SUBORDINATED PROCESSES AND CAUCHY PROBLEMS

by

ERKAN NANE

DEPARTMENT OF STATISTICS AND PROBABILITY
MICHIGAN STATE UNIVERSITY

OUTLINE

- Introduction and history
- Brownian subordinators
- Cauchy problems on bounded domains
- Other subordinators
- Scaling Limits
- Conclusion and open problems

INTRODUCTION AND HISTORY

In recent years, starting with the articles of Burdzy (1993) and (1994), researchers had interest in iterated processes in which one changes the time parameter with one-dimensional Brownian motion.

To define **iterated Brownian motion** Z_t, due to Burdzy (1993), started at $z \in \mathbb{R}$, let X_t^+, X_t^- and Y_t be three independent one-dimensional Brownian motions, all started at 0. **Two-sided Brownian motion** is defined to be

$$X_t = \begin{cases} X_t^+, & t \geq 0 \\ X_{(-t)}^-, & t < 0. \end{cases}$$

Then iterated Brownian motion started at $z \in \mathbb{R}$ is

$$Z_t = z + X_{(Y_t)}, \quad t \geq 0.$$

BM versus IBM: This process has many properties analogous to those of Brownian motion; we list a few

(1) Z_t has stationary (but not independent) increments, and is a **self-similar process** of index $1/4$.

(2) **Laws of the iterated logarithm (LIL)** holds: usual LIL by Burdzy (1993)

$$\limsup_{t\to\infty} \frac{Z(t)}{t^{1/4}(\log\log(1/t))^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \quad a.s.$$

Chung-type LIL by Khoshnevisan and Lewis (1996) and Hu et al. (1995).

(3) Khoshnevisan and Lewis (1999) extended results of Burdzy (1994), to develop a **stochastic calculus** for iterated Brownian motion.

(4) In 1998, Burdzy and Khoshnevisan showed that IBM can be used to model diffusion in a crack.

(5) Local time of this process was studied by Burdzy and Khoshnevisan (1995), Csáki, Csörgö, Földes, and Révész (1996), Shi and Yor (1997), Xiao (1998), and Hu (1999).

(6) Bañuelos and DeBlassie (2006) studied the **distribution of exit place** for iterated Brownian motion in cones.

(7) DeBlassie (2004) studied the lifetime asymptotics of iterated Brownian motion in cones and Bounded domains. Nane (2006), in a series of papers, extended some of the results of DeBlassie. He also studied the lifetime asymtotics of iterated Brownian motion in several unbounded domains(parabola-shaped domains, twisted domains...).

(8) Khoshnevisan and Lewis (1996) established the modulus of continuity for iterated Brownian motion: with probability one

$$\lim_{\delta \to 0} \sup_{0 \le s,t \le 1} \sup_{0 \le |s-t| \le \delta} \frac{|Z(s) - Z(t)|}{\delta^{1/4}(\log(1/\delta))^{3/4}} = 1.$$

