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Statistics & Probability Letters 71 (2005) 15–22
STATISTICS & PROBABILITY LETTERS
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Limit theorems for continuous time random walks with slowly varying waiting times

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Received 14 February 2004
Available online 23 November 2004

Abstract

Continuous time random walks incorporate a random waiting time between random jumps. They are used in physics to model particle motion. When the time between particle jumps has a slowly varying probability tail, the resulting plume disperses at a slowly varying rate. The limiting stochastic process is useful for modeling ultraslow diffusion in physics.

MSC: primary 60F17; 60J60; 26A33

Keywords: Anomalous diffusion; Operator stable law; Continuous time random walk; Fractional derivative

1. Introduction

Continuous time walks, in which the waiting time between jumps is random, are a useful model for particle motion in physics (Metzler and Klafter, 2000; Montroll and Weiss, 1965; Scher and Lax, 1973). An interesting and previously unexplored case is where the waiting times between particle jumps have slowly varying probability tails. In this case, none of the moments of the

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[1]MMM was partially supported by NSF grants DES-9980484, DMS-0139927 and DMS-0417869.

0167-7152/$ - see front matter
doi:10.1016/j.spl.2004.10.030

waiting times exist, so the usual arguments based on stable domains of attraction (Becker-Kern et al., 2004, 2003; Meerschaert and Scheffler, 2004) do not apply. Instead we find that the time process (sums of IID waiting times) converges to an extremal process, since the largest summand dominates. If the simple random walk of particle jumps converges to some Lévy process $A(t)$, then the continuous time random walk will be shown to converge to a subordinated process $A(E(t))$ where $E(t)$ is the inverse or hitting time process of the extremal process. The continuous time random walk limit process $A(E(t))$ is useful in physics as a model for ultraslow diffusion (Chechkin et al., 2002, 2003), where a dispersing plume of particles spreads at a slowly varying rate. In the case of finite variance particle jumps, this limit process has a Laplace distribution at every time, with increments that are all defective Laplace distributed.

Let $J_1, J_2, \ldots$ be nonnegative independent and identically distributed (i.i.d.) random variables that model the waiting times between jumps of a particle. We set $T(0) = 0$ and $T(n) = \sum_{j=1}^{n} J_j$, the time of the nth jump. The particle jumps are given by i.i.d. random vectors $Y_1, Y_2, \ldots$ on $\mathbb{R}^d$ which are assumed independent of (J_i). Let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} Y_i$, the position of the particle after the nth jump. For $t \geqslant 0$ let

$$N_t = \max\{n \geqslant 0 : T(n) \leqslant t\} \tag{1.1}$$

the number of jumps up to time t and define

$$X(t) = S_{N_t} = \sum_{i=1}^{N_t} Y_i \tag{1.2}$$

the position of a particle at time t. The stochastic process $\{X(t)\}_{t \geqslant 0}$ is called a *continuous time random walk* (CTRW). In Section 2, we will establish the limiting behavior of the waiting time process N_t, and then in Section 3, we will apply a transfer theorem from Becker-Kern et al. (2003) to prove a limit theorem for the CTRW process $X(t)$. The limiting process represents the behavior of a random particle in the long-time limit.

2. Waiting time process

Let $L(t) > 0$ be a strictly increasing continuous function of $t \geqslant t_0$ with $L(t_0) = 1$. Assume L is slowly varying, so that $L(\lambda t)/L(t) \to 1$ as $t \to \infty$ for all $\lambda > 0$, and suppose that $P\{J_i > t\} = 1/L(t)$. For $t \geqslant 0$ let $T(t) = \sum_{j=1}^{[t]} J_j$. Then Theorem 2.1 in Kasahara (1986) (see also Watanabe (1980)) implies that

$$\{c^{-1}L(T(ct))\}_{t \geqslant 0} \overset{\text{f.d.}}{\Rightarrow} \{D(t)\}_{t \geqslant 0} \quad \text{as } c \to \infty, \tag{2.1}$$

where $\overset{\text{f.d.}}{\Rightarrow}$ denotes convergence in distribution of all finite dimensional marginal distributions, and the extremal process $\{D(t)\}$ has finite dimensional distributions given for any $0 \leqslant t_1 < \cdots < t_m$ and $0 \leqslant x_1 < \cdots < x_m$ by

$$P\{D(t_1) \leqslant x_1, \ldots, D(t_n) \leqslant x_n\} = \mathrm{e}^{-t_1/x_1}\mathrm{e}^{-(t_2-t_1)/x_2} \cdots \mathrm{e}^{-(t_n-t_{n-1})/x_n}. \tag{2.2}$$

