have

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(2-\alpha)} \int_0^\infty f''(x-y) y^{1-\alpha} dy.$$

If $g(x) = f(-x)$ then $\hat g(k) = -\hat f(-k)$, and then it follows by an easy computation that the negative fractional derivative $d^\alpha f(x)/d(-x)^\alpha = d^\alpha g(-x)/dx^\alpha$ so that

$$\frac{d^\alpha f(x)}{d(-x)^\alpha} = \frac{-1}{\Gamma(1-\alpha)} \int_0^\infty f'(x+y) y^{-\alpha} dy$$

for $0 < \alpha < 1$ and

$$\frac{d^\alpha f(x)}{d(-x)^\alpha} = \frac{1}{\Gamma(2-\alpha)} \int_0^\infty f''(x+y) y^{1-\alpha} dy$$

for $1 < \alpha < 2$. Then it follows from Propositions 4.2 and 4.3 that the generator $A = A_\nu$ of the convolution semigroup (T_ν) is of the form

$$A = \tilde a \frac{\partial}{\partial x} + qa \frac{\partial^\alpha}{\partial(-x)^\alpha} + pa \frac{\partial^\alpha}{\partial x^\alpha} \tag{5.1}$$

for some real constant a. If $0 < \alpha < 1$ then $a = -(\bar\theta_1 + \bar\theta_2)\Gamma(1-\alpha) < 0$, and if $1 < \alpha < 2$ then $a = -(\bar\theta_1 + \bar\theta_2)\Gamma(2-\alpha)/(1-\alpha) > 0$. In both cases $p = \bar\theta_1/(\bar\theta_1 + \bar\theta_2)$ and $q = 1 - p$.

Given ν infinitely divisible, let $\{Y(t)\}$ be the associated Lévy process, so that $Y(t)$ has distribution ν^t. The distribution functions $F(x,t) = P(Y(t) \le x)$ for $t > 0$ solve the abstract Cauchy problem

$$\frac{\partial F(x,t)}{\partial t} = AF(x,t); \quad F(x,0) = I(x \ge 0) \tag{5.2}$$

where $A = A_\nu$ is the generator given by (4.2), see Hille and Phillips [10]. If ν is stable with index $0 < \alpha < 1$ or $1 < \alpha < 2$ then (5.1) holds, and (5.2) is called the fractional diffusion equation. This equation is useful in physics and hydrology to describe anomalous diffusion, in which particles spread faster than the classical Brownian motion model predicts [3, 2, 25].

Now suppose that the situation of Theorem 3.1 pertains, so that ν_c is strictly $(c^{1/\alpha}, c)$ semistable law for each $c > 1$, and $\nu_c \Rightarrow \nu$ nondegenerate as $c \to 1$. Then ν is stable with index α, and if A_c, A are the generators of ν_c, ν respectively, then $A_c f(x) \to Af(x)$ as $c \to 1$ for all $x \in \mathbb{R}$, see [8]. This result can be used to obtain an approximation to the fractional diffusion equation, which may be useful for numerical solutions. Now we develop the simplest such approximation.

Lemma 5.1. *Let $0 < \alpha < 1$, $c > 1$, and $b = c^{1/\alpha}$. Define a measure ϕ_c concentrated on $\{b^n : n \in \mathbb{Z}\}$ by setting $\phi\{b^n\} = c^{-n}(c-1)$. Then ϕ_c is a Lévy measure, and if we let*

$$a_c = \int \left(\frac{y}{1+y^2} \right) \phi_c(dy)$$

then the infinitely divisible law ν_c with Lévy representation $[a_c, 0, \phi_c]$ is strictly (b, c) semistable.

Proof. For any $r > 0$ we have

$$\begin{aligned} \phi_c(r, \infty) &= \sum_{n : b^n > r} c^{-n}(c-1) = \sum_{n : c^{n/\alpha} > r} c^{-n}(c-1) \\ &= \sum_{n = [\alpha \log r / \log c] + 1}^{\infty} c^{-n}(c-1) = c^{-[\alpha \log r / \log c]} = r^{-\alpha} \theta_1^{(c)}(\log r) \end{aligned}$$

where $\theta_1^{(c)}(t) = c^{\frac{\alpha t}{\log c} - \left[\frac{\alpha t}{\log c}\right]}$ is periodic with period $\log b = \alpha^{-1} \log c$. Then ϕ_c is the Lévy measure of a unique (b, c) strictly semistable law ν_c with index $\alpha = \log c / \log b$. Furthermore, since $1 \le \theta_1^{(c)}(t) \le c$ for all $t > 0$ we also have $\theta_1^{(c)}(t) \to 1$ so that $\phi_c(r, \infty) \to r^{-\alpha}$ as $c \to 1$.

Now

$$a_c = \int \left(\frac{y}{1+y^2} \right) \phi_c(dy) = I_1 + I_2$$

where

$$I_1 = \int_{\varepsilon}^{\infty} \left(\frac{y}{1+y^2} \right) \phi_c(dy)$$

exists since the integrand is bounded and $\phi_c(\varepsilon,\infty)$ is finite. Also since $\alpha = \log c/\log b < 1$ we have $1 < c < b = c^{1/\alpha}$ so that

$$
\begin{aligned}
I_2 = \int_0^{\varepsilon} \left(\frac{y}{1+y^2}\right) \phi_c(dy) &\leq \int_0^{\varepsilon} y \ \phi_c(dy) \\
&= \sum_{b^n<\varepsilon} b^n c^{-n}(c-1) \\
&= (c-1) \sum_{n>-\log\varepsilon/\log b} (c/b)^n \\
&\leq (c/b)^{-\log\varepsilon/\log b} \frac{c-1}{1-(c/b)} \\
&= \varepsilon^{1-\alpha} \frac{c-1}{1-c^{1-1/\alpha}} \\
&\to \varepsilon^{1-\alpha} \left(\frac{\alpha}{1-\alpha}\right) \quad \text{as } c \downarrow 1.
\end{aligned}
\tag{5.3}
$$