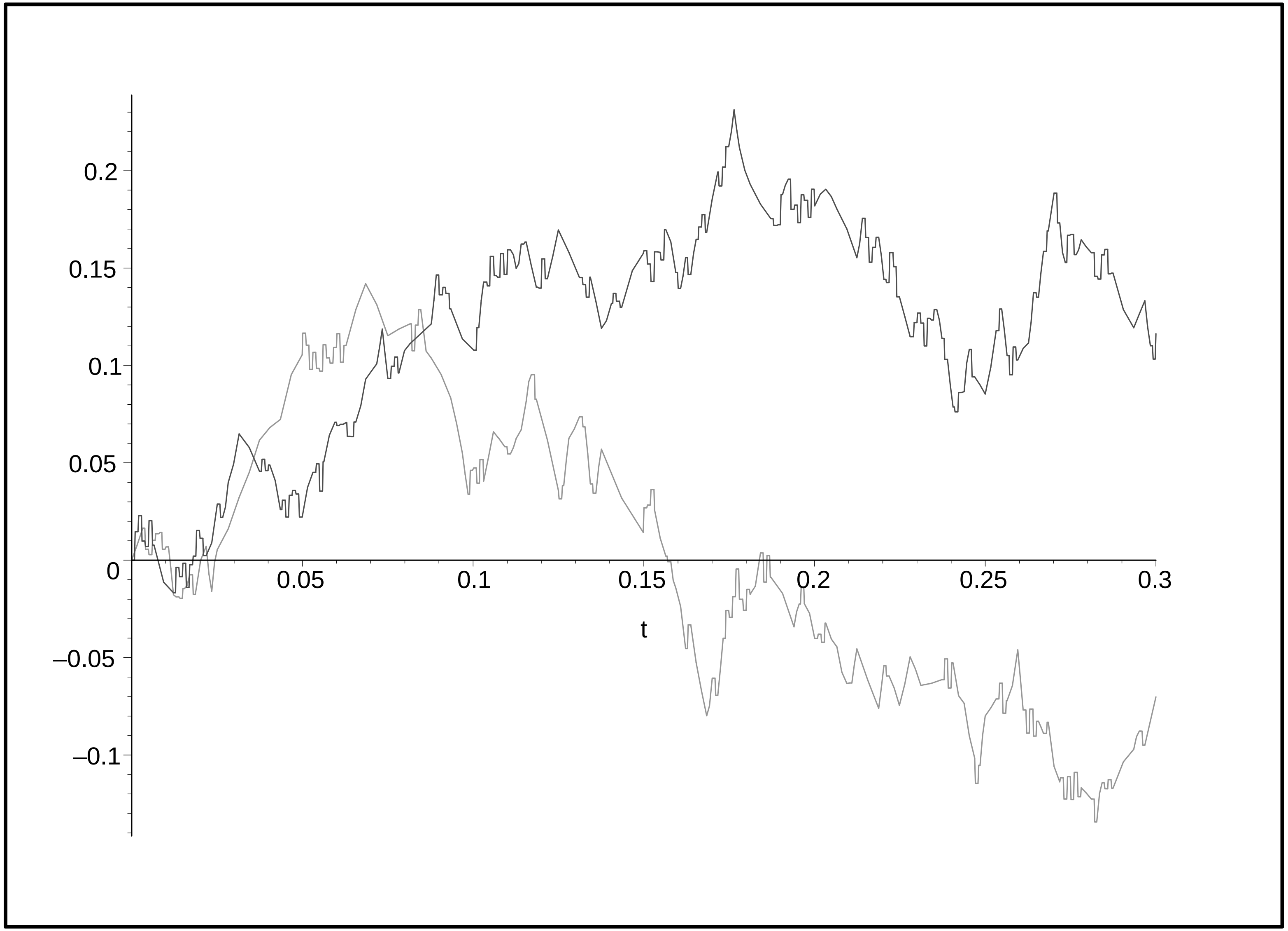

Figure 1: Simulations of two Brownian motions

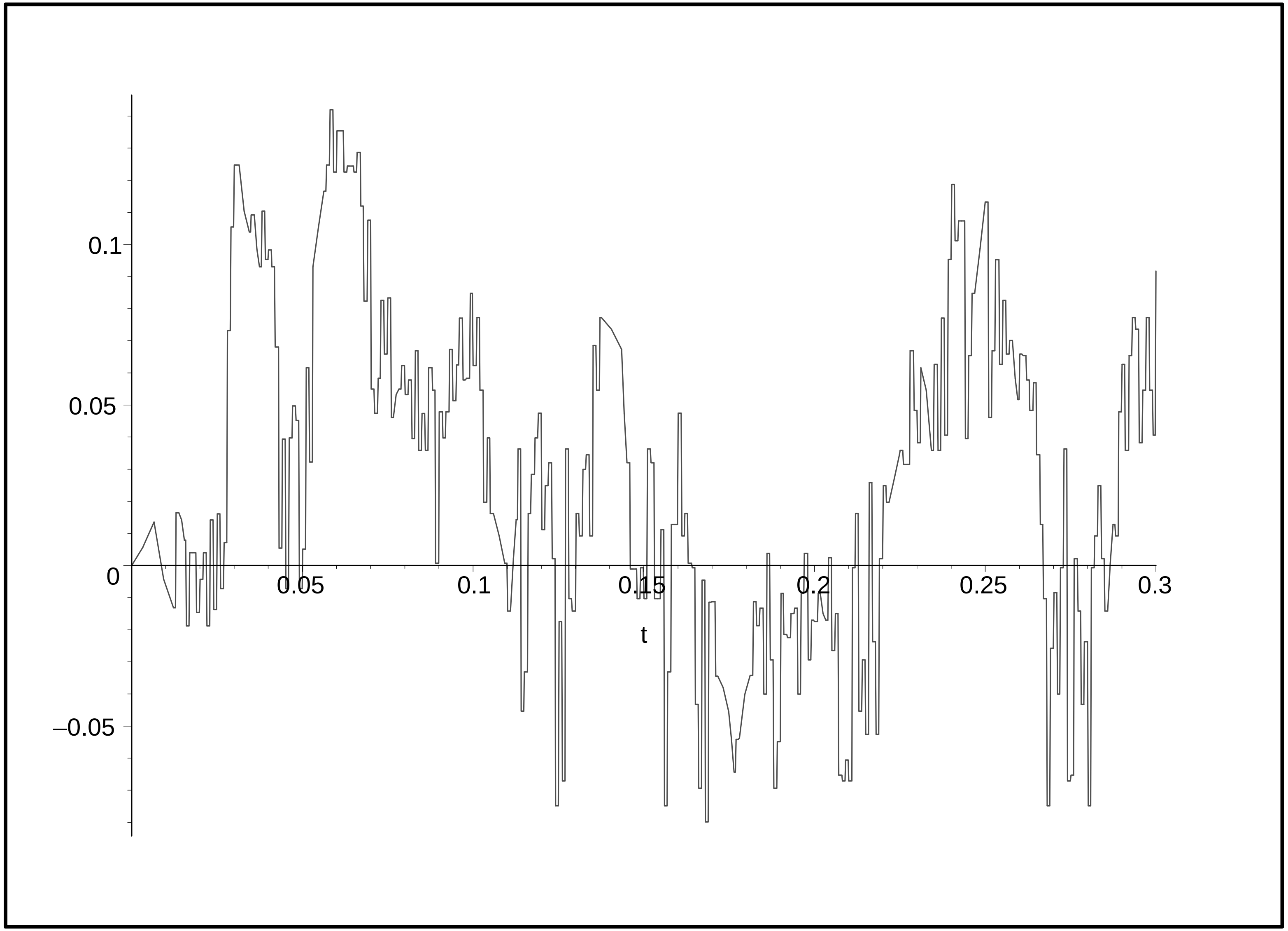

Figure 2: Simulation of IBM $Z_t^1 = X(|Y_t|)$

PDE connection

The classical well-known connection of a PDE and a stochastic process is the Brownian motion and heat equation connection. Let $X_t \in \mathbb{R}^d$ be Brownian motion started at x. Then the function

$$u(t,x) = E_x[f(X_t)]$$

solves the Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \Delta u(t,x), \qquad t > 0, \;\; x \in \mathbb{R}^d \\ u(0,x) &= f(x), \qquad x \in \mathbb{R}^d. \end{aligned}$$

In addition to the above properties of IBM there is an interesting connection between iterated Brownian motion and the **biharmonic operator** Δ^2; the function

$$u(t,x) = E_x[f(Z_t)]$$

solves the Cauchy problem (Allouba and Zheng (2001) and DeBlassie (2004))

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \frac{\Delta f(x)}{\sqrt{\pi t}} + \Delta^2 u(t,x); \\ u(0,x) &= f(x). \end{aligned} \tag{1}$$

for $t > 0$ and $x \in \mathbb{R}^d$. The non-Markovian property of IBM is reflected by the appearance of the initial function $f(x)$ in the PDE.

Let $q(t,s) = \frac{2}{\sqrt{4\pi t}} \exp\left(-\frac{s^2}{4t}\right)$ be the transition density of reflected one-dimensional Brownian motion, $|B_t|$.