The inverse or hitting time process of the extremal process $D(t)$ is defined by

$$E(t) = \inf\{x : D(x) > t\}. \tag{2.3}$$

Since $D(t)$ has strictly increasing sample paths (this follows from the representation of $D(t)$ in Eq. (2.1) of Kasahara (1986)) we have for any $0 \leqslant t_1 < \cdots < t_m$ and $x_1, \ldots, x_m \geqslant 0$ that

$$\{E(t_i) \leqslant x_i \text{ for } i = 1, \ldots, m\} = \{D(x_i) \geqslant t_i \text{ for } i = 1, \ldots, m\}. \tag{2.4}$$

Then it follows from (2.2) that for any $t > 0$ and $x \geqslant 0$ we have

$$P\{E(t) \leqslant x\} = P\{D(x) \geqslant t\} = 1 - P\{D(x) < t\} = 1 - \mathrm{e}^{-x/t}$$

so that $E(t)$ has an exponential distribution with mean t. Note that by (2.2) for any $c > 0$ we have $P\{D(ct_i) \leqslant cx_i \ \forall i\} = P\{D(t_i) \leqslant x_i \ \forall i\}$ so that

$$\begin{aligned} P\{D(ct_i) \leqslant x_i \ \forall i\} &= P\{D(ct_i) \leqslant cc^{-1}x_i \ \forall i\} \\ &= P\{D(t_i) \leqslant c^{-1}x_i \ \forall i\} \\ &= P\{cD(t_i) \leqslant x_i \ \forall i\} \end{aligned} \tag{2.5}$$

and hence $D(t)$ is self-similar with index $H = 1$. This means that $\{D(t)\}_{t \geqslant 0}$ is continuous in law with $\{D(ct)\}_{t \geqslant 0} \stackrel{\mathrm{d}}{=} \{c^H D(t)\}_{t \geqslant 0}$ for all $c > 0$, where $\stackrel{\mathrm{d}}{=}$ indicates equality of all finite dimensional distributions, see for example Embrechts and Maejima (2002) and Meerschaert and Scheffler (2001). Then we also have

$$\begin{aligned} P\{E(ct_i) \leqslant x_i \ \forall i\} &= P\{D(x_i) \geqslant ct_i \ \forall i\} \\ &= P\{c^{-1}D(x_i) \geqslant t_i \ \forall i\} \\ &= P\{D(c^{-1}x_i) \geqslant t_i \ \forall i\} \\ &= P\{E(t_i) \leqslant c^{-1}x_i \ \forall i\} \\ &= P\{cE(t_i) \leqslant x_i \ \forall i\} \end{aligned} \tag{2.6}$$

so that $E(t)$ is self-similar with the same index $H = 1$.

Theorem 2.1. *Under the assumptions at the beginning of this section, letting L^{-1} denote the inverse function, we have*

$$\{c^{-1}N_{L^{-1}(ct)}\}_{t \geqslant 0} \stackrel{\text{f.d.}}{\Rightarrow} \{E(t)\}_{t \geqslant 0} \quad \text{as } c \to \infty. \tag{2.7}$$