Then a_c exists and we can define an infinitely divisible law ν_c with Lévy representation $[a_c, 0, \phi_c]$. In view of (4.1) the infinitely divisible law ν_c has characteristic function $\hat{\nu}_c(k) = e^{\psi_c(k)}$ where

$$\psi_c(k) = \int_{x\neq 0} \left(e^{ikx} - 1\right) \phi_c(dx)$$

and since $\phi_c(b^{-1}dx) = c \cdot \phi_c(dx)$ it is easy to check that ν_c is strictly (b, c) semistable. □

Lemma 5.2. *Let $0 < \alpha < 1$ and let ν_c be the semistable law from Lemma 5.1. Then $\nu_c \Rightarrow \nu$ as $c \downarrow 1$ where ν is stable with characteristic function $e^{\psi(k)}$ with $\psi(k) = -\Gamma(1-\alpha)(-ik)^{\alpha}$.*

Proof. Let ϕ be a Lévy measure on $(0,\infty)$ with $\phi(r,\infty) = r^{-\alpha}$ for all $r > 0$, and let ν be infinitely divisible with Lévy representation $[a, 0, \phi]$ where

$$a = \int \left(\frac{y}{1+y^2}\right) \phi(dy).$$

Since the integrand is $O(y)$ as $y \to 0$ and $O(y^{-1})$ as $y \to \infty$ it is easy to check that a exists. In view of (4.1) the infinitely divisible law ν has characteristic function $\hat{\nu}(k) = e^{\psi(k)}$ where

$$\psi(k) = \int_{x\neq 0} \left(e^{ikx} - 1\right) \phi(dx) = -\Gamma(1-\alpha)(-ik)^{\alpha}$$

by a straightforward computation, see for example [15] Lemma 7.3.7. Then it is easy to check that ν is strictly stable with index α.

We have already shown in the proof of Lemma 5.1 that $\phi_c \to \phi$ as $c \to 1$. Now we want to show that $a_c \to a$ as $c \to 1$. Proposition 1.2.19 in [15], the Portmanteau theorem for Lévy measures, shows that

$$\int_\varepsilon^\infty \left(\frac{y}{1+y^2}\right)\phi_c(dy) \to \int_\varepsilon^\infty \left(\frac{y}{1+y^2}\right)\phi(dy)$$

as $c \to 1$. Then using (5.3) and the fact that a exists, it follows easily that $a_c \to a$. Then Theorem 3.1.16 of [15], the standard convergence criteria for infinitely divisible laws, implies that $\nu_c \Rightarrow \nu$ as $c \to 1$. □

Theorem 5.3. *If* $0 < \alpha < 1$ *then*

$$\lim_{c\downarrow 1} \frac{1-c}{\Gamma(1-\alpha)} \sum_{n=-\infty}^{\infty} \big(f(x - c^{n/\alpha}) - f(x)\big)\, c^{-n} = \frac{d^\alpha f(x)}{dx^\alpha}. \tag{5.4}$$

Proof. Let ν_c, ν be as in Lemma 5.2. It follows from (5.1) with $\bar\theta_1 = 1$ and $\bar\theta_2 = 0$ that the generator $A = A_\nu$ is given by

$$Af(x) = -\Gamma(1-\alpha)\frac{d^\alpha f(x)}{dx^\alpha}.$$

Then

$$\begin{aligned} A_c f(x) &= \int_0^\infty (f(x-y) - f(x))\,\phi_c(dy) \\ &= \sum_{n=-\infty}^{\infty} (f(x-b^n) - f(x))\, c^{-n}(c-1) \\ &\to Af(x) = -\Gamma(1-\alpha)\frac{d^\alpha f(x)}{dx^\alpha} \end{aligned}$$

as $c \to 1$, and (5.4) follows easily. □

Corollary 5.4. *If* $0 < \alpha < 1$ *then*

$$\lim_{c\downarrow 1} \frac{1-c}{\Gamma(1-\alpha)} \sum_{n=-\infty}^{\infty} \big(f(x + c^{n/\alpha}) - f(x)\big)\, c^{-n} = \frac{d^\alpha f(x)}{d(-x)^\alpha}. \tag{5.5}$$

Proof. Since the negative fractional derivative $d^\alpha f(x)/d(-x)^\alpha = d^\alpha g(-x)/dx^\alpha$ where $g(x) = f(-x)$, the result follows immediately from Theorem 5.3. □

Remark 5.5. Since $\nu_c \Rightarrow \nu$ in Lemma 5.2, Theorem 3.1 implies that (3.1) also holds, so that the approximation of the fractional derivative is intimately related to the approximation of a Lévy motion by a semistable Lévy motion. We could also obtain the formula (5.5) by repeating the arguments above with $\phi\{-b^n\} = c^{-n}(c-1)$ so that ϕ_c is supported on $(-\infty, 0)$. Then we again have $\nu_c \Rightarrow \nu$ so that (5.5) also relates to the approximation of a Lévy motion by a sequence of semistable Lévy motions.

