Available online at www.sciencedirect.com
SCIENCE DIRECT®
Stochastic Processes and their Applications 116 (2006) 1215–1235
stochastic processes and their applications
www.elsevier.com/locate/spa

Stochastic model for ultraslow diffusion

Mark M. Meerschaert[a,*], Hans-Peter Scheffler[b]

[a] *Department of Mathematics and Statistics, University of Otago, New Zealand*
[b] *Department of Mathematics, University of Siegen, Germany*

Received 2 September 2004; received in revised form 16 December 2005; accepted 18 January 2006
Available online 10 February 2006

Abstract

Ultraslow diffusion is a physical model in which a plume of diffusing particles spreads at a logarithmic rate. Governing partial differential equations for ultraslow diffusion involve fractional time derivatives whose order is distributed over the interval from zero to one. This paper develops the stochastic foundations for ultraslow diffusion based on random walks with a random waiting time between jumps whose probability tail falls off at a logarithmic rate. Scaling limits of these random walks are subordinated random processes whose density functions solve the ultraslow diffusion equation. Along the way, we also show that the density function of any stable subordinator solves an integral equation (5.15) that can be used to efficiently compute this function.

MSC: primary 60G50, 60F17; secondary 60H30, 82C31

Keywords: Continuous time random walk; Slowly varying tails; Anomalous diffusion; Stable subordinator

1. Introduction

The classical diffusion equation $\partial c/\partial t = \partial^2 c/\partial x^2$ governs the scaling limit of a random walk where i.i.d. particle jumps have zero mean and finite variance. The probability density $c(x, t)$ of the Brownian motion scaling limit $B(t)$ solves the diffusion equation, and represents the relative concentration of a cloud of diffusing particles. Self-similarity $B(ct) \stackrel{d}{=} c^{1/2} B(t)$ implies that particles spread at the rate $t^{1/2}$ in this classical model. In many practical applications the diffusion is anomalous: the spreading rate is slower (subdiffusion) or faster (superdiffusion) than

* Corresponding address: Department of Mathematics and Statistics, University of Otago, P.O. Box 56, Dunedin 9001, New Zealand. Tel.:+64 3 479 7889; fax: +64 3 479 8427.
E-mail addresses: mcubed@maths.otago.ac.nz (M.M. Meerschaert), peter.scheffler@mac.com (H.-P. Scheffler).
URL: http://www.maths.otago.ac.nz/~mcubed/ (M.M. Meerschaert).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2006.01.006

the classical model predicts, and/or the plume shape is non-Gaussian. Anomalous superdiffusion can be modeled using infinite variance particle jumps that lead to space-fractional derivatives in the governing partial differential equation [5,10,26,27]. Anomalous subdiffusion can be modeled using i.i.d. infinite mean waiting times between particle jumps, leading to a fractional time derivative in the governing partial differential equation [30,38,44,51]. Continuous time random walks (CTRW) with i.i.d. waiting times between i.i.d. particle jumps were introduced in [33, 42]. Some recent surveys of their wide application in physics and connections with fractional governing equations are given in [19,24,32,49].

Ultraslow subdiffusion occurs when the spreading rate of a plume is logarithmic. Several examples from polymer physics, particles in a quenched force field, random walks in random media, and nonlinear dynamics are given in [20,15,36,43,46]. Recently a connection has been established between ultraslow kinetics and distributed-order time-fractional derivatives in the diffusion equation [11–13,48]. In this model, the first time derivative in the classical diffusion equation is replaced by a fractional derivative of order $0 < \beta < 1$ as in the usual subdiffusive model, and then the order β of the fractional time derivative is randomized according to some probability density $p(\beta)$ on $0 < \beta < 1$. When β is fixed and nonrandom, the relevant CTRW model has waiting times in the domain of attraction of a β-stable subordinator, and CTRW scaling limits involve subordination to the inverse stable subordinator [30,29]. Randomizing β leads to waiting times with a slowly varying probability tail. Limit theorems for these random walks were developed in [31] using nonlinear scaling, the usual approach for slowly varying tails [16,23,50].

In this paper, using a triangular array approach instead of the nonlinear scaling used in [31], we give a more detailed description of possible scaling limits together with asymptotic behavior of moments. Furthermore we show that our approach actually gives a stochastic solution to the distributed-order time-fractional diffusion equations and we provide explicit formulas for the solutions of those equations. Those solutions are density functions of subordinated stochastic process, where the subordinator is the inverse of the limit process of the triangular array that governs waiting times between particle jumps. We also show that, complementary to results in [14], a renewal process in which the waiting time between jumps has a slowly varying probability tail can be analyzed in much more detail. These results may be of independent interest. Finally we note that the general stochastic solutions to distributed-order time-fractional diffusion equations that we develop here may be useful in other contexts [47].

This paper is organized as follows. In Section 2 we define a generalization of the classical continuous time random walk (CTRW) model, using a triangular array of waiting times. In Section 3 a special triangular array with slowly varying tails is considered and the limiting Lévy process together with its hitting time process is analyzed. These results are then used in Section 4 to derive a limit theorem for generalized CTRWs with slowly varying waiting times and jumps in some generalized domain of attraction, and we derive various properties of the limiting process. Finally in Section 5 we show that, under certain technical conditions, the density of this limiting process solves a variant of the distributed order time-fractional diffusion equation considered in [11,12]. Along the way, we also show that the density function of any stable subordinator solves an integral equation (5.15) that can be used to efficiently compute this function.

2. Generalized CTRW

Given any scale $c \geq 1$, let $J_1^{(c)}, J_2^{(c)}, \ldots$ be nonnegative and independent and identically distributed (i.i.d.) random variables, modelling the waiting times between particle jumps at scale c. Let

$$T^{(c)}(0) = 0 \quad \text{and} \quad T^{(c)}(t) = \sum_{i=1}^{[t]} J_i^{(c)}, \tag{2.1}$$

so that $T^{(c)}(n)$ is the time of the nth jump at scale c. Let

$$N_t^{(c)} = \max\{n \geq 0 : T^{(c)}(n) \leq t\} \tag{2.2}$$

be the number of jumps by time $t \geq 0$ at scale c.

To model the particle jumps let $Y_1, Y_2, \ldots$ be i.i.d. $\mathbb{R}^d$-valued random vectors. Let $S(0) = 0$ and $S(t) = \sum_{i=1}^{[t]} Y_i$, so that $S(n)$ is the position of the particle after n jumps at scale $c = 1$. We assume that Y_1 belongs to the strict generalized domain of attraction of some full operator stable law with exponent E. This means that for some linear operators L_n we have $L_n S(n) \Rightarrow A$, where the distribution of the limit A is not supported on any lower dimensional hyperplane, see [28] Definition 3.3.24. In this case, the limit distribution is called operator stable, since there exists at least one linear operator E called an exponent of A such that if $A_1, \ldots, A_n$ are i.i.d. copies of A then $n^{-E}(A_1 + \cdots + A_n)$ is identically distributed with A for each n. Operator stable laws and their exponents are characterized in [28] Section 7.2, while generalized domains of attraction are described in [28] Chapter 8. Then there exists a norming function $B \in \mathrm{RV}(-E)$, meaning that $B(c) \in \mathrm{GL}(\mathbb{R}^d)$ for all $c > 0$ and $B(\lambda c)B(c)^{-1} \to \lambda^{-E}$ as $c \to \infty$ for any $\lambda > 0$, such that

$$\{B(c)S(ct)\}_{t\geq 0} \overset{f.d.}{\Longrightarrow} \{A(t)\}_{t\geq 0} \quad \text{as } c \to \infty \tag{2.3}$$

where $\{A(t)\}_{t\geq 0}$ is an operator Lévy motion with $A(t) \overset{d}{=} t^E A(1)$. Here $\overset{f.d.}{\Longrightarrow}$ denotes convergence in distribution of all finite dimensional marginals. See [28], Example 11.2.18 for details.

At scale $c \geq 1$ the jumps are given by $B(c)Y_i$ and hence $B(c)S(n)$ is the position of a particle after n jumps at scale c. Therefore

$$X^{(c)}(t) = B(c)S(N_t^{(c)}) \tag{2.4}$$

describes the position of a particle at time $t \geq 0$ and scale c. We call $\{X^{(c)}(t)\}_{t\geq 0}$ a *generalized continuous time random walk.*

3. The time process

In this section we construct and analyze a class of specific triangular arrays $\{J_i^{(c)} : i \geq 1, c \geq 1\}$ which corresponds to waiting times with slowly varying tails. It is shown that the corresponding partial sum processes $\{T^{(c)}(t)\}_{t\geq 0}$ defined by (2.1) converge to a class of Lévy processes complementing β-stable subordinators to the limiting case $\beta = 0$.

Our approach gives a much larger class of possible limiting processes than the nonlinear scaling of a random walk with slowly varying tails considered in [14,23,50]. There only one process, the so-called extremal process, can appear. See [16] for details on extremal processes. Our approach decomposes the case $\beta = 0$ of slowly varying tails into a family of different processes described by an additional parameter $\alpha > 0$, where any positive α is possible.

Stimulated by [11], our approach is based on the following idea. Given a measurable nonnegative function $p :]0, 1[\to \mathbb{R}_+$ with $0 < \int_0^1 p(\beta) d\beta < \infty$ and some constant $C > 0$

let

$$L(t) = C \int_0^1 t^{-\beta} p(\beta) \mathrm{d}\beta \tag{3.1}$$

for $t > 0$. In the following we always assume that the function p is defined on $\mathbb{R}_+$ but vanishes outside $]0, 1[$. Observe that L is decreasing and continuous. Moreover L is a mixture of the tail functions $Ct^{-\beta}$ with respect to $p(\beta)$. The following lemma describes the behavior of L near infinity in terms of regular variation of p. Recall that a function R is regularly varying at infinity with exponent $\gamma \in \mathbb{R}$, if R is measurable, eventually positive and $R(\lambda t)/R(t) \to \lambda^\gamma$ as $t \to \infty$ for any $\lambda > 0$. We write $R \in \mathrm{RV}_\infty(\gamma)$ in this case. Similarly, R is called regularly varying at zero with exponent $\gamma \in \mathbb{R}$, if R is measurable, positive in some neighborhood $(0, t_0)$ of the origin and $R(\lambda t)/R(t) \to \lambda^\gamma$ as $t \to 0$. We write $R \in \mathrm{RV}_0(\gamma)$ in this case. Note that $R(t) \in \mathrm{RV}_0(\gamma)$ if and only if $R(1/t) \in \mathrm{RV}_\infty(-\gamma)$.