Proof. Fix $0 \leqslant t_1 < \cdots < t_m$ and $x_1, \ldots, x_m \geqslant 0$ and note that $\{N_t \geqslant x\} = \{T(\lceil x \rceil) \leqslant t\}$ where $\lceil x \rceil$ is the smallest integer greater than or equal to x. Then using (2.1) as $c \to \infty$

$$\begin{aligned} P\{c^{-1}N_{L^{-1}(ct_i)} < x_i \ \forall i\} &= P\{N_{L^{-1}(ct_i)} < cx_i \ \forall i\} \\ &= P\{T(\lceil cx_i \rceil) > L^{-1}(ct_i) \ \forall i\} \\ &= P\{c^{-1}L(T(\lceil cx_i \rceil)) > c^{-1}L(L^{-1}(ct_i)) \ \forall i\} \end{aligned}$$

$$\begin{aligned} &\to P\{D(x_i) > t_i \ \forall i\} \\ &= P\{D(x_i) \geqslant t_i \ \forall i\} \\ &= P\{E(t_i) \leqslant x_i \ \forall i\} \\ &= P\{E(t_i) < x_i \ \forall i\} \end{aligned} \tag{2.8}$$

since by (2.2) and (2.4) both $E(t)$ and $D(x)$ have a density. □

See Dwass (1964) for more information on the structure of the extremal process $D(t)$ in (2.1). The inverse extremal process $E(t)$ also has some interesting structure that we will now investigate. Recall that $E(t)$ has an exponential distribution with mean t for every $t > 0$. Assume $0 \leqslant x_1 \leqslant x_2$ and $0 < t_1 < t_2$. Then

$$\begin{aligned} P\{E(t_1) \leqslant x_1, E(t_2) \leqslant x_2\} &= P\{D(x_1) \geqslant t_1, D(x_2) \geqslant t_2\} \\ &= 1 - P\{D(x_1) < t_1 \text{ or } D(x_2) < t_2\} \\ &= 1 - (P\{D(x_1) < t_1\} + P\{D(x_2) < t_2\} - P\{D(x_1) < t_1, D(x_2) < t_2\}) \\ &= 1 - \mathrm{e}^{-x_1/t_1} - \mathrm{e}^{-x_2/t_2} + \mathrm{e}^{-x_1/t_1}\mathrm{e}^{-(x_2 - x_1)/t_2}. \end{aligned} \tag{2.9}$$

Now take U_1 exponential with rate $\lambda_1 = 1/t_1 - 1/t_2$, U_2 exponential with rate $\lambda_2 = 1/t_2$ and independent of U_1, and let $U = \min(U_1, U_2)$ so that U is exponential with rate $\lambda_1 + \lambda_2 = 1/t_1$.

Proposition 2.2. $(E(t_1), E(t_2)) \stackrel{\mathrm{d}}{=} (U, U_2)$.

Proof. Assume $0 \leqslant x_1 \leqslant x_2$ and let $A = \{U \leqslant x_1\}$ and $B = \{U_2 \leqslant x_2\}$. Then

$$\begin{aligned} P\{U \leqslant x_1, U_2 \leqslant x_2\} &= P(A \cap B) \\ &= P(A) + P(B) - P(A \cup B) \\ &= P(A) + P(B) - 1 + P(A^c \cap B^c), \end{aligned}$$

where $P(A) = 1 - \mathrm{e}^{-x_1/t_1}$, $P(B) = 1 - \mathrm{e}^{-x_2/t_2}$, and

$$\begin{aligned} P(A^c \cap B^c) &= P(U > x_1, U_2 > x_2) \\ &= P(U_1 > x_1, U_2 > x_1, U_2 > x_2) \\ &= \mathrm{e}^{-\lambda_1 x_1}\mathrm{e}^{-\lambda_2 x_2} \\ &= \mathrm{e}^{-x_1/t_1}\mathrm{e}^{-(x_2 - x_1)/t_2} \end{aligned}$$

and then it follows that

$$P\{U \leqslant x_1, U_2 \leqslant x_2\} = 1 - \mathrm{e}^{-x_1/t_1} - \mathrm{e}^{-x_2/t_2} + \mathrm{e}^{-x_1/t_1}\mathrm{e}^{-(x_2 - x_1)/t_2}$$

and comparing with (2.9) completes the proof. □

Proposition 2.3. *For $0 < t_1 < t_2$ the increment $E(t_2) - E(t_1)$ has a defective exponential distribution with $P\{E(t_2) - E(t_1) > x\} = (1 - t_1/t_2)\mathrm{e}^{-x/t_2}$ and $P\{E(t_2) - E(t_1) = 0\} = t_1/t_2$.*