Remark 5.6. The following heuristic argument illustrates how the new fractional derivative approximation developed in this section can be viewed as an approximating sum for the integral defining the fractional derivative. Partition $(0,\infty)$ at the points $y_n = c^{-n}$ so that $\Delta y_n = y_{n-1} - y_n = c^{-n}(c-1)$. Then the left hand side of (5.4) is the Riemann sum approximation

$$\begin{aligned}
&\frac{-1}{\Gamma(1-\alpha)} \sum_{n=-\infty}^{\infty} \big(f(x - c^{n/\alpha}) - f(x)\big)\, c^{-n}(c-1) \\
&= \frac{-1}{\Gamma(1-\alpha)} \sum_{n=-\infty}^{\infty} \big(f(x - y_n^{-1/\alpha}) - f(x)\big)\, \Delta y_n \\
&\approx \frac{-1}{\Gamma(1-\alpha)} \int_0^\infty \big(f(x - y^{-1/\alpha}) - f(x)\big)\, dy \\
&= \frac{-1}{\Gamma(1-\alpha)} \int_0^\infty (f(x-y) - f(x))\, \alpha y^{-\alpha-1} dy \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty f'(x-y) y^{-\alpha} dy
\end{aligned}$$

which is an integral form of the fractional derivative $d^\alpha f(x)/dx^\alpha$.

Now we develop a similar approximation formula for the case $1 < \alpha < 2$.

Lemma 5.7. *Let $1 < \alpha < 2$, $c > 1$, and $b = c^{1/\alpha}$. Define a measure ϕ_c concentrated on $\{b^n : n \in \mathbb{Z}\}$ by setting $\phi\{b^n\} = c^{-n}(c-1)$. Then ϕ_c is a Lévy measure, and if we let*

$$a_c = \int \left(\frac{y}{1+y^2} - y \right) \phi_c(dy)$$

then the infinitely divisible law ν_c with Lévy representation $[a_c, 0, \phi_c]$ is strictly (b,c) semistable.

Proof. It follows exactly as in the proof of Lemma 5.1 that ϕ_c is the Lévy measure of a unique (b,c) strictly semistable law ν_c with index $\alpha = \log c/\log b$, and that $\phi_c(r,\infty) \to r^{-\alpha}$ as $c \to 1$.

Now

$$-a_c = \int \left(y - \frac{y}{1+y^2} \right) \phi_c(dy) = I_1 + I_2 + I_3$$

where

$$I_1 = \int_\varepsilon^M \left(y - \frac{y}{1+y^2} \right) \phi_c(dy)$$

exists since the integrand is bounded and $\phi_c(\varepsilon, M)$ is finite. Also since $\alpha = \log c/\log b > 1$ we have $1 < b = c^{1/\alpha} < c < b^3$ so that

$$\begin{aligned}
I_2 &= \int_0^\varepsilon \left(y - \frac{y}{1+y^2} \right) \phi_c(dy) \\
&= \int_0^\varepsilon \left(\frac{y^3}{1+y^2} \right) \phi_c(dy) \\
&\leq \int_0^\varepsilon y^3 \phi_c(dy) \\
&= \sum_{b^n < \varepsilon} b^{3n} c^{-n}(c-1) \\
&= (c-1) \sum_{n > -\log\varepsilon/\log b} (c/b^3)^n \\
&\leq (c/b^3)^{-\log\varepsilon/\log b} \frac{c-1}{1-(c/b^3)} \\
&= \varepsilon^{3-\alpha} \frac{c-1}{1-c^{1-3/\alpha}} \\
&\to \varepsilon^{3-\alpha} \left(\frac{\alpha}{3-\alpha} \right) \quad \text{as } c \downarrow 1.
\end{aligned} \tag{5.6}$$

Finally

$$\begin{aligned}
I_3 &= \int_M^\infty \left(y - \frac{y}{1+y^2} \right) \phi_c(dy) \\
&\leq \int_M^\infty y \, \phi_c(dy) \\
&= \sum_{b^n > M} b^n c^{-n}(c-1) \\
&= (c-1) \sum_{n > \log M/\log b} (b/c)^n \\
&\leq (b/c)^{\log M/\log b} \frac{c-1}{1-(b/c)} \\
&= M^{1-\alpha} \frac{c-1}{1-c^{1/\alpha-1}} \\
&\to M^{1-\alpha} \left(\frac{\alpha}{\alpha-1} \right) \quad \text{as } c \downarrow 1.
\end{aligned} \tag{5.7}$$

Then a_c exists and we can define an infinitely divisible law ν_c with Lévy representation $[a_c, 0, \phi_c]$. In view of (4.1) the infinitely divisible law ν_c has characteristic function $\hat{\nu}_c(k) = e^{\psi_c(k)}$ where

$$\psi_c(k) = \int_{x\neq 0} \left(e^{ikx} - 1 - ikx\right) \phi_c(dx)$$

and since $\phi_c(b^{-1}dx) = c \cdot \phi_c(dx)$ it is easy to check that ν_c is strictly (b, c) semistable. □

Lemma 5.8. *Let $1 < \alpha < 2$ and let ν_c be the semistable law from Lemma 5.7. Then $\nu_c \Rightarrow \nu$ as $c \downarrow 1$ where ν is stable with characteristic function $e^{\psi(k)}$ with $\psi(k) = (-ik)^\alpha \Gamma(2-\alpha)/(\alpha-1)$.*

Proof. Let ϕ be a Lévy measure on $(0, \infty)$ with $\phi(r, \infty) = r^{-\alpha}$ for all $r > 0$, and let ν be infinitely divisible with Lévy representation $[a, 0, \phi]$ where

$$a = \int \left(y - \frac{y}{1+y^2}\right) \phi(dy).$$

Since the integrand is $O(y^3)$ as $y \to 0$ and $O(y)$ as $y \to \infty$ it is easy to check that a exists. In view of (4.1) the infinitely divisible law ν has characteristic function $\hat{\nu}(k) = e^{\psi(k)}$ where

$$\psi(k) = \int_{x\neq 0} \left(e^{ikx} - 1 - ikx\right) \phi(dx) = (-ik)^\alpha \Gamma(2-\alpha)/(\alpha-1)$$

by a straightforward computation, see for example [15] Lemma 7.3.8. Then it is easy to check that ν is strictly stable with index α.