Lemma 3.1. *For $\alpha > 0$ let $p \in \mathrm{RV}_0(\alpha - 1)$ and define $L(t)$ by (3.1). Then there exists a function $L^* \in \mathrm{RV}_\infty(0)$ such that*

$$L(t) = (\log t)^{-\alpha} L^*(\log t). \tag{3.2}$$

Especially $L(t) = R(\log t)$ for some $R \in \mathrm{RV}_\infty(-\alpha)$ and $L \in \mathrm{RV}_\infty(0)$, so L is slowly varying at infinity. Conversely, if for L defined by (3.1) we have $L(t) = R(\log t)$ for some $R \in \mathrm{RV}_\infty(-\alpha)$ and $\alpha > 0$, then $p \in \mathrm{RV}_0(\alpha - 1)$.

Proof. First note that since $p \in \mathrm{RV}_0(\alpha - 1)$ with $\alpha > 0$, we have for any $\delta > 0$ there exists a $\beta_0 > 0$ and some constant K such that $p(\beta) \le K\beta^{\alpha-1-\delta}$ for all $0 < \beta \le \beta_0$ (see, e.g., [45] p. 18). Hence $\int_0^1 p(\beta)\mathrm{d}\beta$ is finite and positive. Moreover

$$L(t) = C \int_0^1 \mathrm{e}^{-\beta \log t} p(\beta) \mathrm{d}\beta = C\tilde{p}(\log t)$$

where $\tilde{p}(s) = \int_0^1 \mathrm{e}^{-s\beta} p(\beta)\mathrm{d}\beta$ denotes the Laplace transform of a function p with $\mathrm{supp}(p) \subset [0, 1]$. Since p vanishes outside the interval $[0, 1]$, it is ultimately monotone in the sense of Feller [17], p. 446. Then, since $p \in \mathrm{RV}_0(\alpha - 1)$ by Theorem 4 on p. 446 of [17] we know $\tilde{p} \in \mathrm{RV}_\infty(-\alpha)$, so $\tilde{p}(s) = s^{-\alpha} L^*(s)$ for some $L^* \in \mathrm{RV}_\infty(0)$. Hence (3.2) holds.

Conversely, if $L(t) = R(\log t)$ for some $R \in \mathrm{RV}_\infty(-\alpha)$ and some $\alpha > 0$, since $L(t) = C\tilde{p}(\log t)$, we have $\tilde{p}(u) = C^{-1} R(u)$. Using Theorem 4 on p. 446 of [17] again, we conclude $p \in \mathrm{RV}_0(\alpha - 1)$ and the proof is complete. □

We now construct a triangular array $\{J_i^{(c)} : i \ge 1, c \ge 1\}$ with i.i.d. rows $J_1^{(c)}, J_2^{(c)}, \ldots$ of nonnegative random variables. In the following we assume that $p \in \mathrm{RV}_0(\alpha - 1)$ for some $\alpha > 0$ is supported in $[0, 1]$. Then we can take $C^{-1} = C^{-1}(p) = \int_0^1 p(\beta)\mathrm{d}\beta$ is finite and positive, so Cp is a probability density. We will assume without loss of generality, that $C = 1$ so p is a probability density on $[0, 1]$. Note that by Lemma 3.1 the function $L(t) = \int_0^1 t^{-\beta} p(\beta)\, \mathrm{d}\beta$ is in $\mathrm{RV}_\infty(0)$ with $L(t) = (\log t)^{-\alpha} L^*(\log t)$ for $t > 1$. We do need an additional integrability condition on $p(\beta)$ for $\beta \to 1$. This condition does not change the asymptotic behavior of $L(t)$ near infinity, but is necessary for our analysis. We assume that p also fulfills

$$\int_0^1 \frac{p(\beta)}{1 - \beta} \mathrm{d}\beta < \infty. \tag{3.3}$$

Note that (3.3) trivially holds true, if p vanishes in some open neighborhood of one.

Now let $B_1, B_2, \ldots$ be i.i.d. with density p. Given any scale $c \geq 1$ let $J_1^{(c)}, J_2^{(c)}, \ldots$ be nonnegative i.i.d. random variables such that for any $0 < \beta < 1$ we have

$$P\{J_i^{(c)} > u | B_i = \beta\} = \begin{cases} 1 & 0 \leq u < c^{-1/\beta} \\ c^{-1} u^{-\beta} & u \geq c^{-1/\beta}. \end{cases} \tag{3.4}$$

Then the density $\psi_c(u|\beta)$ of $J_i^{(c)}$ given $B_i = \beta$ is

$$\psi_c(u|\beta) = \begin{cases} 0 & 0 \leq u < c^{-1/\beta} \\ c^{-1} \beta u^{-\beta-1} & u \geq c^{-1/\beta}. \end{cases} \tag{3.5}$$

Remark 3.2. If we define for $0 < \beta < 1$

$$P\{J_1 > t | B_1 = \beta\} = \begin{cases} 1 & 0 \leq t < 1 \\ t^{-\beta} & t \geq 1 \end{cases}$$

we get by letting $u = c^{-1/\beta} t$ that

$$P\{c^{-1/\beta} J_1 > u | B_1 = \beta\} = \begin{cases} 1 & 0 \leq u < c^{-1/\beta} \\ c^{-1} u^{-\beta} & u \geq c^{-1/\beta} \end{cases}$$

so conditionally on $B_1 = \beta$ we have $J_1^{(c)} \overset{d}{=} c^{-1/\beta} J_1$. Moreover, for $t \geq 1$

$$P\{J_1 > t\} = \int_0^1 t^{-\beta} p(\beta) \mathrm{d}\beta$$

so by Lemma 3.1 J_1 has a slowly varying tail.

Remark 3.3. An application of ultraslow diffusion to disordered systems in [13] illustrates the physical meaning of the generalized CTRW model described here. The parameter $\beta = B_i$ relates to the shallowness of a potential well from which a particle must escape, and the waiting time J_i until escape from the well has a probability tail that falls off like a power law with exponent β. The probability density $p(\beta)$ governs the depth distribution for potential wells, and the index α indicates the scarcity of very deep wells. It should be noted, however, that the trapping in the ultraslow CTRW model is somewhat different than the model in Sinai [46] (random walk in a random environment), since in the Sinai model the deep traps are at a fixed location in space for any realization of the random environment. Hence the localization phenomena seen in these models [18,36] do not occur in the CTRW formulation [8].

Theorem 3.4. *Given $p \in \mathrm{RV}_0(\alpha - 1)$ for some $\alpha > 0$ as above and define the triangular array $\{J_i^{(c)} : 1 \leq i \leq [ct], c \geq 1\}$ by (3.4). Assume that (3.3) holds. Then for the partial sum process $\{T^{(c)}(t)\}_{t \geq 0}$ defined by (2.1) we have*

$$\{T^{(c)}(ct)\}_{t \geq 0} \overset{f.d.}{\Longrightarrow} \{D(t)\}_{t \geq 0} \quad \text{as } c \to \infty, \tag{3.6}$$

where $\{D(t)\}_{t \geq 0}$ is a subordinator such that $D(1)$ has Lévy–Khinchin representation $[0, 0, \phi]$ and the Lévy measure ϕ assigns to intervals (u, ∞) for any $u > 0$ the measure

$$\phi(u, \infty) = L(u) \tag{3.7}$$

where L is given by (3.1) with $C = 1$.

Proof. Since $T^{(c)}(ct) = \sum_{i=1}^{[ct]} J_i^{(c)}$ is a sum of i.i.d. random variables, the convergence of all finite dimensional marginals follows from the convergence for one fixed $t > 0$ by considering increments. See [28], Example 11.2.18 for details. Fix any $t > 0$ and observe that $\{J_i^{(c)} : 1 \le i \le [ct], c \ge 1\}$ is an infinitesimal triangular array. By standard convergence criteria for triangular arrays, see e.g. [28], Theorem 3.2.2, we know that

$$T^{(c)}(ct) - a_{[ct]} \Rightarrow D(t) \quad \text{as } c \to \infty \tag{3.8}$$

where

$$a_{[ct]} = [ct] \int_0^R x \, \mathrm{d}P_{J_1^{(c)}}(x) \tag{3.9}$$

for some $R > 0$ and $D(t)$ has Lévy–Khinchin representation $[0, 0, t \cdot \phi]$ if

$$[ct] \cdot P_{J_1^{(c)}} \to t \cdot \phi \quad \text{as } c \to \infty \tag{3.10}$$

and

$$\lim_{\varepsilon \downarrow 0} \limsup_{c \to \infty} \; [ct] \int_0^\varepsilon u^2 \, \mathrm{d}P_{J_1^{(c)}}(u) = 0. \tag{3.11}$$

