Iterated Brownian Motion: Lifetime Asymptotics and Isoperimetric-type Inequalities

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Introduction

To define **iterated Brownian motion** Z_t, due to Burdzy (1993), started at $z \in \mathbb{R}$, let X_t^+, X_t^- and Y_t be three independent one-dimensional Brownian motions, all started at 0. **Two-sided Brownian motion** is defined to be

$$X_t = \begin{cases} X_t^+, & t \geq 0 \\ X_{(-t)}^-, & t < 0. \end{cases}$$

Then iterated Brownian motion started at $z \in \mathbb{R}$ is

$$Z_t = z + X(Y_t), \quad t \geq 0.$$

BM versus IBM: This process has many properties analogous to those of Brownian motion; we list a few

(1) Z_t has stationary (but not independent) increments, and is a **self-similar process** of index 1/4.

(2) **Laws of the iterated logarithm (LIL)** holds: usual LIL by Burdzy (1993)

$$\limsup_{t\to\infty} \frac{Z(t)}{t^{1/4}(\log\log(1/t))^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \quad a.s.$$

Chung-type LIL by Khoshnevisan and Lewis (1996) and Hu et al. (1995).

(3) Khoshnevisan and Lewis (1999) extended results of Burdzy (1994), to develop a **stochastic calculus** for iterated Brownian motion.

(4) In 1998, Burdzy and Khosnevisan showed that IBM can be used to model diffusion in a crack.

(5) Local times of this process was studied by Burdzy and Khosnevisan (1995), Csáki, Csörgö, Földes, and Révész (1996), Shi and Yor (1997), Xiao (1998), and Hu (1999).

(6) Bañuelos and DeBlassie (2006) studied the **distribution of exit place** for iterated Brownian motion in cones.

PDE-connection:

In addition to the above properties there is an interesting connection between iterated Brownian motion and the **biharmonic operator** Δ^2; the function

$$u(t,x) = E_x[f(Z_t)]$$

solves the Cauchy problem (Allouba and Zheng (2001) and DeBlassie (2004))

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \frac{\Delta f(x)}{\sqrt{2\pi t}} + \frac{1}{2}\Delta^2 u(t,x), \\ & \qquad t > 0,\ x \in \mathbb{R}^n \\ u(0,x) &= f(x), \qquad x \in \mathbb{R}^n. \end{aligned}$$

Let $\tau_D(Z)$ be the first exit time of iterated Brownian motion from a domain D, started at $z \in D$. Then $P_z[\tau_D(Z) > t]$ **provides a measure of the lifetime of iterated Brownian motion in** D**.**

In light of the above PDE connection to bi-harmonic operator, there is hope the function $u(t,x) = P_x[\tau_D(Z) > t]$ solves

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \frac{1}{2}\Delta^2 u(t,x), \ t > 0, \ x \in D, \\ u(0,x) &= 1, \qquad x \in D, \\ u(t,x) &= 0, \qquad x \in \partial D. \end{aligned}$$

But, DeBlassie (2004) established that the function

$$u(t,x) = P_x[\tau_{(0,1)}(Z) > t]$$

does not satisfy

$$\frac{\partial}{\partial t} u(t,x) = a \frac{\partial^4}{\partial x^4} u(t,x)$$

for any $a > 0$.

We follow another path to study the distribution of the exit time of iterated Brownian motion in domains in $\mathbb{R}^n$.

ITERATED PROCESSES IN UNBOUNDED DOMAINS

Let D be a domain in $\mathbb{R}^n$. Let

$$\tau_D(Z) = \inf\{t \geq 0: \; Z_t \notin D\}$$

be the first exit time of Z_t from D. Write

$$\tau_D^{\pm}(z) = \inf\{t \geq 0: \quad X_t^{\pm} + z \notin D\},$$

and if $I \subset \mathbb{R}$ is an open interval, write

$$\eta_I = \eta(I) = \inf\{t \geq 0: \quad Y_t \notin I\}.$$

By continuity of the paths of $Z_t = z + X(Y_t)$ (for f the pdf of $\tau_D^{\pm}(z)$)