We have already shown in the proof of Lemma 5.7 that $\phi_c \to \phi$ as $c \to 1$. Now we want to show that $a_c \to a$ as $c \to 1$. Proposition 1.2.13 in [15], the Portmanteau theorem for finite measures, shows that

$$\int_\varepsilon^M \left(y - \frac{y}{1+y^2}\right) \phi_c(dy) \to \int_\varepsilon^\infty \left(y - \frac{y}{1+y^2}\right) \phi(dy)$$

as $c \to 1$. Then using (5.6) and (5.7) along with the fact that a exists, it follows easily that $a_c \to a$. Then Theorem 3.1.16 of [15], the standard convergence criteria for infinitely divisible laws, implies that $\nu_c \Rightarrow \nu$ as $c \to 1$. □

Theorem 5.9. *If $1 < \alpha < 2$ then*

(5.8)
$$\lim_{c\downarrow 1} \frac{\alpha-1}{\Gamma(2-\alpha)} \sum_{n=-\infty}^{\infty} \left(f(x - c^{n/\alpha}) - f(x) + c^{n/\alpha} f'(x)\right) c^{-n}(c-1) = \frac{d^\alpha f(x)}{dx^\alpha}.$$

Proof. Let ν_c, ν be as in Lemma 5.8. It follows from (5.1) that the generator $A = A_\nu$ is given by

$$Af(x) = \frac{\Gamma(2-\alpha)}{\alpha-1} \frac{d^\alpha f(x)}{dx^\alpha}.$$

Then

$$\begin{aligned} A_c f(x) &= \int_0^\infty \left(f(x-y) - f(x) + y f'(x) \right) \phi_c(dy) \\ &= \sum_{n=-\infty}^{\infty} \left(f(x-b^n) - f(x) + b^n f'(x) \right) c^{-n}(c-1) \\ &\to Af(x) = \frac{\Gamma(2-\alpha)}{\alpha-1} \frac{d^\alpha f(x)}{dx^\alpha} \end{aligned}$$

as $c \to 1$, and (5.8) follows. □

Corollary 5.10. *If* $1 < \alpha < 2$ *then*

(5.9)

$$\lim_{c \downarrow 1} \frac{\alpha-1}{\Gamma(2-\alpha)} \sum_{n=-\infty}^{\infty} \left(f(x+c^{n/\alpha}) - f(x) - c^{n/\alpha} f'(x) \right) c^{-n}(c-1) = \frac{d^\alpha f(x)}{d(-x)^\alpha}.$$

Proof. The proof is the same as Corollary 5.4 □

Remark 5.11. As in the case of Theorem 5.3, the convergence in (5.8) corresponds to the convergence (3.1) of semistable Lévy motions to a Lévy motion. We could also obtain (5.9) by starting with $\phi\{-b^n\} = c^{-n}(c-1)$ so that ϕ_c is supported on $(-\infty, 0)$, and then (3.1) again holds. In view of Proposition 4.1 we also have $cA_c f(x) = A_c(\delta_{c^{1/\alpha}} f)(c^{-1/\alpha} x)$, so that all of the fractional derivative approximations in this section have a nice scaling property. If A is the generator of a strictly stable law with index α, then $tAf(x) = A(\delta_{t^{1/\alpha}} f)(t^{-1/\alpha} x)$ for all $t > 0$. In other words, the approximations developed here have the same sort of scaling as the fractional derivative operator itself, but only along a geometric sequence of scales.

Remark 5.12. When A is the generator of a semistable semigroup, we call (5.2) the semistable diffusion equation. Solutions to the semistable diffusion equation are of the form $F(x,t) = P(Y(t) \le x)$ for $t > 0$ where $\{Y(t)\}$ is a semistable Lévy motion. The semistable diffusion equation extends the fractional diffusion equation for Lévy motion, and may be useful in physics as a more flexible model of anomalous diffusion. The fractional diffusion equation has recently been extended to several dimensions [14, 16]. Solutions to the multidimensional diffusion equation are vector stable Lévy motions, or more generally, operator stable Lévy motions. It should be possible to extend the results of this paper to multiple dimensions as well, using vector semistable or operator semistable Lévy motions. The basic theory of (operator) semistable random vectors can be found in [15].

6. Fractional differences

Another discrete approximation to the fractional derivative $d^\alpha f(x)/dx^\alpha$ is given by the fractional difference quotient $\Delta_h^\alpha f(x)/h^\alpha$ where

$$\Delta_h^\alpha f(x) = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} f(x - mh)$$

and

$$\binom{\alpha}{m} = \frac{(-1)^{m-1}\alpha\Gamma(m-\alpha)}{\Gamma(1-\alpha)\Gamma(m+1)}, \tag{6.1}$$

see for example [21]. To illuminate the connection with (4.2) we write the difference operator in integral form.