Fix any $u > 0$. Then, for all large c we obtain from (3.4) that

$$\begin{aligned} [ct] P\{J_1^{(c)} > u\} &= [ct] \int_0^1 P\{J_1^{(c)} > u | B_1 = \beta\} p(\beta) \mathrm{d}\beta \\ &= \frac{[ct]}{c} \int_0^1 u^{-\beta} p(\beta) \mathrm{d}\beta \\ &\to t \cdot L(u) = t \cdot \phi(u, \infty) \end{aligned}$$

as $c \to \infty$. Hence (3.10) holds and by Lemma 3.1 the Lévy measure ϕ has the form (3.7). Moreover, for the Gaussian part we compute using (3.5) that

$$\begin{aligned} [ct] \int_0^\varepsilon u^2 \, \mathrm{d}P_{J_1^{(c)}}(u) &= [ct] \int_0^\varepsilon u^2 \int_0^1 \psi_c(u|\beta) p(\beta) \mathrm{d}\beta \, \mathrm{d}u \\ &= [ct] \int_0^1 \int_0^\varepsilon u^2 \psi_c(u|\beta) \mathrm{d}u \; p(\beta) \mathrm{d}\beta \\ &= [ct] \int_0^1 \int_{c^{-1/\beta}}^\varepsilon u^2 \beta c^{-1} u^{-\beta-1} \, \mathrm{d}u \; p(\beta) \mathrm{d}\beta \\ &= \frac{[ct]}{c} \int_0^1 \frac{\beta}{2-\beta} (\varepsilon^{2-\beta} - c^{1-2/\beta}) p(\beta) \mathrm{d}\beta \\ &= \frac{[ct]}{c} \int_0^1 \varepsilon^{2-\beta} \frac{\beta}{2-\beta} p(\beta) \mathrm{d}\beta - \frac{[ct]}{c} \int_0^1 c^{1-2/\beta} \frac{\beta}{2-\beta} p(\beta) \mathrm{d}\beta. \end{aligned}$$

Observe that $\beta/(2-\beta) \le 1$ and $1 - 2/\beta \le -1$. Then dominated convergence yields

$$\limsup_{c \to \infty} [ct] \int_0^\varepsilon u^2 \mathrm{d}P_{J_1^{(c)}}(u) = t \int_0^1 \varepsilon^{2-\beta} \frac{\beta}{2-\beta} p(\beta) \mathrm{d}\beta \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence (3.11) holds and therefore (3.8) holds true. Note that since ϕ has a Lebesgue density any $R > 0$ in (3.9) is possible. We show now that the shifts $a_{[ct]}$ can be made arbitrarily small for all large c, by choosing $R > 0$ small enough. This implies that we can choose $a_{[ct]} = 0$ for all $c \ge 1$

and then (3.8) holds without $a_{[ct]}$. For $R > 0$ we get from (3.5) that

$$\begin{aligned}
[ct] \int_0^R x \, \mathrm{d}P_{J_1^{(c)}}(x) &= [ct] \int_0^R x \int_0^1 \psi_c(x|\beta) p(\beta) \mathrm{d}\beta \, \mathrm{d}x \\
&= \frac{[ct]}{c} \int_0^1 \int_{c^{-1/\beta}}^R x^{-\beta} \, \mathrm{d}x \, \beta p(\beta) \mathrm{d}\beta \\
&= \frac{[ct]}{c} \int_0^1 \frac{\beta}{1-\beta} R^{1-\beta} p(\beta) \mathrm{d}\beta - \frac{[ct]}{c} \int_0^1 c^{1-1/\beta} \frac{\beta}{1-\beta} p(\beta) \mathrm{d}\beta \\
&= I(c, R) - J(c).
\end{aligned}$$

Now, since $1 - 1/\beta < 0$ we get from (3.3) and dominated convergence that $J(c) \to 0$ as $c \to \infty$. Moreover, by the same argument we see that $I(c, R) \to 0$ as $R \to 0$ uniformly in $c \geq 1$. Hence $a_{[ct]}$ can be made arbitrary small for all large c by choosing $R > 0$ small enough. This concludes the proof. $\square$

Corollary 3.5. *Under the assumptions of Theorem 3.4 we also have*

$$\{T^{(c)}(ct)\}_{t \geq 0} \Rightarrow \{D(t)\}_{t \geq 0} \quad \text{as } c \to \infty$$

in the J_1-topology on $D([0, \infty), [0, \infty))$.

Proof. Note that the sample paths of $\{T^{(c)}(ct)\}_{t \geq 0}$ and $\{D(t)\}_{t \geq 0}$ are nondecreasing. Moreover, as a Lévy-process, $\{D(t)\}_{t \geq 0}$ is stochastically continuous. Then Theorem 3.4 together with Theorem 3 of [7] yields the assertion. $\square$

Corollary 3.6. *Assume that $\{D(t)\}_{t \geq 0}$ is the limit process obtained in (3.6) with Lévy measure of the form (3.7) for some $p \in \mathrm{RV}_0(\alpha - 1)$ and some $\alpha > 0$. Let $\log_+(x) = \max(\log x, 0)$. Then for $\rho \geq 0$ and any $t > 0$ we have*

$$\mathbb{E}((\log_+ D(t))^\rho) \begin{cases} < \infty & \rho < \alpha \\ = \infty & \rho > \alpha. \end{cases}$$

Proof. Let $g(x) = (\log(\max(x, e)))^\rho$. Then it is easy to see that the assertion follows if we can show that $\mathbb{E}(g(D(t))) < \infty$ if $\rho < \alpha$ and $\mathbb{E}(g(D(t))) = \infty$ if $\rho > \alpha$. Since the function g is submultiplicative (this is easy to check, see Proposition 25.4 of [40]), by Theorem 25.3 of [40] the assertion follows if $\int_1^\infty g(x) \mathrm{d}\phi(x) < \infty$ for $\rho < \alpha$ and $\int_1^\infty g(x) \mathrm{d}\phi(x) = \infty$ for $\rho > \alpha$. By definition of g this is equivalent to

$$\int_e^\infty (\log x)^\rho \, \mathrm{d}\phi(x) \begin{cases} < \infty & \rho < \alpha \\ = \infty & \rho > \alpha. \end{cases} \tag{3.12}$$

Note that by (3.7) the Lévy measure ϕ has density $x \mapsto \int_0^1 x^{-\beta-1} \beta p(\beta) \mathrm{d}\beta$. Then, by Tonelli's theorem and a change of variable we obtain

$$\begin{aligned}
\int_e^\infty (\log x)^\rho \, \mathrm{d}\phi(x) &= \int_0^1 \int_e^\infty (\log x)^\rho x^{-\beta-1} \, \mathrm{d}x \, \beta p(\beta) \mathrm{d}\beta \\
&= \int_0^1 \int_1^\infty y^\rho \mathrm{e}^{-\beta y} \, \mathrm{d}y \, \beta p(\beta) \mathrm{d}\beta \\
&= \int_0^1 \left(\int_\beta^\infty s^\rho \mathrm{e}^{-s} \, \mathrm{d}s \right) \beta^{-\rho} p(\beta) \mathrm{d}\beta.
\end{aligned}$$

Since $\int_\beta^\infty s^\rho e^{-s}\,ds \to \Gamma(\rho+1)$ as $\beta \to 0$, it is easy to see that (3.12) follows from

$$\int_0^1 \beta^{-\rho} p(\beta)d\beta \begin{cases} < \infty & \rho < \alpha \\ = \infty & \rho > \alpha. \end{cases} \tag{3.13}$$

Since $p \in \mathrm{RV}(\alpha - 1)$, for any $\delta > 0$ there exist constants $C_1, C_2 > 0$ such that $C_1\beta^{\alpha-1+\delta} \le p(\beta) \le C_2\beta^{\alpha-1-\delta}$ for all $0 < \beta < 1$, a simple calculation shows that (3.13) holds true and the proof is complete. $\square$

Corollary 3.7. *Assume that $\{D(t)\}_{t\ge0}$ is the limit process obtained in (3.6) with Lévy measure of the form (3.7) for some $p \in \mathrm{RV}_0(\alpha-1)$ and some $\alpha > 0$. Then every $D(t)$ has a C^∞-density $g(t, y)$ and all derivatives of the density with respect to y vanish at infinity.*

Proof. We use the following sufficient condition due to Orey, see [40], Proposition 28.3. It says that, if there exists any $0 < \rho < 2$ such that

$$\liminf_{r\downarrow0} r^{\rho-2} \int_{|x|\le r} x^2 d\phi(x) > 0 \tag{3.14}$$

then $D(t)$ has a C^∞ density with the desired property. Since the Lévy measure of $D(t)$ is $t\cdot\phi$ it suffices to show the assertion for $D(1)$.

Note that $u \mapsto \int_0^1 u^{-\beta-1}\beta p(\beta)d\beta$ is the density of ϕ. Note that since $p \in \mathrm{RV}_0(\alpha-1)$ with $\mathrm{supp}(p) \subset [0, 1]$ we know that for some $0 < \rho_0 < 1/2$ we have $p(\beta) > 0$ for all $0 < \beta < 2\rho_0$. By Tonelli's theorem we have

$$\begin{aligned}\int_{|x|\le r} x^2\, d\phi(x) &= \int_0^1 \int_0^r x^{1-\beta}\, dx\, \beta p(\beta)d\beta \\ &= \int_0^1 r^{2-\beta} \frac{\beta}{2-\beta} p(\beta)d\beta.\end{aligned}$$

Now, for $\rho = \rho_0$, we obtain

$$r^{\rho_0-2}\int_{|x|\le r} x^2\, d\phi(x) = \int_0^1 r^{\rho_0-\beta}\frac{\beta}{2-\beta}p(\beta)d\beta \ge \int_{\rho_0}^{2\rho_0} r^{\rho_0-\beta}\frac{\beta}{2-\beta}p(\beta)d\beta$$

and hence, by Fatou's lemma

$$\begin{aligned}\liminf_{r\downarrow0} r^{\rho_0-2}\int_{|x|\le r} x^2\, d\phi(x) &\ge \liminf_{r\downarrow0}\int_{\rho_0}^{2\rho_0} r^{\rho_0-\beta}\frac{\beta}{2-\beta}p(\beta)d\beta \\ &\ge \int_{\rho_0}^{2\rho_0}\left(\liminf_{r\downarrow0} r^{\rho_0-\beta}\right)\frac{\beta}{2-\beta}p(\beta)d\beta \\ &= \infty.\end{aligned}$$

Therefore (3.14) holds with $\rho = \rho_0$ and the proof is complete. $\square$

In view of the form of the Lévy measure ϕ of $\{D(t)\}_{t\ge0}$ in (3.7), this process is not a stable process. However, our next result shows that $\{D(t)\}_{t\ge0}$ is a *selfdecomposable* process in the sense of Definition 15.6 of [40]. For an introduction to selfdecomposable laws see Section 3.15 in [40] or Chapter 2 in [22]. A random variable X is selfdecomposable if for any $0 < a < 1$ there exists another random variable Y independent of X such that $aX + Y$ is identically distributed with X. Selfdecomposable distributions are the weak limits of normalized sums of independent (but not necessarily identically distributed) random variables, see for example Theorem 15.3 in [40]. Hence they extend the class of stable laws in a natural way. We do not need this property in our analysis of CTRWs but we include it for the sake of completeness.