Proof. Using Proposition 2.2 we have $P\{E(t_2) - E(t_1) = 0\} = P\{U_2 < U_1\} = \lambda_2/(\lambda_1 + \lambda_2) = t_1/t_2$. Also $P\{E(t_2) - E(t_1) > x\} = P\{U_2 - U > x\}$ and since $U = \min(U_1, U_2)$ this equals

$P\{U_2 - U_1 > x, U_2 > U_1\} = \mathrm{e}^{-x/t_2} \cdot \lambda_1/(\lambda_1 + \lambda_2)$ which finishes the proof since $\lambda_1/(\lambda_1+\lambda_2) = (1 - t_1/t_2)$. □

Example 2.4. Take $\rho > 0$ and let $L(t) = (\log t)^\rho$ for $t \geqslant t_0 = e$. Then $L^{-1}(t) = \exp(t^{1/\rho})$ and in this case, we can write (2.7) in the form

$$\{c^{-\rho} N_{s^c}\}_{s \geqslant 1} \overset{\text{f.d.}}{\Rightarrow} \{\tilde{E}(s)\}_{s \geqslant 1} \quad \text{as } c \to \infty, \tag{2.10}$$

where $\tilde{E}(s) = E(L(s))$. Note that $\tilde{E}(s)$ has an exponential distribution with mean $L(s) = (\log s)^\rho$. To see this, let $t = L(s)$ so that $t^{1/\rho} = \log s$ and $L^{-1}(ct) = s^{c^{1/\rho}}$ and hence (2.7) becomes

$$\{c^{-1} N_{s^{c^{1/\rho}}}\}_{s \geqslant 1} \overset{\text{f.d.}}{\Rightarrow} \{\tilde{E}(s)\}_{s \geqslant 1} \quad \text{as } c \to \infty.$$

The substitute c^ρ for c to get (2.10).

3. CTRW limit theorem

Assume that (Y_i) are i.i.d. $\mathbb{R}^{\mathrm{d}}$-valued random variables independent of (J_i) and assume that Y_1 belongs to the strict generalized domain of attraction of some full operator stable law ν, where full means that ν is not supported on any proper hyperplane of $\mathbb{R}^{\mathrm{d}}$, and strictly operator stable means that there exists a linear operator E on $\mathbb{R}^{\mathrm{d}}$ such that $\nu^t = t^E \nu$ for all $t > 0$, where ν^t denotes the t-fold convolution power of the infinitely divisible law ν, and $t^E \nu(\mathrm{d}x) = \nu(t^{-E}\,\mathrm{d}x)$ is the image measure of ν under the linear operator $t^E = \exp(E \log t)$. By Theorem 8.1.5 of Meerschaert and Scheffler (2001) there exists a function $B \in \mathrm{RV}(-\mathrm{E})$ (that is, $B(c)$ is invertible for all $c > 0$ and $B(\lambda c)B(c)^{-1} \to \lambda^{-E}$ as $c \to \infty$ for any $\lambda > 0$) such that

$$B(n) \sum_{i=1}^{n} Y_i \Rightarrow A \quad \text{as } n \to \infty, \tag{3.1}$$

where A has distribution ν. Note that by Theorem 7.2.1 of Meerschaert and Scheffler (2001) the real parts of the eigenvalues of E are greater than or equal to $1/2$.