Lemma 6.1. *For any $h > 0$ and $0 < \alpha < 1$ we have*

$$h^{-\alpha}\Delta_h^\alpha f(x) = -\int (f(x-y) - f(x))\, \phi_h(dy) \tag{6.2}$$

where the measure ϕ_h is concentrated on $\{mh : m \geq 1\}$ with

$$\phi_h\{mh\} = (-1)^{m-1}\binom{\alpha}{m} h^{-\alpha}. \tag{6.3}$$

Proof. Note that

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{m} = (-1)^m \frac{(-\alpha)(-\alpha+1)\cdots(-\alpha+m-1)}{m!}$$

so that

$$(-1)^{m-1}\binom{\alpha}{m} > 0 \quad \text{for all } m \geq 0.$$

It is well known that

$$(1+z)^\alpha = \sum_{m=0}^{\infty}\binom{\alpha}{m} z^m \tag{6.4}$$

for any complex $|z| \leq 1$ and any $\alpha > 0$. Using (6.4) with $z = -1$ we get

$$\begin{aligned} \sum_{m=1}^{\infty}\phi_h\{mh\} &= -h^{-\alpha}\sum_{m=1}^{\infty}\binom{\alpha}{m}(-1)^m \\ &= -h^{-\alpha}\left(\sum_{m=0}^{\infty}\binom{\alpha}{m}(-1)^m - 1\right) \\ &= h^{-\alpha} \end{aligned} \tag{6.5}$$

and the result follows easily. □

Now we will construct an infinitely divisible law ν_h such that $A_h = -h^{-\alpha}\Delta_h^\alpha$ is the generator of the semigroup associated with ν_h. Then we will show that $\nu_h \Rightarrow \nu$ as $h \to 0$, where ν is stable with index α, and $A = -d^\alpha/dx^\alpha$ is the generator of the semigroup associated with ν.

Lemma 6.2. *For any $0 < \alpha < 1$ the measure ϕ_h defined by (6.3) is Lévy measure, and if we let*

$$a_h = \int \left(\frac{y}{1+y^2}\right) \phi_h(dy) \tag{6.6}$$

then there exists a unique infinitely divisible law ν_h with Lévy representation $[a_h, 0, \phi_h]$.

Proof. In view of (6.5), ϕ_h is a finite measure on $(0,\infty)$ and hence a Lévy measure. Furthermore, we have

$$\begin{aligned} a_h &= \int \left(\frac{y}{1+y^2}\right) \phi_h(dy) \\ &\le \frac{1}{2}\int \phi_h(dy) \\ &= \frac{1}{2}h^{-\alpha} \end{aligned} \tag{6.7}$$

so that a_h exists. Then the result follows from the Lévy representation theorem, see for example [15] Theorem 3.1.11. □

In view of (4.1) the infinitely divisible law ν_h has characteristic function $\hat{\nu}_h(k) = e^{\psi_h(k)}$ where

$$\psi_h(k) = \int_{x\neq 0} \left(e^{ikx} - 1\right) \phi(dx)$$

and the generator $A_h = A_{\nu_h}$ is

$$A_h f(x) = \int \left(f(x-y) - f(x)\right) \phi_h(dy)$$

so that $A_h = -h^{-\alpha}\Delta_h^\alpha$ by Lemma 6.1.

Theorem 6.3. *Let $0 < \alpha < 1$ and let ν_h be the infinitely divisible law defined in Lemma 6.2. Then $\nu_h \Rightarrow \nu$ as $h \downarrow 0$, where ν is stable with index α and characteristic function $\hat{\nu}(k) = e^{\psi(k)}$ where $\psi(k) = -(-ik)^\alpha$.*

Proof. Let ϕ be concentrated on $(0,\infty)$ with $\phi(r,\infty) = r^{-\alpha}/\Gamma(1-\alpha)$. Then it is easy to check that ϕ is a Lévy measure. Now let

$$a = \int \left(\frac{y}{1+y^2}\right) \phi(dy).$$

Since the integrand is $O(y)$ as $y \to 0$ and $O(y^{-1})$ as $y \to \infty$ it is easy to check that a exists. Then there exists a unique infinitely divisible law ν with Lévy representation $[a, 0, \phi]$. In view of (4.1) the infinitely divisible law ν has characteristic function $\hat{\nu}(k) = e^{\psi(k)}$ where

$$\psi(k) = \int_{x \neq 0} \left(e^{ikx} - 1\right) \phi(dx) = -(-ik)^{\alpha}$$

by a straightforward computation, see for example [15] Lemma 7.3.7. Then it is easy to check that ν is strictly stable with index α.

Now in order to show that $\nu_h \Rightarrow \nu$ we could show that $a_h \to a$ and $\phi_h \to \phi$, but in this case it is easier to verify directly that

$$\begin{aligned} \psi_h(k) &= \int_{x \neq 0} \left(e^{ikx} - 1\right) \phi_h(dx) \\ &= \sum_{m=1}^{\infty} e^{ikmh} (-1)^{m-1} \binom{\alpha}{m} h^{-\alpha} - \int_{x \neq 0} \phi_h(dx) \\ &= -\sum_{m=1}^{\infty} e^{ikmh} (-1)^{m} \binom{\alpha}{m} h^{-\alpha} - h^{\alpha} \\ &= -h^{-\alpha} \sum_{m=0}^{\infty} (-1)^{m} \binom{\alpha}{m} (e^{ikh})^{m} \\ &= -h^{-\alpha}(1 - e^{ikh})^{\alpha} \\ &\to -(-ik)^{\alpha} = \psi(k) \quad \text{as } h \to 0 \end{aligned} \tag{6.8}$$

using (6.5), (6.4) with $z = -e^{ikh}$, and the fact that $e^{ikh} = 1 + ikh + O(h^2)$ as $h \to 0$. Then $\hat{\nu}_h(k) \to \hat{\nu}(k)$ for all k, and hence $\nu_h \Rightarrow \nu$ by the Lévy continuity theorem. □