Corollary 3.8. *The limiting process $\{D(t)\}_{t\geq 0}$ obtained in Theorem 3.4 above is selfdecomposable. That is, the distribution of any $D(t)$ is selfdecomposable.*

Proof. It suffices to show the assertion for $D(1)$. Since the Lévy measure ϕ of $D(1)$ has the density $\tilde{k}(x) = \int_0^1 x^{-\beta-1}\beta p(\beta)\mathrm{d}\beta$ it follows from the Lévy–Khinchin representation (see, e.g., Theorem 8.1 in [40]) that the log-characteristic function ψ of the distribution of $D(1)$ has the form

$$\begin{aligned}\psi(\xi) &= \mathrm{i}a\xi + \int_0^\infty (\mathrm{e}^{\mathrm{i}\xi x} - 1 - \mathrm{i}\xi x I(|x| \leq 1))\tilde{k}(x)\mathrm{d}x \\ &= \mathrm{i}a\xi + \int_0^\infty (\mathrm{e}^{\mathrm{i}\xi x} - 1 - \mathrm{i}\xi x I(|x| \leq 1))\frac{x\tilde{k}(x)}{x}\,\mathrm{d}x.\end{aligned}$$

Since $k(x) = x\tilde{k}(x) = \int_0^1 x^{-\beta}\beta p(\beta)\mathrm{d}\beta$ is decreasing on $(0,\infty)$, it follows from Corollary 15.11 of [40] that $D(1)$ has a selfdecomposable distribution. □

Let $\{D(u)\}_{u\geq 0}$ be the Lévy process obtained in Theorem 3.4. Note that, by Theorem 21.3 of [40] and the fact that the integral in (3.7) tends to infinity as $u \to 0$, the sample paths are strictly increasing. Note also that, by Theorem 48.1 in [40] and the fact that the Lévy measure (3.7) is concentrated on the positive reals, $D(u) \to \infty$ as $u \to \infty$ almost surely. Define the *hitting time* process by

$$E(t) = \inf\{x \geq 0 : D(x) > t\}. \tag{3.15}$$

Then it is easy to see that for $t, x \geq 0$

$$\{E(t) \leq x\} = \{D(x) \geq t\}. \tag{3.16}$$

Later we do need the asymptotic behavior of $\mathbb{E}(E(t))$ as $t \to \infty$. We present a more general result on the asymptotic behavior of all moments of $E(t)$. We write $f(x) \sim g(x)$ if $f(x)/g(x) \to 1$.

Theorem 3.9. *Let $E(t)$ be the hitting time of the subordinator $\{D(u)\}_{u\geq 0}$ obtained in Theorem 3.4 for $p \in \mathrm{RV}_0(\alpha-1)$ and some $\alpha > 0$. Then there exists a function $\bar{L} \in \mathrm{RV}_\infty(0)$ such that for any $\gamma > 0$*

$$\mathbb{E}(E(t)^\gamma) \sim (\log t)^{\alpha\gamma}\bar{L}(\log t)^{-\gamma} \quad \text{as } t \to \infty.$$

Proof. Since $p \in \mathrm{RV}_0(\alpha-1)$ and $\Gamma(1) = 1$ it follows that $q(\beta) = \Gamma(1-\beta)p(\beta) \in \mathrm{RV}_0(\alpha-1)$ as well. Note that by (3.3) and the fact that $\Gamma(x) \sim 1/x$ as $x \to 0$ we have $\int_0^1 q(\beta)\mathrm{d}\beta < \infty$. Then, by Lemma 3.1, there exists a function $\bar{L} \in \mathrm{RV}_\infty(0)$ such that

$$\int_0^1 t^{-\beta}q(\beta)\mathrm{d}\beta = (\log t)^{-\alpha}\bar{L}(\log t) \quad \text{as } t \to \infty.$$

Hence

$$I(s) = \int_0^1 s^\beta q(\beta)\mathrm{d}\beta = (\log(1/s))^{-\alpha}\bar{L}(\log(1/s)) \quad \text{as } s \to 0. \tag{3.17}$$

Next observe that the well-known formula for the Laplace transform of a subordinator (see, e.g., Theorem 30.1 of [40]) together with (3.7) yield

$$\begin{aligned}\psi(s) &= \int_0^\infty (e^{-su}-1)d\phi(u)\\ &= \int_0^1 \left(\int_0^\infty (e^{-su}-1)\beta u^{-\beta-1}\,du\right)p(\beta)d\beta\\ &= -\int_0^1 \Gamma(1-\beta)s^\beta p(\beta)d\beta = -I(s). \end{aligned}\tag{3.18}$$

Fix any $\gamma > 0$. Then, for $t > 0$ we have by (3.16) and a well-known formula for fractional moments (see, e.g., Lemma 1 on p. 150 of [17]) that

$$\begin{aligned} h_\gamma(t) = \mathbb{E}(E(t)^\gamma) &= \gamma\int_0^\infty x^{\gamma-1}P\{E(t)>x\}\,dx\\ &= \gamma\int_0^\infty x^{\gamma-1}P\{D(x)<t\}\,dx. \end{aligned}\tag{3.19}$$

Now let $F(t) = P\{D(x) < t\}$ denote the distribution function of $D(x)$ for some fixed $x > 0$. Then, by Theorem 30.1 of [40] together with (3.18), we get $\int_0^\infty e^{-st}\,dF(t) = e^{-xI(s)}$. Moreover, by integration by parts $\int_0^\infty e^{-st}\,dF(t) = s\int_0^\infty e^{-st}P\{D(x) < t\}\,dt$ and hence

$$\int_0^\infty e^{-st}P\{D(x)<t\}\,dt = \frac{1}{s}e^{-xI(s)}. \tag{3.20}$$

Using (3.20) and Tonelli's theorem we therefore compute

$$\tilde{h}_\gamma(s) = \int_0^\infty e^{-st}h_\gamma(t)dt = \frac{\gamma}{s}\int_0^\infty x^{\gamma-1}e^{-xI(s)}\,dx = \Gamma(\gamma+1)s^{-1}I(s)^{-\gamma}.$$

In view of (3.17) this implies

$$\tilde{h}_\gamma(s) = \Gamma(\gamma+1)s^{-1}(\log(1/s))^{\alpha\gamma}\bar{L}(\log(1/s))^{-\gamma} \quad \text{as } s\to 0.$$

By a Tauberian theorem (see [17], Theorem 4 on p. 446) we conclude

$$h_\gamma(t) \sim \frac{\Gamma(\gamma+1)}{\Gamma(\gamma)}(\log t)^{\alpha\gamma}\bar{L}(\log t)^{-\gamma} \quad \text{as } t\to\infty.$$

Note that in view of (3.19) the function $h_\gamma(t)$ is ultimately monotone. □

After investigating the hitting time process $\{E(t)\}_{t\geq 0}$ we now show that the rescaled counting process $\{N_t^{(c)}\}_{t\geq 0}$ defined by (2.2) converges to $\{E(t)\}_{t\geq 0}$.

Theorem 3.10. *Suppose that we are given a probability density $p \in \mathrm{RV}_0(\alpha-1)$ for some $\alpha > 0$ such that (3.3) holds as in Theorem 3.4. Define the triangular array $\Delta = \{J_i^{(c)} : 1 \leq i \leq [ct], c \geq 1\}$ by (3.4) and the counting process $\{N_t^{(c)}\}_{t\geq 0}$ by (2.2). Then*

$$\left\{\frac{1}{c}N_t^{(c)}\right\}_{t\geq 0} \stackrel{f.d.}{\Longrightarrow} \{E(t)\}_{t\geq 0} \quad \text{as } c\to\infty, \tag{3.21}$$

where $\{E(t)\}_{t\geq 0}$ is the hitting time process defined by (3.15) of the subordinator $\{D(u)\}_{u\geq 0}$ corresponding to the triangular array Δ obtained in Theorem 3.4.

Proof. Observe that for $t \geq 0$ and $c \geq 1$ we have $\{N_t^{(c)} \geq x\} = \{T^{(c)}(\lceil x \rceil) \leq t\}$ where $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x \geq 0$. Note that by Corollary 3.7, $D(u)$ has a density with respect to Lebesgue measure. Fix any $0 \leq t_1 < \cdots < t_m$ and $x_1, \ldots, x_m \geq 0$ and let $\forall i$ mean for $i = 1, \ldots, m$. Since $T^{(c)}(x)$ has nondecreasing sample paths, Theorem 3.4 together with (3.16) imply

$$\begin{aligned} P\{c^{-1} N_{t_i}^{(c)} < x_i \ \forall i\} &= P\{N_{t_i}^{(c)} < c x_i \ \forall i\} \\ &= P\{T^{(c)}(\lceil c x_i \rceil) > t_i \ \forall i\} \\ &\geq P\{T^{(c)}(c x_i) > t_i \ \forall i\} \\ &\to P\{D(x_i) > t_i \ \forall i\} \\ &= P\{E(t_i) < x_i \ \forall i\} \end{aligned}$$

as $c \to \infty$. Also for any $\varepsilon > 0$ for all $c > 0$ sufficiently large we have

$$\begin{aligned} P\{c^{-1} N_{t_i}^{(c)} < x_i \ \forall i\} &= P\{N_{t_i}^{(c)} < c x_i \ \forall i\} \\ &= P\{T^{(c)}(\lceil c x_i \rceil) > t_i \ \forall i\} \\ &\leq P\{T^{(c)}(c(1+\varepsilon) x_i) > t_i \ \forall i\} \\ &\to P\{D((1+\varepsilon) x_i) > t_i \ \forall i\} \\ &= P\{E(t_i) < (1+\varepsilon) x_i \ \forall i\} \end{aligned}$$

as $c \to \infty$. Now let $\varepsilon \to 0$ and use the fact that D_x is stochastically continuous to complete the proof. $\square$

Corollary 3.11. *Under the assumptions of Theorem 3.10 we have*

$$\left\{ \frac{1}{c} N_t^{(c)} \right\}_{t \geq 0} \Rightarrow \{E(t)\}_{t \geq 0} \quad \text{as } c \to \infty$$

in the J_1-topology on $D([0, \infty), [0, \infty))$.