The essential argument, using integration by parts twice, is that

$$
\begin{aligned}
&\tfrac{\partial}{\partial t}u(t,x) \\
&\qquad = \int_0^\infty T(s)f(x)\,\frac{\partial}{\partial t}q(t,s)ds \\
&\qquad = \int_0^\infty T(s)f(x)\,\frac{\partial^2}{\partial s^2}q(t,s)ds \\
&\qquad = q(t,s)\frac{\partial}{\partial s}\left[T(s)f(x)\right]\Big|_{s=0} \\
&\qquad + \int_0^\infty \frac{\partial^2}{\partial s^2}\left[T(s)f(x)\right]q(t,s)ds \\
&\qquad = q(t,0)L_x[T(0)f(x)] + \int_0^\infty L_x^2\left[T(s)f(x)\right]q(t,s)ds \\
&\qquad = \frac{1}{\sqrt{\pi t}}L_x f(x) + L_x^2\int_0^\infty T(s)f(x)\,q(t,s)ds
\end{aligned}
$$

Fractional Diffusion

Nigmatullin (1986) gave a Physical derivation of fractional diffusion

$$\frac{\partial^\beta}{\partial t^\beta} u(t,x) = L_x u(t,x); \quad u(0,x) = f(x) \tag{2}$$

where $0 < \beta < 1$ and L_x is the generator of some continuous Markov process $X_0(t)$ started at $x = 0$. Here $\partial^\beta g(t)/\partial t^\beta$ is the Caputo fractional derivative in time, which can be defined as the inverse Laplace transform of $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$, with $\tilde{g}(s) = \int_0^\infty e^{-st} g(t) dt$ the usual Laplace transform.