Since $\nu_h \Rightarrow \nu$, we also have $A_h f(x) \to Af(x)$ for all x as $h \to 0$, or in other words

$$\lim_{h \downarrow 0} \frac{\Delta_h^{\alpha} f(x)}{h^{\alpha}} = \frac{d^{\alpha} f(x)}{dx^{\alpha}}. \tag{6.9}$$

This result is well known, see for example [21] p.373. In addition, if $\{Y_h(t)\}$ is a Lévy process such that $Y_h(t)$ has distribution ν_h^t, and $\{Y(t)\}$ is a Lévy motion such that $Y(t)$ has distribution ν^t, then

$$\{Y_h(t)\} \stackrel{f.d.}{\Longrightarrow} \{Y(t)\} \quad \text{as } h \to 0 \tag{6.10}$$

by an argument similar to Theorem 3.1. In summary, the usual difference approximation to the fractional derivative corresponds to a Lévy process generator, and these processes converge to the Lévy motion connected with the fractional derivative. However, these difference operators do not have the nice scaling properties of the semistable generators.

As in Remark 5.6, fractional difference approximations may also be viewed as Riemann sum approximations to the integral defining the fractional derivative. This heuristic argument depends on the following lemma.

Lemma 6.4. *Let* $0 < \alpha < 1$ *and define*

$$g_h(mh) = (-1)^{m-1} \binom{\alpha}{m} h^{-\alpha-1}$$

for any $h > 0$ *and any positive integer* m*. Then for any* $h > 0$ *we have*

$$(mh)^{\alpha+1} g_h(mh) \to \frac{\alpha}{\Gamma(1-\alpha)}$$

as $m \to \infty$*, and furthermore, this convergence is uniform in* $h > 0$*.*

Proof. In view of (6.1) we have

$$(mh)^{\alpha+1} g_h(mh) = \frac{\alpha}{\Gamma(1-\alpha)} m^{\alpha+1} \frac{\Gamma(m-\alpha)}{\Gamma(m+1)}$$

which does not depend on $h > 0$, so it will suffice to show that

$$m^{\alpha+1} \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} \to 1 \quad \text{as } m \to \infty.$$

Using Stirling's formula $\Gamma(x+1) \sim \sqrt{2\pi x}\; x^x e^{-x}$ as $x \to \infty$, we have

$$\begin{aligned} \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} &\sim \frac{\sqrt{2\pi(m-1-\alpha)}\; (m-1-\alpha)^{(m-1-\alpha)} e^{-(m-1-\alpha)}}{\sqrt{2\pi m}\; m^m e^{-m}} \\ &= e^{\alpha+1} \sqrt{\frac{m-1-\alpha}{m}}\; \frac{(m-1-\alpha)^{(m-1-\alpha)}}{m^m} \end{aligned}$$

where $\sqrt{(m-1-\alpha)/m} \to 1$ and

$$m^{\alpha+1} \frac{(m-1-\alpha)^{(m-1-\alpha)}}{m^m} = \left(1 - \frac{\alpha+1}{m}\right)^m \left(\frac{m}{m-1-\alpha}\right)^{\alpha+1} \to e^{-\alpha-1}$$

as $m \to \infty$, which finishes the proof. □

Now take $y_m = mh$ so that $\Delta y_m = y_{m+1} - y_m = h$, and write

$$\begin{aligned} h^{-\alpha}\Delta_h^\alpha f(x) &= \sum_{m=0}^{\infty}(-1)^m \binom{\alpha}{m} f(x-mh)h^{-\alpha} \\ &= -\sum_{m=1}^{\infty} \left(f(x-mh) - f(x)\right) g_h(mh)h \\ &= -\sum_{m=1}^{\infty} \left(f(x-y_m) - f(x)\right) g_h(y_m)\Delta y_m \\ &\approx \int_0^\infty \left(f(x-y) - f(x)\right) g_h(y)dy \\ &\approx \frac{-1}{\Gamma(1-\alpha)} \int_0^\infty \left(f(x-y) - f(x)\right) \alpha y^{-\alpha-1}dy \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty f'(x-y)y^{-\alpha}dy \end{aligned}$$

which is an integral expression for $d^\alpha f(x)/dx^\alpha$.

For the sake of completeness, we conclude this section with an analysis of the fractional difference approximation in the case $1 < \alpha < 2$. Now we have

$$\binom{\alpha}{0} = 1, \ \binom{\alpha}{1} = \alpha, \ \text{and} \ (-1)^m \binom{\alpha}{m} > 0 \quad \text{for all } m \geq 2$$

and we define

$$\phi_h\{mh\} = (-1)^m \binom{\alpha}{m} h^{-\alpha} \quad \text{for } m \geq 2. \tag{6.11}$$

Lemma 6.5. *For any $1 < \alpha < 2$ the measure ϕ_h defined by (6.11) is Lévy measure, and if we let*

$$a_h = \int \left(\frac{y}{1+y^2} - y\right) \phi_h(dy) \tag{6.12}$$

then there exists a unique infinitely divisible law ν_h with Lévy representation $[a_h, 0, \phi_h]$.