$$
\begin{aligned}
& P_z[\tau_D(Z) > t] \\
&= P_z[Z_s \in D \text{ for all } s \le t] \\
&= P[z + X^+(0 \vee Y_s) \in D \text{ and } \\
&\quad z + X^-(0 \vee (-Y_s)) \in D \text{ for all } s \le t] \\
&= P[\tau_D^+(z) > 0 \vee Y_s \text{ and } \tau_D^-(z) > 0 \vee (-Y_s) \\
&\quad \text{for all } s \le t] \\
&= P[-\tau_D^-(z) < Y_s < \tau_D^+(z) \text{for all } s \le t] \\
&= P[\eta(-\tau_D^-(z), \tau_D^+(z)) > t], \\
&= \int_0^\infty \int_0^\infty P_0[\eta_{(-u,v)} > t] f(u) f(v) dv du.
\end{aligned}
$$

Let τ_D be the first exit time of the Brownian motion X_t from D. In the case of Brownian motion in **general cones**, this has been done by several people including Bañuelos and Smits (1997), Burkholder (1977) and DeBlassie (1987): for $x \in D$,

$$P_x[\tau_D > t] \sim C(x) t^{-p(D)}, \text{ as } t \to \infty.$$

When D is a generalized cone, using the results of Bañuelos and Smits, DeBlassie obtained;

Theorem. 1 (DeBlassie (2004)) *For $z \in D$, as $t \to \infty$,*

$$P_z[\tau_D(Z) > t] \approx \begin{cases} t^{-p(D)}, & p(D) < 1 \\ t^{-1} \ln t, & p(D) = 1 \\ t^{-(p(D)+1)/2}, & p(D) > 1. \end{cases}$$

Here $f \approx g$ means that for some positive C_1 and C_2, $C_1 \leq f/g \leq C_2$.

For **parabola-shaped domains** the study of exit time asymptotics for Brownian motion was initiated by Bañuelos, DeBlassie and Smits.

Theorem. 2 (Bañuelos, et al. (2001)) *Let*

$$\mathcal{P} = \{(x, y) : x > 0, |y| < \sqrt{x}\}.$$

Then for $z \in \mathcal{P}$,

$$\log \ P_z[\tau_{\mathcal{P}} > t] \approx -t^{\frac{1}{3}}$$

Subsequently, Lifshits and Shi found that the above **limit exists** for parabola-shaped domains $P_\alpha = \{(x,Y) \in \mathbb{R} \times \mathbb{R}^{n-1} : x > 0, |Y| < Ax^\alpha\}$, $0 < \alpha < 1$ and $A > 0$ in any dimension;

Theorem. 3 (Lifshits and Shi (2002)) *For* $z \in P_\alpha$,

$$\lim_{t\to\infty} t^{-(\frac{1-\alpha}{1+\alpha})} \log P_z[\tau_{P_\alpha} > t] = -l, \qquad (1)$$

where

$$l = (\frac{1+\alpha}{\alpha}) \left(L \frac{\Gamma^2(\frac{1-\alpha}{2\alpha})}{\Gamma^2(\frac{1}{2\alpha})} \right)^{\frac{\alpha}{(\alpha+1)}}. \qquad (2)$$

where

$$L = \frac{\pi \jmath_{(n-3)/2}^{2/\alpha}}{A^2 2^{(3\alpha+1)/\alpha}((1-\alpha)/\alpha)^{(1-\alpha)/\alpha}}.$$

Here $\jmath_{(n-3)/2}$ denotes the smallest positive zero of the Bessel function $J_{(n-3)/2}$ and Γ is the Gamma function.

By integration by parts $P_z[\tau_D(Z) > t]$ equals to

$$\int_0^\infty \int_0^\infty \Big(\frac{\partial}{\partial u}\frac{\partial}{\partial v} P_0[\eta_{(-u,v)} > t]\Big) .P[\tau_D(z) > u] P[\tau_D(z) > v] dv du.$$

Theorem. 4 *Let* $0 < \alpha < 1$, $A > 0$ *and let*

$$P_\alpha = \{(x, Y) \in \mathbb{R} \times \mathbb{R}^{n-1} : x > 0, |Y| < Ax^\alpha\}.$$

Then for $z \in P_\alpha$,

$$\lim_{t\to\infty} t^{-(\frac{1-\alpha}{3+\alpha})} \log P_z[\tau_{P_\alpha}(Z) > t] = -C_\alpha,$$

where for l *as in the limit given by (2)*

$$C_\alpha = (\frac{3+\alpha}{2+2\alpha})(\frac{1+\alpha}{1-\alpha})^{(\frac{1-\alpha}{3+\alpha})} \pi^{(\frac{2-2\alpha}{3+\alpha})} l^{(\frac{2+2\alpha}{3+\alpha})}.$$