Proof. Note that the sample paths of $\{N_t^{(c)}\}_{t \geq 0}$ and $\{E(t)\}_{t \geq 0}$ are nondecreasing. Moreover, since the sample path of $\{E(t)\}_{t \geq 0}$ are continuous, the process $\{E(t)\}_{t \geq 0}$ is stochastically continuous. Then Theorem 3.10 together with Theorem 3 in [7] yields the assertion. $\square$

4. CTRW limit theorem

Assume that (Y_i) are i.i.d. $\mathbb{R}^d$-valued random vectors independent of the triangular array $\{J_i^{(c)} : 1 \leq i \leq [ct], c \geq 1\}$ of waiting times defined by (3.4). We assume that (3.3) holds. Moreover it is assumed that Y_1 belongs to the strict generalized domain of attraction of some full operator stable law with exponent E and (2.3) holds.

Theorem 4.1. *Under the assumptions of the beginning of this section we have for the generalized CTRW process $\{X^{(c)}(t)\}_{t \geq 0}$ defined in (2.4) that*

$$\{X^{(c)}(t)\}_{t \geq 0} \stackrel{f.d.}{\Longrightarrow} \{A(E(t))\}_{t \geq 0} \quad \text{as } c \to \infty.$$

Here $\{A(t)\}_{t \geq 0}$ is the operator Lévy motion corresponding to the jumps (Y_i) and $\{E(t)\}_{t \geq 0}$ is the hitting time process corresponding to the subordinator $\{D(u)\}_{u \geq 0}$ obtained in Theorem 3.4.

Proof. The proof is similar to the proof of Theorem 4.2 in [4], so we only sketch the argument. Fix any $0 < t_1 < \cdots < t_m$ and let $\forall i$ mean for $i = 1, \ldots, m$. Note that by Theorem 3.10

$$\left(\frac{1}{c} N_{t_i}^{(c)} \ \forall i\right) \Rightarrow (E(t_i) \ \forall i) \quad \text{as } c \to \infty.$$

Moreover, for any $x_1, \ldots, x_m \geq 0$ we know that

$$(B(c)S(cx_i) \ \forall i) \Rightarrow (A(x_i) \ \forall i) \quad \text{as } c \to \infty$$

uniformly on compact sets of $\mathbb{R}_+^m$ as was established in the proof of Theorem 4.2 in [4]. Independence of (Y_i) and $\{N_t^{(c)}\}$ yields

$$\begin{aligned} P_{(X^{(c)}(t_i) \ \forall i)} &= P_{(B(c)S(N_{t_i}^{(c)}) \ \forall i)} \\ &= \int_{\mathbb{R}_+^m} P_{(B(c)S(cx_i) \ \forall i)} \mathrm{d}P_{(c^{-1}N_{t_i}^{(c)} \ \forall i)}(x_1, \ldots, x_m) \\ &\Rightarrow \int_{\mathbb{R}_+^m} P_{(A(x_i) \ \forall i)} \mathrm{d}P_{(E(t_i) \ \forall i)}(x_1, \ldots, x_m) \\ &= P_{(A(E(t_i)) \ \forall i)} \end{aligned}$$

as $c \to \infty$, by a transfer theorem, Proposition 4.1 in [4]. □

Corollary 4.2. *Under the assumptions of Theorem 4.1, if $A(1)$ has no normal component, for every $t > 0$ the distribution λ_t of $M(t) = A(E(t))$ belongs to the domain of normal attraction of $A(1)$. That is, if $m(t) = \mathbb{E}(E(t))$, then for some sequence (b_n) of shifts*

$$(m(t)n)^{-E} \lambda_t^{*n} * \varepsilon_{b_n} \Rightarrow \nu \quad \text{as } n \to \infty,$$

where ν is the distribution of $A(1)$ and E is an exponent of ν.

Proof. Since by Theorem 3.9 we know that $m(t) = \mathbb{E}(E(t))$ is finite and ν is assumed to be a strictly operator stable law with exponent E having no normal component, the assertion follows from Corollary 4.2 of [25]. □

Remark 4.3. It follows from Theorem 4.1 of [25] that, under the additional condition that $A(1)$ has no normal component, the distribution λ_t of $M(t) = A(E(t))$ varies regularly with exponent E. See [28] for a comprehensive introduction to regularly varying measures on $\mathbb{R}^d$. Therefore various results on the tail and moment behavior of λ_t can be obtained from [28]. Let $a_1 < \cdots < a_p$ denote the real parts of the eigenvalues of E. Then Theorem 8.2.14 in [28] implies that there exists a function $\rho : \Gamma \to \{a_p^{-1}, \ldots, a_1^{-1}\}$ such that for all $\theta \in \Gamma = \mathbb{R}^d \setminus \{0\}$ the radial moments

$$\int |\langle y, \theta \rangle|^\gamma \lambda_t(\mathrm{d}y) = \mathbb{E}(|\langle M(t), \theta \rangle|^\gamma) \tag{4.1}$$

exist for $0 \leq \gamma < \rho(\theta)$ and diverge for $\gamma > \rho(\theta)$. Corollary 8.2.15 in [28] implies that

$$\int \|y\|^\gamma \lambda_t(\mathrm{d}y) = \mathbb{E}(\|M(t)\|^\gamma) \tag{4.2}$$

exists if $\gamma < 1/a_p$ and is infinite if $\gamma > 1/a_p$. Also, Theorem 6.4.15 in [28] gives the power law tail behavior of the truncated moments and tail moments

$$\int_{|\langle y,\theta\rangle|\le r} |\langle y,\theta\rangle|^{\zeta}\lambda_t(\mathrm{d}y) \quad \text{and} \quad \int_{|\langle y,\theta\rangle|>r} |\langle y,\theta\rangle|^{\eta}\lambda_t(\mathrm{d}y) \tag{4.3}$$

in terms of multivariable R–O variation. Roughly speaking, this result says that the tail $P(|\langle M(t),\theta\rangle| > r)$ falls off like $r^{-\rho(\theta)}$ as $r \to \infty$.

5. Distributed-order fractional evolution equations

In this section we outline some interesting connections between fractional calculus, anomalous diffusion models in physics, and the stochastic processes studied in this paper. These considerations also lead to a new integral equation (5.15) for stable subordinators, which may be interesting in its own right.

For suitable functions $h : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ we define the *Fourier–Laplace transform* (FLT) by

$$\bar{h}(s,k) = \int_{\mathbb{R}^d}\int_0^\infty \mathrm{e}^{\mathrm{i}\langle k,x\rangle}\mathrm{e}^{-st}h(t,x)\mathrm{d}t\,\mathrm{d}x \tag{5.1}$$

where $(s,k) \in (0,\infty) \times \mathbb{R}^d$. It follows from a general theory of FLTs on semigroups, that this transform has properties similar to the usual Fourier or Laplace transform. See [6] and Theorem 1 in [37] for details. Recall from [9,35] that for $0 < \beta < 1$ and suitable functions g the Caputo derivative $(\frac{\partial}{\partial t})^\beta g(t)$ has Laplace transform $s^\beta\tilde{g}(s) - s^{\beta-1}g(0)$ where $\tilde{g}(s) = \int_0^\infty \mathrm{e}^{-st}g(t)\mathrm{d}t$ denotes the usual Laplace transform. We say that a function $h(t,x)$ is a *mild solution* to a time-fractional partial differential equation, if the FLT $\bar{h}(s,k)$ solves the equivalent algebraic equation in Fourier–Laplace space. This is somewhat different from the standard usage for integer-order time derivative equations (e.g., see Pazy [34] Definition 2.3 p. 106) where a mild solution is defined as a solution to the corresponding integral equation. For time-fractional equations, there is no standard concept of a mild solution, and the usage here is consistent with [3]. Some deeper questions regarding strong solutions of these equations are also interesting, but beyond the scope of this paper.