Proof. Since

$$\begin{aligned} \int \phi_h(dy) &= \sum_{m=2}^{\infty}(-1)^m \binom{\alpha}{m} h^{-\alpha} \\ &= h^{-\alpha}\left(\sum_{m=0}^{\infty}(-1)^m \binom{\alpha}{m} - 1 + \alpha\right) \\ &= (\alpha-1)h^{-\alpha} \end{aligned} \tag{6.13}$$

then ϕ_h is a finite measure on $(0,\infty)$ and hence a Lévy measure. Taking derivatives in (6.4) we have

$$\alpha(1+z)^{\alpha-1} = \sum_{m=1}^{\infty} \binom{\alpha}{m} m z^{m-1} \tag{6.14}$$

for $\alpha > 1$ and $|z| \le 1$. Letting $z=-1$ yields

$$\sum_{m=1}^{\infty} m(-1)^m \binom{\alpha}{m} = 0.$$

Then we have

$$\begin{aligned} \int y \ \phi_h(dy) &= \sum_{m=2}^{\infty} (mh)(-1)^m \binom{\alpha}{m} h^{-\alpha} \\ &= h^{1-\alpha}\left(\sum_{m=1}^{\infty} m(-1)^m \binom{\alpha}{m} + \alpha\right) \\ &= \alpha h^{1-\alpha} \end{aligned} \tag{6.15}$$

and so

$$\begin{aligned} -a_h &= \int \left(\frac{y^3}{1+y^2}\right) \phi_h(dy) \\ &\le \int_0^1 \phi_h(dy) + \int_1^{\infty} y\phi_h(dy) \\ &\le (\alpha-1)h^{-\alpha} + \alpha h^{1-\alpha} \end{aligned}$$

so that a_h exists. Then the result follows from the Lévy representation theorem. □

In view of (4.1) the infinitely divisible law ν_h has characteristic function $\hat{\nu}_h(k) = e^{\psi_h(k)}$ where

$$\psi_h(k) = \int_{x\neq 0} \left(e^{ikx} - 1 - ikx\right) \phi_h(dx)$$

and in view of (6.13) and (6.15) the generator $A_h = A_{\nu_h}$ is

$$\begin{aligned}
A_h f(x) &= \int (f(x-y) - f(x) + yf'(x))\,\phi_h(dy) \\
&= \sum_{m=2}^{\infty} (-1)^m \binom{\alpha}{m} h^{-\alpha} f(x-mh) - (\alpha-1)h^{-\alpha} f(x) + \alpha h^{1-\alpha} f'(x) \\
&= h^{-\alpha} \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} h^{-\alpha} f(x-mh) - h^{-\alpha} f(x) + \alpha h^{-\alpha} f(x-h) \\
&\qquad - (\alpha-1)h^{-\alpha} f(x) + \alpha h^{1-\alpha} f'(x) \\
&= h^{-\alpha} \Delta_h^{\alpha} f(x) + \alpha h^{-\alpha} (f(x-h) - f(x) + hf'(x))
\end{aligned}$$

where the last term tends to zero as $h \to 0$ since $\alpha < 2$.

Theorem 6.6. *Let $1 < \alpha < 2$ and let ν_h be the infinitely divisible law defined in Lemma 6.5. Then $\nu_h \Rightarrow \nu$ as $h \downarrow 0$, where ν is stable with index α and characteristic function $\hat{\nu}(k) = e^{\psi(k)}$ where $\psi(k) = -(-ik)^{\alpha}$.*

Proof. Let ϕ be concentrated on $(0, \infty)$ with $\phi(r, \infty) = (\alpha - 1)r^{-\alpha}/\Gamma(2-\alpha)$. Then it is easy to check that ϕ is a Lévy measure. Now let

$$a = \int \left(\frac{y}{1+y^2} - y \right) \phi(dy).$$

Since the integrand is $O(y^3)$ as $y \to 0$ and $O(y)$ as $y \to \infty$ it is easy to check that a exists. Then there exists a unique infinitely divisible law ν with Lévy representation $[a, 0, \phi]$. In view of (4.1) the infinitely divisible law ν has characteristic function $\hat{\nu}(k) = e^{\psi(k)}$ where

$$\psi(k) = \int_{x \neq 0} \left(e^{ikx} - 1 - ikx \right) \phi(dx) = -(-ik)^{\alpha}$$

by a straightforward computation, see for example [15] Lemma 7.3.8. Then it is easy to check that ν is strictly stable with index α.

Write

$$\psi_h(k) = \int \left(e^{ikx} - 1 - ikx \right) \phi_h(dx) = I_1 + I_2 + I_3$$

where

$$\begin{aligned}
I_1 &= \sum_{m=2}^{\infty} e^{ikmh} (-1)^m \binom{\alpha}{m} h^{-\alpha} \\
&= h^{-\alpha} \left(-1 + \alpha e^{ikh} + \sum_{m=0}^{\infty} \binom{\alpha}{m} (-e^{ihk})^m \right) \qquad (6.16) \\
&= h^{-\alpha} \left(-1 + \alpha e^{ikh} + (1 - e^{ikh})^{\alpha} \right).
\end{aligned}$$

Using (6.13) we have

$$I_2 = -\int \phi_h(dx) = (1-\alpha)h^{-\alpha}$$

and similarly (6.15) yields

$$I_3 = -ik\int x\phi_h(dx) = -ik\alpha h^{1-\alpha}$$

so that

$$\psi_h(k) = h^{-\alpha}(1-e^{ikh})^\alpha + \alpha h^{-\alpha}\left(e^{ikh}-1-ikh\right) \to (-ik)^\alpha = \psi(k)$$

using $e^{ikh} = 1 + ikh + O(h^2)$ twice. Then $\hat\nu_h(k) \to \hat\nu(k)$ for all k, and hence $\nu_h \Rightarrow \nu$ by the Lévy continuity theorem. □

Since $\nu_h \Rightarrow \nu$, we also have $A_h f(x) \to Af(x)$ for all x as $h \to 0$. Since $\alpha h^{-\alpha}\left(f(x-h) - f(x) + hf'(x)\right) \to 0$ for all x it follows once again that (6.9) holds, as well as (6.10).