Moreover, if we define the stochastic process $\{S(t)\}_{t \geqslant 0}$ by $S(t) = \sum_{i=1}^{[t]} Y_i$ it follows from Example 11.2.18 in Meerschaert and Scheffler (2001) that

$$\{B(c)S(ct)\}_{t \geqslant 0} \overset{\text{f.d.}}{\Rightarrow} \{A(t)\}_{t \geqslant 0} \quad \text{as } c \to \infty, \tag{3.2}$$

where $A(t)$ has stationary independent increments with $A(0) = 0$ almost surely and $P_{A(t)} = \nu^t = t^E \nu$ for all $t > 0$; P_X denoting the distribution of X. Then $A(t)$ is continuous in law, and it follows that

$$\{A(ct)\}_{t \geqslant 0} \overset{\text{f.d.}}{=} \{c^E A(t)\}_{t \geqslant 0} \tag{3.3}$$

so, by Definition 11.1.2 of Meerschaert and Scheffler (2001), $A(t)$ is operator selfsimilar with exponent E. The stochastic process $A(t)$ is called an *operator Lévy motion*. If the exponent $E = aI$ a constant multiple of the identity, then ν is a stable law with index $\alpha = 1/a$, and $A(t)$ is a classical d-dimensional Lévy motion. In the special case $a = 1/2$ the process $A(t)$ is a d-dimensional Brownian motion.

Theorem 3.1. *Let $X(t)$ be the continuous time random walk defined in* (1.2). *Under the assumptions on (Y_i) described at the beginning of this section and the assumptions on (J_i) described at the beginning of Section* 2 *we have*

$$\{B(c)X(L^{-1}(ct))\}_{t\geqslant 0} \overset{\text{f.d.}}{\Rightarrow} \{A(E(t))\}_{t\geqslant 0} \quad \textit{as } c \to \infty. \tag{3.4}$$

Proof. The proof is similar to Theorem 4.2 in Becker-Kern et al. (2003). Fix any $0<t_1<\cdots<t_m$. Let ρ_c be the distribution of $(c^{-1}N_{L^{-1}(ct_i)} : 1\leqslant i\leqslant m)$ and let ρ be the distribution of $(E(t_i) : 1\leqslant i\leqslant m)$. Then ρ_c, ρ are probability measures on $\mathbb{R}^m$ and it follows from Theorem 2.1 that $\rho_c \Rightarrow \rho$ as $c \to \infty$. Furthermore, for $x=(x_1,\ldots,x_m)\in\mathbb{R}^m_+$ let

$$\mu_c(x) = P_{(B(c)S(cx_i):1\leqslant i\leqslant m)},$$
$$\nu(x) = P_{(A(x_i):1\leqslant i\leqslant m)}.$$

Then $\mu_c(x), \nu(x)$ are probability measures on $(\mathbb{R}^{\mathrm{d}})^m$ and since $\{A(t)\}_{t\geqslant 0}$ as a Lévy process is stochastically continuous the mapping $x\mapsto\nu(x)$ is weakly continuous. Then in view of the independence of (J_i) and (Y_i) we obtain

$$\begin{aligned}
P_{(B(c)X(L^{-1}(ct_i))\ \forall i)} &= P_{(B(c)S(N_{L^{-1}(ct_i)})\ \forall i)} \\
&= P_{(B(c)S(c\,c^{-1}N_{L^{-1}(ct_i)})\ \forall i)} \\
&= \int P_{(B(c)S(cx_i)\ \forall i)}\, \mathrm{d}P_{(c^{-1}N_{L^{-1}(ct_i)}\ \forall i)}(x_1,\ldots,x_n) \\
&= \int \mu_c(x)\, \mathrm{d}\rho_c(x) \\
&\Rightarrow \int \nu(x)\, \mathrm{d}\rho(x) \\
&= \int P_{(A(x_i)\ \forall i)}\, \mathrm{d}P_{(E(t_i)\ \forall i)}(x_1,\ldots,x_n) \\
&= P_{(A(E(t_i))\ \forall i)}
\end{aligned} \tag{3.5}$$

as $c \to \infty$ by a transfer theorem, Proposition 4.1 in Becker-Kern et al. (2003), along with the fact that $\mu_c(x^{(c)}) \Rightarrow \nu(x)$ as $c\to\infty$ whenever $x^{(c)} \to x \in \mathbb{R}^m_+$, which was established during the proof of Theorem 4.2 in Becker-Kern et al. (2003). □