Next we argue that, under certain technical conditions, the hitting time process $E(t)$ has probability densities that solve a distributed-order time-fractional evolution equation. Let $\{D(u)\}_{u\ge 0}$ be the Lévy process with Lévy measure given by (3.7), and note that by Corollary 3.7 for $x > 0$ the density $g(x,\cdot)$ of $D(x)$ is a bounded C^∞-function. For $t > 0$ let $F(t,x) = P\{E(t) \le x\}$ denote the distribution function of $E(t)$. Note that in view of (3.16) we know $F(t,x) = P\{D(x) \ge t\} = \int_t^\infty g(x,y)\mathrm{d}y$. Then the Laplace transform in $t > 0$ of this family of distribution functions is

$$\begin{aligned}
\tilde{F}(s,x) &= \int_0^\infty \mathrm{e}^{-st}F(t,x)\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-st}\int_t^\infty g(x,y)\mathrm{d}y\,\mathrm{d}t \\
&= \int_0^\infty \left(\int_0^y \mathrm{e}^{-st}\,\mathrm{d}t\right) g(x,y)\mathrm{d}y \\
&= \frac{1}{s}\int_0^\infty (1-\mathrm{e}^{-sy})g(x,y)\mathrm{d}y \\
&= \frac{1}{s}(1-\mathrm{e}^{x\psi(s)}).
\end{aligned} \tag{5.2}$$

Now we define

$$f(t,x)=\int_0^1\int_0^t (t-y)^{-\beta}g(x,y)\mathrm{d}y\,p(\beta)\mathrm{d}\beta \tag{5.3}$$

which under certain technical conditions will be shown to be the density function of $E(t)$. For $s>0$ and $0<\beta<1$, by changing the order of integration, we get

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-st}\int_0^t (t-y)^{-\beta}g(x,y)\mathrm{d}y\,\mathrm{d}t &= \int_0^\infty\left(\int_y^\infty \mathrm{e}^{-st}(t-y)^{-\beta}\,\mathrm{d}t\right)g(x,y)\mathrm{d}y\\ &= s^{\beta-1}\Gamma(1-\beta)\int_0^\infty \mathrm{e}^{-sy}g(x,y)\mathrm{d}y\\ &= s^{\beta-1}\Gamma(1-\beta)\mathrm{e}^{x\psi(s)}.\end{aligned} \tag{5.4}$$

Then, by (5.4) and (3.18) and the Tonelli theorem we obtain

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-st}f(t,x)\mathrm{d}t &= \int_0^1\int_0^\infty \mathrm{e}^{-st}\int_0^t (t-y)^{-\beta}g(x,y)\mathrm{d}y\,\mathrm{d}t\;\; p(\beta)\mathrm{d}\beta\\ &= \mathrm{e}^{x\psi(s)}\int_0^1 s^{\beta-1}\Gamma(1-\beta)p(\beta)\mathrm{d}\beta\\ &= -\frac{1}{s}\psi(s)\mathrm{e}^{x\psi(s)}.\end{aligned} \tag{5.5}$$

Now write $L(t,x)=\int_0^x f(t,y)\mathrm{d}y$ and compute the Laplace transform in $t>0$:

$$\begin{aligned}\tilde{L}(s,x) &= \int_0^\infty \mathrm{e}^{-st}L(t,x)\mathrm{d}t\\ &= \int_0^x\int_0^\infty \mathrm{e}^{-st}f(t,y)\mathrm{d}t\,\mathrm{d}y\\ &= -\frac{1}{s}\psi(s)\int_0^x \mathrm{e}^{y\psi(s)}\,\mathrm{d}y\\ &= \frac{1}{s}(1-\mathrm{e}^{x\psi(s)}).\end{aligned} \tag{5.6}$$

which shows that $L(t,x)$ and $F(t,x)$ have the same Laplace transform in $t>0$ for any fixed $x>0$. Clearly $F(t,x)$ is continuous in $t>0$. Then under the technical condition that

$$L(t,x)=\int_0^x f(t,y)\mathrm{d}y \text{ is a continuous function of } t>0 \text{ for any fixed } x>0, \tag{5.7}$$

it follows from the uniqueness theorem for Laplace transforms (e.g., the corollary on p. 433 of Feller [17]) that $L(t,x)=F(t,x)$, and then the function $f(t,x)$ in (5.3) is the density function of $E(t)$ for every $t>0$.

In this case, the density $f(t,x)$ of $E(t)$ is the mild solution of the distributed-order time-fractional partial differential equation

$$\int_0^1\left(\frac{\partial}{\partial t}\right)^\beta f(t,x)\Gamma(1-\beta)p(\beta)\mathrm{d}\beta=-\frac{\partial}{\partial x}f(t,x),\qquad f(0,x)=\delta(x). \tag{5.8}$$

To see this, let $\bar{f}(s,k)$ be the FLT of $f(t,x)$ for $s>0$ and $k\in\mathbb{R}$. Then it follows from (5.5) that

$$\bar{f}(s,k) = -\frac{1}{s}\psi(s)\int_0^\infty \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{x\psi(s)}\,\mathrm{d}x = \frac{1}{s}\frac{\psi(s)}{\mathrm{i}k+\psi(s)}.$$

Take Laplace transforms in (5.8) to get

$$\int_0^1 (s^\beta \tilde{f}(s,x) - s^{\beta-1}\delta(x))\Gamma(1-\beta)p(\beta)\mathrm{d}\beta = -\frac{\partial}{\partial x}\tilde{f}(s,x)$$

and use (3.18) to obtain

$$-\psi(s)\tilde{f}(s,x) + \frac{1}{s}\psi(s)\delta(x) = -\frac{\partial}{\partial x}\tilde{f}(s,x).$$

Then take Fourier transforms, using the fact that if $g(x)$ has Fourier transform $\mathcal{F}(g)(k)$ then $\mathcal{F}(g')(k) = (-\mathrm{i}k)\mathcal{F}(g)(k)$, to get

$$-\psi(s)\bar{f}(s,k) + \frac{1}{s}\psi(s) = \mathrm{i}k\bar{f}(s,k).$$

Then it follows easily that $f(t,x)$ is the mild solution of (5.8).

Now we argue that, under the technical condition (5.7), the CTRW scaling limit random vector $M(t) = A(E(t))$ also has a density that solves a distributed-order evolution equation. Recall that as an infinitely divisible law, the operator stable random vector $A(t)$ has log-characteristic function $t\cdot\psi_A(k)$ so that $\mathbb{E}(\mathrm{e}^{\mathrm{i}\langle k,A(t)\rangle}) = \mathrm{e}^{t\psi_A(k)}$. It is well known that, under some regularity conditions, the log-characteristic function of an infinitely divisible distribution is the symbol of the pseudo-differential operator defined by the generator

$$Lf(x) = \lim_{t\downarrow 0}\frac{T(t)f(x)-f(x)}{t}$$

of the corresponding C_0-semigroup $T(t)f(x) = \mathbb{E}[f(x-A(t))]$. In particular, for a C^∞ function $u:\mathbb{R}^d\to\mathbb{R}$ with compact support we define the *pseudo-differential operator* $L=\psi_A(\mathrm{i}D_x)$ with *symbol* $\psi_A(k)$ by requiring $Lu(x)$ to have Fourier transform $\psi_A(k)\hat{u}(k)$. Since $\hat{u}(k)$ is rapidly decreasing it follows that, since ψ_A grows at a polynomial rate at infinity, the function $\psi_A(\mathrm{i}D_x)u(x)$ is pointwise defined. Furthermore, it usually can be extended to larger spaces of functions (or even distributions), where the extension is also denoted by $\psi_A(\mathrm{i}D_x)$. For example, a one-dimensional Brownian motion $A(t)$ with variance $2t$ has symbol $\psi_A(k) = -k^2$ and $L = \partial^2/\partial x^2$. For a one-dimensional α-stable Lévy motion, L is a fractional space derivative of order α, and for a d-dimensional operator stable Lévy motion, L is a multivariable fractional space derivative. For more details see [1,2,21,26,27,29,41]. Recall the definition of the Fourier–Laplace transform (FLT) from (5.1).

Recall from Theorem 7.2.7 of [28] that the full operator stable random vector $A(t)$ has a density $p(t,x)$ for any $t>0$. Assume also that $E(t)$ has density function $f(t,x)$ given by (5.3). Then under the assumptions of Theorem 4.1, for every $t>0$ the random vector $M(t) = A(E(t))$ has the density

$$h(t,x) = \int_0^\infty p(u,x)f(t,u)\mathrm{d}u. \tag{5.9}$$

This is a simple conditioning argument using the fact that $\{A(t)\}_{t\geq 0}$ and $\{E(t)\}_{t\geq 0}$ are independent. Then h has FLT

$$\bar{h}(s,k) = \frac{1}{s}\frac{I(s)}{I(s)-\psi_A(k)}, \quad (s,k) \in (0,\infty)\times\mathbb{R}^d, \tag{5.10}$$

where

$$I(s) = -\psi(s) = \int_0^1 s^\beta \Gamma(1-\beta)p(\beta)\mathrm{d}\beta. \tag{5.11}$$

Moreover, h is the mild solution of the distributed-order time-fractional partial differential equation

$$\int_0^1 \left(\frac{\partial}{\partial t}\right)^\beta h(t,x)\Gamma(1-\beta)p(\beta)\mathrm{d}\beta = \psi_A(iD_x)h(t,x),\ h(0,x)=\delta(x). \tag{5.12}$$

To see this, note that since $|\mathrm{e}^{\psi_A(k)}| \leq 1$ we know $\operatorname{Re}\psi_A(k) \leq 0$, and then in view of (5.5) we get

$$\begin{aligned}\bar{h}(s,k) &= \int_0^\infty \hat{p}(u,k)\tilde{f}(s,u)\mathrm{d}u\\ &= \frac{1}{s}I(s)\int_0^\infty \mathrm{e}^{-u(I(s)-\psi_A(k))}\,\mathrm{d}u\\ &= \frac{1}{s}\frac{I(s)}{I(s)-\psi_A(k)},\end{aligned}$$

so (5.10) holds true. Equivalently $I(s)\bar{h}(s,k) - s^{-1}I(s) = \psi_A(k)\bar{h}(s,k)$ and in view of (3.18) this is equivalent to

$$\int_0^1 (s^\beta \bar{h}(s,k) - s^{\beta-1})\Gamma(1-\beta)p(\beta)\,\mathrm{d}\beta = \psi_A(k)\bar{h}(s,k).$$

Taking Laplace and then Fourier transforms in (5.12) as before yields the same equation. Hence $h(t,x)$ is the mild solution of (5.12).