7. Conclusion

Semistable Lévy motions are the most general class of useful approximations for simple random walks. These limit results are in terms of stochastic compactness, with weak convergence along a geometric sequence of time scales. Suitably chosen sequences of semistable Lévy motions converge to a stable Lévy motion. Stable generators can be written in terms of fractional derivatives. The abstract Cauchy problem for a stable generator is called the fractional diffusion equation. This equation is useful in physics to model anomalous diffusion, where a cloud of particles spreads faster than the classical diffusion equation predicts. Semistable generators are discrete convolutions, and provide a new approximation to fractional derivatives. These discrete convolutions have power law weights on a geometric grid, and nicer scaling properties than the usual finite difference approximation. Semistable diffusion equations may also be useful in physics as a more flexible model for anomalous diffusion. Finite difference approximations to the fractional derivative are also generators of infinitely divisible semigroups, and the associated Lévy processes also converge to Lévy motions. These laws are similar to semistable laws, but they do not pertain to any limit theorem. Their generators are discrete convolutions, with asymptotically power law weights on an arithmetic grid, but they do not have the nice scaling properties of semistable generators. The stochastic processes and approximations in this paper have natural generalizations to multiple dimensions, which may prove a fruitful direction for further research.

References

[1] D. Benson, S. Wheatcraft and M. Meerschaert (2000) The fractional-order governing equation of Lévy motion. *Water Resources Research*, **36**, 1413–1424.

[2] D. Benson, S. Wheatcraft and M. Meerschaert (2000) Application of a fractional advection-dispersion equation. *Water Resources Research*, **36**, 1403–1412.

[3] D. Benson, R. Schumer, M. Meerschaert and S. Wheatcraft (2001) Fractional dispersion, Lévy motions, and the MADE tracer tests. *Transp. Porous Media* **42**, 211–240.

[4] J. Bertoin (1996) *Lévy processes*. Cambridge University Press.

[5] A. Blumen, G. Zumofen and J. Klafter (1989) Transport aspects in anomalous diffusion: Lévy walks. *Phys Rev. A* **40**, 3964–3973.

[6] A. Chaves (1998) A fractional diffusion equation to describe Lévy flights. *Phys. Lett. A* **239**, 13–16.

[7] W. Feller (1971) *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd Ed., Wiley, New York.

[8] W. Hazod and H.P. Scheffler (1993) The domains of partial attraction of Probabilities on Groups and on Vector Spaces. *J. Theoretical Probab.*, **6**, 175–186.

[9] Heyer, H. (1977) *Probability Measures on Locally Compact Groups*. Springer-Verlag, Berlin.

[10] E. Hille and R. Phillips (1957) Functional Analysis and Semi-Groups. *Colloquium Publications* **31**, American Mathematical Society, Providence, Rhode Island.

[11] J. Klafter, A. Blumen and M. Shlesinger (1987) Stochastic pathways to anomalous diffusion. *Phys. Rev. A* **35** 3081–3085.

[12] V. Kruglov (1972) On the extension of the class of stable distributions. *Theory Probab. Appl.* **17**, 685–694.

[13] P. Lévy (1954) *Théorie de l'Addition des Variables Aléatoires*. 2nd Ed., Gauthier, Paris.

[14] M. Meerschaert, D. Benson and B. Baeumer (1999) Multidimensional advection and fractional dispersion. *Phys. Rev. E* **59**, 5026–5028.

[15] M. Meerschaert and H.P. Scheffler (2001) *Limit Theorems for Sums of Independent Random Vectors*. Wiley, New York.

[16] M. Meerschaert, D. Benson and B. Baeumer (2001) Operator Lévy motion and multiscaling anomalous diffusion. *Phys. Rev. E* **63**, 1112–1117.

[17] D. Mejzler (1973) On a certain class of infinitely divisible distributions. *Israel J. Math.* **16**, 1–19.

[18] K. Miller and B. Ross (1993) *An Introduction to Fractional Calculus and Fractional Differential Equations*. Wiley, New York.

[19] R. Pillai (1971) Semi stable laws as limit distributions. *Ann. Math. Statist.* **42**, 780–783.

[20] K.I. Sato (1999) *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

[21] S. Samko, A. Kilbas and O. Marichev (1993) *Fractional Integrals and derivatives: Theory and Applications*. Gordon and Breach, London.

[22] R. Shimuzu (1970) On the domain of partial attraction of semi–stable distributions. *Ann. Inst. Statist. Math.* **22**, 245–255.

[23] M. Shlesinger, G. Zaslavsky and U. Frisch, Eds. (1995) Lévy flights and related topics in physics. Proceedings of the International Workshop held in Nice, June 27–30, 1994. *Lecture Notes in Physics*, **450**, Springer-Verlag, Berlin.

[24] E. Seneta (1976) Regularly Varying Functions. *Lecture Notes in Mathematics* **508**, Springer-Verlag, Berlin.

[25] V.V. Uchaikin and V.M. Zolotarev (1999) *Chance and stability. Stable distributions and their applications.* VSP, Utrecht.

Mark M. Meerschaert, Department of Mathematics, University of Nevada, Reno, USA

E-mail address: mcubed@unr.edu

Hans-Peter Scheffler, Fachbereich Mathematik, University of Dortmund, 44221 Dortmund, Germany

E-mail address: hps@math.uni-dortmund.de