Zaslavsky (1994) used this to model Hamiltonian chaos.

Stochastic solution

Baeumer and Meerschaert (2001) and Meerschaert and Scheffler (2004) shows that, in the case $p(t, x) = T(t)f(x)$ is a bounded continuous semigroup on a Banach space (with corresponding process X_t, $E_t = \inf\{u : D_u > t\}$, D_t is a stable subordinator with index β) , the formula

$$u(t, x) = E_x(f(X_{E_t})) = \frac{t}{\beta} \int_0^{\infty} p(s, x) g_\beta(\frac{t}{s^{1/\beta}}) s^{-1/\beta - 1} ds$$

yields a solution to the **fractional Cauchy problem**:

$$\frac{\partial^\beta}{\partial t^\beta} u(t, x) = L_x u(t, x); \quad u(0, x) = f(x) \tag{3}$$

Here $g_\beta(t)$ is the smooth density of the stable subordinator, such that the Laplace transform $\tilde{g}_\beta(s) = \int_0^\infty e^{-st} g_\beta(t)\, dt = e^{-s^\beta}$.

BROWNIAN SUBORDINATORS

(Baeumer, Meerschaert and Nane (2007)) show that, taking $\beta = 1/2$ in the time-fractional diffusion yields exactly 1-D distributions $X_{|B_t|} \stackrel{d}{=} X_{E_t}$.

If L_x is the generator of the semigroup $T(t)f(x) = E_x[(f(X_t))]$ on $L^1(\mathbb{R}^d)$, then for any $f \in D(L_x)$

$$\frac{\partial}{\partial t}u(t,x) = \frac{L_x f(x)}{\sqrt{\pi t}} + {L_x}^2 u(t,x); \quad u(0,x) = f(x), \qquad (4)$$

and the fractional Cauchy problem (3) with $\beta = 1/2$ have the same solution

$$u(t,x) = E_x[f(X(|B_t|))] = \frac{2}{\sqrt{4\pi t}} \int_0^\infty T(s)f(x) \exp\left(-\frac{s^2}{4t}\right) ds.$$

Fourier-Laplace method

The Lévy process $X_0(t)$ has characteristic function

$$E[\exp(ik \cdot X_0(t))] = \exp(t\psi(k))$$

with

$$\psi(k) = ik \cdot a - \frac{1}{2} k \cdot Qk + \int_{y \neq 0} \left(e^{ik \cdot y} - 1 - \frac{ik \cdot y}{1 + ||y||^2} \right) \nu(dy),$$

where $a \in \mathbb{R}^d$, Q is a nonnegative definite matrix, and ν is a σ-finite Borel measure on $\mathbb{R}^d$ such that

$$\int_{y \neq 0} \min\{1, ||y||^2\} \nu(dy) < \infty.$$

Denote the Fourier transform by

$$\hat{f}(k) = \int_{\mathbb{R}^d} e^{-ik \cdot x} f(x)\, dx$$

Meerschaert and Scheffler (2001) shows that $L_x f(x)$ is the inverse Fourier transform of $\psi(k)\hat{f}(k)$ for all $f \in D(L_x)$, where

$$D(L_x) = \{f \in L^1(\mathbb{R}^d) : \ \psi(k)\hat{f}(k) = \hat{h}(k) \, \exists \, h \in L^1(\mathbb{R}^d)\},$$

and

$$\begin{aligned} L_x f(x) &= a \cdot \nabla f(x) + \frac{1}{2}\nabla \cdot Q \nabla f(x) \\ &\quad + \int_{y \neq 0} \left(f(x+y) - f(x) - \frac{\nabla f(x) \cdot y}{1+y^2} \right) \nu(dy) \end{aligned} \tag{5}$$

for all $f \in W^{2,1}(\mathbb{R}^d)$, the Sobolev space of L^1-functions whose first and second partial derivatives are all L^1-functions.

We can also write $L_x = \psi(-i\nabla)$ where $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_d)'$.

For example, if $X_0(t)$ is spherically symmetric stable then $\psi(k) = -D\|k\|^\alpha$ and $L_x = -D(-\Delta)^{\alpha/2}$, a fractional derivative in space, using the correspondence $k_j \to -i\partial/\partial x_j$ for $1 \leq j \leq d$.

If X_0 has independent stable marginals, then one possible form is $\psi(k) = D\sum_j (ik_j)^{\alpha_j}$ and $L_x = D\sum_j \partial^{\alpha_j}/\partial x_j^\alpha$ using Riemann-Liouville fractional derivatives in each variable. This form does not coincide with the fractional Laplacian unless all $\alpha_j = 2$.