In particular, for a **planar iterated Brownian motion in a parabola**, the limit $l = 3\pi^2/8$ in equation (2). Then from Theorem 4 for $z \in \mathcal{P}$,

$$\lim_{t\to\infty} t^{-\frac{1}{7}} \log P_z[\tau_{\mathcal{P}}(Z) > t] = -\frac{7\pi^2}{2^{25/7}}.$$

ITERATED PROCESSES IN BOUNDED DOMAINS

For many bounded domains $D \subset \mathbb{R}^n$ the asymptotics of $P_z[\tau_D > t]$ is well-known. For $z \in D$,

$$\lim_{t\to\infty} e^{\lambda_D t} P_z[\tau_D > t] = \psi(z) \int_D \psi(y) dy, \qquad (3)$$

where λ_D is the first eigenvalue of $\frac{1}{2}\Delta$ with Dirichlet boundary conditions and ψ is its corresponding eigenfunction.

DeBlassie proved the following result for iterated Brownian motion in bounded domains;

Theorem. 5 (DeBlassie (2004)) *For $z \in D$,*

$$\lim_{t\to\infty} t^{-1/3} \log P_z[\tau_D(Z) > t] = -\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}. \qquad (4)$$

We have the following theorem which improves the limit in (4).

Theorem. 6 *Let $D \subset \mathbb{R}^n$ be a bounded domain for which (3) holds point-wise and let λ_D and ψ be as above. Then for $z \in D$,*

$$\lim_{t\to\infty} t^{-1/2} \exp\left(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}\right) P_z[\tau_D(Z) > t]$$

$$= \frac{\lambda_D 2^{7/2}}{\sqrt{3\pi}} \left(\psi(z)\int_D \psi(y)dy\right)^2.$$

Ingredients of the proof of Theorem 6 It turns out that the integral over the set A is the dominant one: $K > 0$ and $M > 0$ define A as

$$A = \left\{ (u,v): \quad K \le u \le \frac{1}{2}\sqrt{\frac{t}{M}},\ u \le v \le \sqrt{\frac{t}{M}} - u \right\}.$$

As $t \to \infty$, uniformly for $x \in (0,1)$,

$$P_x[\eta_{(0,1)} > t] \ \sim\ \frac{4}{\pi} e^{-\frac{\pi^2 t}{2}} \sin \pi x.$$

We use **Laplace transform method for integrals** (de Bruijn (1958)). Let h and f be continuous functions on $\mathbb{R}$. Suppose f is non-positive and has a global max at x_0, $f'(x_0) = 0$, $f''(x_0) < 0$ and $h(x_0) \neq 0$ and

$$\int_0^\infty h(x) \exp(\lambda f(x)) < \infty$$

for all $\lambda > 0$. Then as $\lambda \to \infty$,

$$\int_0^\infty h(x) \exp(\lambda f(x)) dx$$

$$\sim\ h(x_0) \exp(\lambda f(x_0)) \sqrt{\frac{2\pi}{\lambda |f''(x_0)|}}.$$

$$
\begin{aligned}
&P_z[\tau_D(Z) > t] \\
&= \int_0^\infty \int_0^\infty P_{\frac{u}{u+v}}[\eta_{(0,1)} > \frac{t}{(u+v)^2}] f(u) f(v)\, dv du \\
&\geq C^1 \int_K^{\frac{1}{2}\sqrt{t/M}} \int_u^{\sqrt{t/M}-u} \sin\left(\frac{\pi u}{(u+v)}\right) \\
&\qquad . \exp(-\frac{\pi^2 t}{2(u+v)^2}) \exp(-\lambda_D (u+v))\, dv du,
\end{aligned}
$$

where $C^1 = C^1(z) = 2(4/\pi) A(z)^2 (1-\epsilon)^3$. Changing the variables $x = u+v, z = u$ the integral is