Remark 3.2. If the random particle jumps (Y_i) are one dimensional with a finite variance, then $A(t)$ is a scalar Brownian motion, and the CTRW limit $A(E(t))$ in (3.4) is a Laplace process, meaning that $A(E(t))$ has a Laplace or double-sided exponential distribution with mean zero and variance $\sigma^2 t$ for every $t>0$, where σ^2 is the variance of $A(1)$. This is due to the well known fact that an exponential scale mixture of a normal law has a Laplace distribution, see for example Proposition 2.2.1 in Kotz et al. (2001). In the special case $L(t) = (\log t)^\rho$ we can reparameterize as in Example 2.4 to get

$$\{B(c^\rho)S(N_{s^c})\}_{s\geqslant 1} \overset{\text{f.d.}}{\Rightarrow} \{A(\tilde{E}(s))\}_{s\geqslant 1} \quad \text{as } c \to \infty, \tag{3.6}$$

where the limit process $A(\tilde{E}(s))$ is Laplace distributed with mean zero and variance $\sigma^2(\log s)^\rho$ for each $s>0$. The limiting process $A(\tilde{E}(s))$ is related to a model for ultraslow diffusion (Chechkin et al., 2002, 2003) in which a diffusing cloud of particles spreads at a logarithmic rate. In ultraslow diffusion, the probability density of particle location solves a diffusion equation

$$\int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\beta h(x,t)p(\beta)\,\mathrm{d}\beta = \frac{\partial^2}{\partial x^2}h(x,t) \tag{3.7}$$

in which the usual first order time derivative is replaced by a fractional derivative of order β, spread over the interval $0<\beta<1$ according to the probability density $p(\beta)$, a so-called distributed order fractional derivative (Caputo, 2001). Here $(\mathrm{d}/\mathrm{d}t)^\beta h(x,t)$ is the Caputo fractional derivative, with Laplace transform $s^\beta \tilde{h}(x,s) - s^{\beta-1}h(x,0)$, where the Laplace transform $\tilde{h}(x,s) = \int_0^\infty \mathrm{e}^{-st}h(x,t)\,\mathrm{d}t$, see for example Caputo (1967). In Chechkin et al. (2003) a Tauberian theorem is used to show that the solution to (3.7) is asymptotic to the Laplace density of $A(\tilde{E}(s))$ as $s \to \infty$. That paper also connects the solution of (3.7) to a certain kind of CTRW scaling limit with slowly varying waiting times, but we have not been able to verify that the two limits are actually equal. If (Y_i) are random vectors with a finite covariance matrix then $A(t)$ is a vector Brownian motion and $A(E(t))$ is a multivariable Laplace process. Reparameterizing as before gives a vector analogue to the ultraslow diffusion process in Chechkin et al. (2002, 2003), which is apparently new. Note that whether or not we reparameterize, Theorem 3.1 implies that particles following a CTRW with slowly varying waiting times spread very slowly. In Eq. (3.4) this is seen by noting that the time scale on the left is very fast, so that it takes a very long time (inverse of a slowly varying function) for the plume to evolve.

Remark 3.3. Since $A(t)$ is a Lévy process independent of the subordinator $E(t)$, the increment $A(E(t_2)) - A(E(t_1)) \stackrel{\mathrm{d}}{=} A(E(t_2) - E(t_1))$ and then Proposition 2.3 can be used to determine the increments of this CTRW scaling limit. If $A(t)$ is a (multivariable) Brownian motion then the increments of $A(E(t))$ have a defective (multivariable) Laplace distribution with $P\{A(E(t_2)) - A(E(t_1)) = 0\} = t_1/t_2$ for $0<t_1<t_2$. In terms of the ultraslow diffusion process $A(\tilde{E}(s))$ with $\tilde{E}(s)$ from Example 2.4, this means that particles rest for random periods of time, and $(\log s_2/\log s_1)^\rho$ is the probability that the particle has not moved between times s_1 and s_2. Unlike the case of waiting times with power law probability tails considered in Becker-Kern et al. (2003, 2004) and Meerschaert and Scheffler (2004), when the waiting times have slowly varying tails, the scaling limit process retains the resting periods implicit in the CTRW model.

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