Remark 5.1. Note that the above arguments also hold true, if we replace integration with respect to $p(\beta)\mathrm{d}\beta$ for some probability density p supported in [0, 1] and satisfying (3.3) by integration with respect to a probability measure $\rho(\mathrm{d}\beta)$ with support in [0, 1] and $\int_0^1 \frac{\rho(\mathrm{d}\beta)}{1-\beta} < \infty$. Hence, if $\{D(u)\}_{u\geq 0}$ is a subordinator with Lévy measure ϕ of the form $\phi(u,\infty) = \int_0^1 u^{-\beta}\rho(\mathrm{d}\beta)$ and having a bounded C^∞ density $g(u,\cdot)$ for $D(u)$, then under the technical condition (5.7) the hitting time $E(t)$ has the density

$$f(t,x) = \int_0^1\int_0^t (t-y)^{-\beta}g(x,y)\mathrm{d}y\,\rho(\mathrm{d}\beta). \tag{5.13}$$

Moreover, in this case it follows that $f(t,x)$ is the mild solution of

$$\int_0^1 \left(\frac{\partial}{\partial t}\right)^\beta f(t,x)\Gamma(1-\beta)\,\rho(\mathrm{d}\beta) = -\frac{\partial}{\partial x}f(t,x), \quad f(0,x)=\delta(x).$$

Especially, if $\rho = \varepsilon_\gamma$ is the point mass in some $0<\gamma<1$, then $\{D(u)\}_{u\geq 0}$ is a γ-stable subordinator and its density is given by $g(u,y) = u^{-1/\gamma}g_0(u^{-1/\gamma}y)$ where g_0 is the bounded

C^∞-density of $D(1)$. It is not hard to verify that the technical condition (5.7) holds in this case, using dominated convergence along with the fact that $\sup\{g_0(y) : y > 0\} < \infty$ and (see, e.g., [39] p. 16) $g_0(y) = O(y^{-\gamma-1})$ as $y \to \infty$. Then the density $f(t,x)$ of the corresponding hitting time $E(t)$ is given by

$$f(t,x) = x^{-1/\gamma} \int_0^t (t-y)^{-\gamma} g_0(x^{-1/\gamma} y) \mathrm{d}y. \tag{5.14}$$

On the other hand, in view of Corollary 3.2 of [30]

$$f(t,x) = \frac{t}{\gamma} x^{-1-1/\gamma} g_0(t x^{-1/\gamma}).$$

Hence, the density g_0 of a γ-stable random variable D solves the integral equation

$$g_0(z) = \frac{\gamma}{z} \int_0^z (z-y)^{-\gamma} g_0(y) \mathrm{d}y. \tag{5.15}$$

To our knowledge this property of the density g_0 of a γ-stable random variable is new and may be of independent interest. For example, it can be used to efficiently compute the function $g_0(x)$, and our brief numerical experiments suggest that the convergence is rather fast. Moreover, the density $f(t,x)$ of $E(t)$ in this case is the mild solution of

$$\Gamma(1-\gamma) \left(\frac{\partial}{\partial t}\right)^\gamma f(t,x) = -\frac{\partial}{\partial x} f(t,x), \quad f(0,x) = \delta(x)$$

which agrees with (3.8) in [4].

Remark 5.2. In the degenerate case $A(t) = t$, under the technical condition (5.7), the process $A(E(t)) = E(t)$ has density $f(t,x)$ given by (5.3). Note that this density solves (5.8) which is formally equivalent to (5.12) if we take $\psi_A(k) = \mathrm{i}k$, which is the symbol of the semigroup generator $-\partial/\partial x$ for the associated semigroup $T(t)f(x) = f(x-t)$.

Remark 5.3. Let $\{A(t)\}_{t\geq 0}$ be a one-dimensional Brownian motion with $\mathrm{Var}(A(t)) = 2t$. Then $\psi_A(k) = -k^2$ and the pseudo-differential operator $\psi_A(\mathrm{i}D_x) = -(\mathrm{i}D_x)^2 = D_x^2$, and the density $h(t,x)$ of $M(t) = A(E(t))$ has FLT

$$\bar{h}(s,k) = \frac{1}{s} \frac{I(s)}{I(s)+k^2}, \tag{5.16}$$

where $I(s)$ is given by (5.11). Furthermore $h(t,x)$ is the mild solution of the distributed-order time-fractional partial differential equation

$$\int_0^1 \left(\frac{\partial}{\partial t}\right)^\beta h(t,x) \Gamma(1-\beta) p(\beta) \mathrm{d}\beta = \frac{\partial^2}{\partial x^2} h(t,x), \quad h(0,x) = \delta(x). \tag{5.17}$$

In this case

$$h(t,x) = \int_0^\infty \frac{1}{\sqrt{4\pi u}} \mathrm{e}^{-x^2/4u} f(t,u) \mathrm{d}u \tag{5.18}$$

where $f(t,u)$ is the density of $E(t)$ given by (5.3). Eq. (5.17) first appeared in [13] together with (5.16). They show that $h(x,t)$ is a probability density for every $t > 0$ by using (5.18) along with the fact that (5.5) is completely monotone. The present paper extends (5.17) to the case of a more general pseudo-differential operator, and identifies the stochastic process for which $h(t,x)$ is a density. A simple conditioning argument along with Theorem 3.12 shows that the mean square displacement of a particle governed by Eq. (5.17) is

$$\mathbb{E}(M(t)^2) = 2\mathbb{E}(E(t)) \sim (\log t)^\alpha \bar{L}(\log t)^{-1}$$

as $t \to \infty$ for some $\bar{L} \in \mathrm{RV}_\infty(0)$. This agrees with [11,12] and shows that (5.17) describes an ultraslow diffusion, where a cloud of diffusing particles spreads at the rate $(\log t)^{\alpha/2}$.

Remark 5.4. Following [11] we note that, in the case where $A(t)$ is a one-dimensional Brownian motion, the Fourier transform formula $\mathcal{F}[(c/2)\mathrm{e}^{-c|x|}] = c^2/(c^2+k^2)$ along with (5.16) implies that

$$\tilde{h}(s,x) = \frac{I(s)^{1/2}}{2s}\mathrm{e}^{-I(s)^{1/2}|x|}.$$

Under the additional assumption that $h(t,x)$ is ultimately monotone, a Tauberian theorem (Theorem 4 on p. 446 of Feller [17]) yields that

$$h(t,x) \sim \frac{I(1/t)^{1/2}}{2}\mathrm{e}^{-I(1/t)^{1/2}|x|} \quad \text{as } t \to \infty.$$

If $p \in \mathrm{RV}_0(\alpha-1)$ then it follows from Lemma 3.1 as in the proof of Theorem 3.9 that $I(1/t) = (\log t)^{-\alpha} L_1(\log t)$ for some $L_1 \in \mathrm{RV}_\infty(0)$. Hence $h(t,x)$ is asymptotically equivalent to a Laplace density whose variance grows like $(\log t)^\alpha$. A different stochastic model for ultraslow diffusion presented in [31], using nonlinear rescaling for the waiting time process, leads exactly to a Laplace limit with density

$$h_1(t,x) = \frac{(\log t)^{-\alpha/2}}{2}\mathrm{e}^{-(\log t)^{-\alpha/2}|x|}.$$

Using the converse of the same Tauberian theorem yields

$$\tilde{h}_1(s,x) = \frac{(\log(1/s))^{-\alpha/2}}{2s}\mathrm{e}^{-(\log(1/s))^{-\alpha/2}|x|}$$

and then the same Fourier transform formula leads to (5.16) with $I(1/t) = (\log t)^{-\alpha}$. Now suppose that

$$I(s) = \int_0^\infty s^\beta q(\beta)\mathrm{d}\beta = (\log(1/s))^{-\alpha}$$

for some function $q(\beta)$, which is equivalent to $\tilde{q}(s) = s^{-\alpha}$. In view of the Laplace transform pair $\mathcal{L}[t^{\alpha-1}/\Gamma(\alpha)] = s^{-\alpha}$ for $\alpha > 0$ this implies that $q(\beta) = \beta^{\alpha-1}/\Gamma(\alpha)$ supported on the positive real line $\beta > 0$. Then the uniqueness theorem for Laplace transforms implies that we cannot write $q(\beta) = \Gamma(1-\beta)p(\beta)$ for any $p(\beta)$ supported on $0 < \beta < 1$. Hence the family of Laplace densities $h_1(t,x)$ cannot be the mild solution of (5.17) for any choice of $p(\beta)$, so the two process densities are only asymptotically equivalent. This resolves an open question in [31].

Remark 5.5. As in Remark 5.1 we can also consider the more general equation

$$\int_0^1 \left(\frac{\partial}{\partial t}\right)^\beta h(t,x)\Gamma(1-\beta)\,\rho(\mathrm{d}\beta) = \psi_A(\mathrm{i}D_x)h(t,x), \quad h(0,x) = \delta(x)$$

whose mild solution is given by (5.9) and (5.13). An application where $\psi_A(iD_x) = \partial^2/\partial x^2$ and $\rho(\mathrm{d}\beta)$ consists of two atoms at $0 < \beta_1 < \beta_2 < 1$ is considered in [48], Section 4.2 and [12]. The results of this paper give a different and perhaps simpler proof that the solutions in these papers are probability distributions, and also illuminate the nature of the stochastic limit. The limit process $\{D(t)\}$ in this case is a sort of mixture of β_1-stable and β_2-stable, with the larger exponent dominating at early time, and the smaller (heavier tail) emerging at a later time. Presumably a similar behavior can be expected whenever the support of the measure ρ is bounded away from zero, but we have not examined this in detail.