$$
\begin{aligned}
= C^1 \int_K^{\frac{1}{2}\sqrt{t/M}} \int_{2z}^{\sqrt{t/M}} \sin\left(\frac{\pi z}{x}\right) \exp(-\frac{\pi^2 t}{2x^2}) \\
. \exp(-\lambda_D x)\, dx dz,
\end{aligned}
$$

and reversing the order of integration

$$
\begin{aligned}
&= C^1 \int_{2K}^{\sqrt{t/M}} \int_K^{\frac{1}{2}x} \sin\left(\frac{\pi z}{x}\right) \exp(-\frac{\pi^2 t}{2x^2}) \\
&\qquad . \exp(-\lambda_D x)\, dz dx \\
&= C^1/\pi \int_{2K}^{\sqrt{t/M}} x \cos\left(\frac{\pi K}{x}\right) \exp(-\frac{\pi^2 t}{2x^2}) \\
&\qquad . \exp(-\lambda_D x)\, dx
\end{aligned}
$$

By Laplace transform method, after making the change of variables $x = (atb^{-1})^{1/3}u$, for $a = \pi^2/2$, $b = \lambda_D$. As $t \to \infty$,

$$\begin{aligned}
&\int_0^\infty x \cos\left(\frac{\pi K}{x}\right) \exp(-\frac{\pi^2 t}{2x^2}) \exp(-\lambda_D x) dx \\
= &\int_0^\infty (atb^{-1})^{1/3} u \cos\left(\frac{\pi K}{(atb^{-1})^{1/3} u}\right) \\
&. \exp\left(-a^{1/3} b^{2/3} t^{1/3} (\frac{1}{u^2} + u)\right) (atb^{-1})^{1/3} du \\
\sim\ & 2\sqrt{\frac{\pi}{3}} (\frac{\pi^2}{2})^{1/2} \lambda_D^{-1} t^{1/2} \exp(-\frac{3}{2} \pi^{2/3} \lambda_D^{2/3} t^{1/3}).
\end{aligned}$$

Above x_0 in the Laplace Transform method is $2^{1/3}$.

ISOPERIMETRIC-TYPE INEQUALITIES

Let $D \subset \mathbb{R}^n$ be a **domain of finite volume**, and denote by D^* the ball in $\mathbb{R}^n$ centered at the origin with same volume as D. **The class of quantities related to the Dirichlet Laplacian in D which are maximized or minimized by the corresponding quantities for D^* are often called generalized isoperimetric-type inequalities (C. Bandle (1980)).**

Probabilistically generalized isoperimetric-type inequalities read as

$$P_z[\tau_D > t] \;\leq\; P_0[\tau_{D^*} > t] \qquad (5)$$

for all $z \in D$ and all $t > 0$, where τ_D is the first exit time of Brownian motion from the domain D and P_z is the associated probability measure when this process starts at z.

Theorem. 7 *Let $D \subset \mathbb{R}^n$ be an open set of finite volume. Then*

$$P_z[\tau_D(Z) > t] \;\leq\; P_0[\tau_{D^*}(Z) > t] \qquad (6)$$

for all $z \in D$ and all $t > 0$.

Proof of Theorem 7

The idea of the proof is to use **integration by parts and the corresponding generalized isoperimetric-type inequalities for Brownian motion.** Let f^* denote the probability density of τ_{D^*}.

$$G_x(u,v,t) = \left(\frac{\partial}{\partial x} P_0[\eta_{(-u,v)} > t]\right).$$

By integration by parts $P_z[\tau_D(Z) > t]$ equals

$$\begin{aligned}
& \int_0^\infty \int_0^\infty P_0[\eta_{(-u,v)} > t] f(u) f(v) dv du. \\
= & \int_0^\infty \int_0^\infty G_v(u,v,t) P[\tau_D(z) > v] f(u) dv du \\
\leq & \int_0^\infty \int_0^\infty G_v(u,v,t) P[\tau_{D^*}(0) > v] f(u) dv du \\
= & \int_0^\infty \int_0^\infty P_0[\eta_{(-u,v)} > t] f(u) f^*(v) dv du \\
= & \int_0^\infty \int_0^\infty G_u(u,v,t) P[\tau_D(z) > u] f^*(v) du dv \\
\leq & \int_0^\infty \int_0^\infty G_u(u,v,t) P[\tau_{D^*}(0) > u] f^*(v) du dv \\
= & P_0[\tau_{D^*}(Z) > t]
\end{aligned}$$