Remark 5.6. The classical continuous time random walk model considered in [30] is a special case of the generalized CTRW model described in Section 2. In fact, assume that $J_1, J_2, \ldots$ are nonnegative and i.i.d. belonging to the domain of attraction of some β-stable law with $0 < \beta < 1$. Then, for some norming function $b \in \mathrm{RV}(-1/\beta)$ we have

$$b(c)\sum_{i=1}^{[ct]} J_i \Rightarrow D(t) \quad \text{as } c \to \infty$$

where $\{D(t)\}_{t\geq 0}$ is a β-stable subordinator. If we set $J_i^{(c)} = b(c)J_i$, then $T^{(c)}(n) = b(c)\sum_{i=1}^{n} J_i$ is the time of the nth jump at scale $c \geq 1$. In this case the generalized CTRW converges as $c \to \infty$ to a limit process $M(t)$ whose density $h(t,x)$ solves the fractional partial differential equation

$$\frac{\partial^\beta h(x,t)}{\partial t^\beta} = Lh(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{5.19}$$

Here $\delta(x)$ is the Dirac delta function, the fractional derivative $\partial^\beta h(x,t)/\partial t^\beta$ is defined as the inverse Laplace transform of $s^\beta \tilde{h}(x,s)$, where $\tilde{h}(x,s) = \int_0^\infty \mathrm{e}^{-st}h(x,t)\mathrm{d}t$ is the usual Laplace transform, and $-L$ is the generator of the continuous convolution semigroup associated with the Lévy process $\{A(t)\}_{t\geq 0}$. For example, if $\{A(t)\}_{t\geq 0}$ is a one-dimensional Brownian motion then $L = \partial^2/\partial x^2$. Here the hitting time process $E(t)$ defined by (3.15) is self-similar with $E(ct) \stackrel{d}{=} c^\beta E(t)$, so that the CTRW scaling limit $A(E(t))$ grows more slowly than $A(t)$, a subdiffusive effect. See [30] for more details. A different norming scheme is used in [4] for $J_i > 0$ belonging to the domain of attraction of some β-stable law with $1 < \beta < 2$. Now for some norming function $b \in \mathrm{RV}(-1/\beta)$ we have

$$c^{-1}\mu[ct] + b(c)\sum_{i=1}^{[ct]}(J_i - \mu) \Rightarrow \bar{D}(t) \quad \text{as } c \to \infty$$

where $\mu = \mathbb{E}J_i$ and $\{\bar{D}(t)\}_{t\geq 0}$ is a totally positively skewed β-stable Lévy motion with drift such that $\mathbb{E}D(t) = \mu t$. Letting $J_i^{(c)} = b(c)(J_i - \mu) + c^{-1}\mu$, the resulting generalized CTRW limit process has a density that solves a fractional partial differential equation similar to (5.19) but with both a first order and a β-order time derivative on the left-hand side.

Acknowledgements

MMM was partially supported by NSF grants DMS-0139927 and DMS-0417869 and by the Marsden Foundation in New Zealand. HPS was partially supported by NSF grant DMS-0417869.

References

[1] W. Arendt, C. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, in: Monographs in Mathematics, Birkhaeuser-Verlag, Berlin, 2001.

[2] B. Baeumer, M. Meerschaert, Stochastic solutions for fractional Cauchy problems, Frac. Calc. Appl. Anal. 4 (2001) 481–500.

[3] B. Baeumer, D.A. Benson, M. Meerschaert, Advection and dispersion in time and space, Phys. A 350 (2005) 245–262.

[4] P. Becker-Kern, M.M. Meerschaert, H.P. Scheffler, Limit theorem for continuous time random walks with two time scales, J. Appl. Probab. 41 (2) (2004) 455–466.

[5] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, The fractional-order governing equation of Lévy motion, Water Resour. Res. 36 (2000) 1413–1424.

[6] C. Berg, J.P. Reus Christensen, P. Ressel, Harmonic Analysis on Semigroups, Springer-Verlag, New York, 1984.

[7] N.H. Bingham, Limit theorems for occupation times of Markov processes, Z. Wharsch. verw. Geb. 17 (1971) 1–22.

[8] J. Bisquert, Fractional diffusion in the multiple-trapping regime and revision of the equivalence with the continuous-time random walk, Phys. Rev. Lett. 91 (1) (2003) 602–605.

[9] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, Part II, Geophys. J. R. Astr. Soc. 13 (1967) 529–539.

[10] A. Chaves, A fractional diffusion equation to describe Lévy flights, Phys. Lett. A 239 (1998) 13–16.

[11] A.V. Chechkin, J. Klafter, I.M. Sokolov, Fractional Fokker-Plank equation for ultraslow kinetics, Europhys. Lett. 63 (3) (2003) 326–332.

[12] A.V. Chechkin, R. Gorenflo, I.M. Sokolov, V.Yu. Gonchar, Distributed order time fractional diffusion equation, Frac. Calc. Appl. Anal. 6 (3) (2003) 259–279.

[13] A.V. Chechkin, R. Gorenflo, I.M. Sokolov, Retarding subdiffusion and accelerating superdiffusion governed by distributed-order fractional diffusion equations, Phys. Rev. E 66 (2002) 046129–046135.

[14] D.A. Darling, The influence of the maximum term in the addition of independent random variable, Trans. Amer. Math. Soc. 73 (1952) 95–107.

[15] J. Dräger, J. Klafter, Strong anomaly in diffusion generated by iterated maps, Phys. Rev. Lett. 84 (2000) 5998–6001.

[16] M. Dwass, Extremal processes, Ann. Math. Statist. 35 (4) (1964) 1718–1725.

[17] W. Feller, An Introduction to Probability Theory and Its Applications, 2nd ed., vol. II, Wiley, New York, 1971.

[18] L.R.G. Fontes, M. Isope, C.M. Newman, Random walks with strongly inhomogeneous rates and singular diffusions: Convergence, localization, and aging in one dimension, Ann. Probab. 30 (2) (2002) 579–604.

[19] R. Gorenflo, F. Mainardi, Fractional diffusion processes: Probability distribution and continuous time random walk, Lecture Notes in Phys. 621 (2003) 148–166.

[20] F. Iglói, Anomalous diffusion in aperiodic environments, Phys. Rev. E 59 (1999) 1465–1474.

[21] N. Jacob, Pseudo-Differential Operators and Markov Processes, Akad. Verl., Berlin, 1996.

[22] Z. Jurek, J.D. Mason, Operator-Limit Distributions in Probability Theory, Wiley, New York, 1993.

[23] Y. Kasahara, A limit theorem for sums of i.i.d. random variables with slowly varying tail probability, J. Math. Kyoto Univ. 26 (3) (1986) 437–443.

[24] J. Klafter, A. Blumen, M.F. Shlesinger, Stochastic pathways to anomalous diffusion, Phys. Rev. A 35 (1987) 3081–3085.

[25] T.J. Kozubowski, M.M. Meerschaert, H.P. Scheffler, ν-operator stable laws, Publ. Math. Debrecen. 63 (4) (2003) 569–585.

[26] M.M. Meerschaert, D.A. Benson, B. Baeumer, Multidimensional advection and fractional dispersion, Phys. Rev. E 59 (1999) 5026–5028.

[27] M.M. Meerschaert, D.A. Benson, B. Baeumer, Operator Lévy motion and multiscaling anomalous diffusion, Phys. Rev. E 63 (2001) 1112–1117.

[28] M.M. Meerschaert, H.P. Scheffler, Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice, Wiley Interscience, New York, 2001.

[29] M.M. Meerschaert, D.A. Benson, H.P. Scheffler, B. Baeumer, Stochastic solution of space-time fractional diffusion equations, Phys. Rev. E 65 (2002) 1103–1106.

[30] M.M. Meerschaert, H.P. Scheffler, Limit theorems for continuous time random walks with infinite mean waiting times, J. Appl. Probab. 41 (3) (2004) 623–638.

[31] M.M. Meerschaert, H.P. Scheffler, Limit theorems for continuous time random walks with slowly varying waiting times, Statist. Probab. Lett. 71 (1) (2005) 15–22.

[32] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. 339 (2000) 1–77.
[33] E.W. Montroll, G.H. Weiss, Random walks on lattices. II, J. Mathematical Phys. 6 (1965) 167–181.
[34] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential equations, in: Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.
[35] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[36] T. Prosen, M. Znidaric, Anomalous diffusion and dynamical localization in polygonal billiards, Phys. Rev. Lett. 87 (2001) 114101–114104.
[37] P. Ressel, Semigroups in probability theory, in: H. Heyer (Ed.), Probability Measures on Groups, in: X. Proceedings of the Tenth Conference held in Oberwolfach, November 4–10, Plenum Press, New York, 1990.
[38] A.I. Saichev, G.M. Zaslavsky, Fractional kinetic equations: Solutions and applications, Chaos 7 (1997) 753–764.
[39] G. Samorodnitsky, M. Taqqu, Stable non-Gaussian Random Processes, Chapman and Hall, New York, 1994.
[40] K.I. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, 1999.
[41] S. Samko, A. Kilbas, O. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, London, 1993.
[42] H. Scher, M. Lax, Stochastic transport in a disordered solid. I. Theory, Phys. Rev. B 7 (1973) 4491–4502.
[43] H. Schiessel, I.M. Sokolov, A. Blumen, Dynamics of a polyampholyte hooked around an obstacle, Phys. Rev. E 56 (1997) R2390–R2393.
[44] M. Shlesinger, J. Klafter, Y.M. Wong, Random walks with infinite spatial and temporal moments, J. Statist. Phys. 27 (1982) 499–512.
[45] E. Seneta, Regularly Varying Functions, in: Lecture Notes Mathematics, vol. 508, Springer-Verlag, Berlin, 1976.
[46] Y.G. Sinai, The limiting behavior of a one-dimensional random walk in a random medium, Theor. Probab. Appl. 27 (1982) 256–268.
[47] I.M. Sokolov, A.V. Chechkin, J. Klafter, Fractional diffusion equation for power-law-truncated Lévy processes, Phys. A 336 (2004) 245–251.
[48] I.M. Sokolov, A.V. Chechkin, J. Klafter, Distributed-order fractional kinetics, Acta Phys. Polon. B 35 (2004) 1323.
[49] V.V. Uchaikin, V.M. Zolotarev, Chance and Stability. Stable Distributions and Their Applications, VSP, Utrecht, 1999.
[50] S. Watanabe, A limit theorem for sums of non-negative i.i.d. random variables with slowly varying tail probabilities, in: Proc. 5th International Conf. Multivariate Anal., North-Holland, Amsterdam, 1980, pp. 249–261.
[51] G. Zaslavsky, Fractional kinetic equation for Hamiltonian chaos. Chaotic advection, tracer dynamics and turbulent dispersion, Phys. D 76 (1994) 110–122.