The proof does not use Theorem 0.1 in Allouba and Zheng (2001), rather it relies on a Laplace-Fourier transform argument.

We will use the following notation for the Laplace, Fourier, and Fourier-Laplace transforms (respectively):

$$\tilde{u}(s,x) = \int_0^\infty e^{-st} u(t,x)\,dt;$$

$$\hat{u}(t,k) = \int_{\mathbb{R}^d} e^{-ik\cdot x} u(t,x)\,dx;$$

$$\bar{u}(s,k) = \int_{\mathbb{R}^d} e^{-ik\cdot x} \int_0^\infty e^{-st} u(t,x)\,dt\,dx.$$

Let ψ be the characteristic exponent of X_t. Take Fourier transforms on both sides of (4) to get

$$\frac{\partial \hat{u}(t,k)}{\partial t} = \frac{1}{\sqrt{\pi t}} \psi(k) \hat{f}(k) + \psi(k)^2 \hat{u}(t,k)$$

using the fact that $\psi(k)\hat{f}(k)$ is the Fourier transform of $L_x f(x)$. Then take Laplace transforms on both sides to get

$$s\bar{u}(s,k) - \hat{u}(t=0,k) = s^{-1/2}\psi(k)\hat{f}(k) + \psi(k)^2 \bar{u}(s,k),$$

using the well-known Laplace transform formula

$$\int_0^\infty \frac{t^{-\beta}}{\Gamma(1-\beta)} e^{-st} dt = s^{\beta-1}, \quad \beta < 1.$$

Since $\hat{u}(t=0,k)=\hat{f}(k)$, collecting like terms yields

$$\bar{u}(s,k)=\frac{(1+s^{-1/2}\psi(k))\hat{f}(k)}{s-\psi(k)^2} \tag{6}$$

for $s>0$ sufficiently large.

On the other hand, taking Fourier transforms on both sides of (3) with $\beta=1/2$ gives

$$\frac{\partial^{1/2}\hat{u}(t,k)}{\partial t^{1/2}}=\psi(k)\hat{u}(t,k)$$

Take Laplace transforms on both sides, using the fact that $s^{\beta}\tilde{g}(s)-s^{\beta-1}g(0)$ is the Laplace transform of the Caputo fractional derivative $\partial^{\beta}g(t)/\partial t^{\beta}$, to get

$$s^{1/2}\bar{u}(s,k)-s^{-1/2}\hat{f}(k)=\psi(k)\bar{u}(s,k)$$

and collect terms to obtain

$$\begin{aligned}\bar{u}(s,k) &= \frac{s^{-1/2}\hat{f}(k)}{s^{1/2}-\psi(k)} \\ &= \frac{s^{-1/2}\hat{f}(k)}{s^{1/2}-\psi(k)} \cdot \frac{s^{1/2}+\psi(k)}{s^{1/2}+\psi(k)} \\ &= \frac{(1+s^{-1/2}\psi(k))\hat{f}(k)}{s-\psi(k)^2}\end{aligned} \tag{7}$$

which agrees with (6). For any fixed $k \in \mathbb{R}^d$, the two formulae are well-defined and equal for all $s > 0$ sufficiently large.

An easy extension of the argument as above shows that, under the same conditions, for any $k = 2, 3, 4, \ldots$ both the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} = \sum_{j=1}^{k-1} \frac{t^{1-j/k}}{\Gamma(j/k)} L_x^j f(x) + L_x^k u(t,x); \qquad (8)$$
$$u(0,x) = f(x)$$

and the fractional Cauchy problem:

$$\frac{\partial^{1/k}}{\partial t^{1/k}} u(t,x) = L_x u(t,x); \quad u(0,x) = f(x), \qquad (9)$$

have the same unique solution given by

$$u(t,x) = \int_0^\infty p((t/s)^\beta, x) g_\beta(s)\, ds$$

with $\beta = 1/k$. Hence the process $Z_t = X(E_t)$ is also the stochastic solution to this higher order Cauchy problem.

Orsingher and Benghin (2004) and (2008) show that for $\beta = 1/2^n$ the solution to

$$\frac{\partial^{1/2^n}}{\partial t^{1/2^n}} u(t,x) = \Delta_x u(t,x); \quad u(0,x) = f(x), \qquad (10)$$

is given by running

$$I_n(t) = B_1(|B_2(|B_3(|\cdots(B_{n+1}(t))\cdots|)|)|)$$

Where B_j's are independent Brownian motions, i.e., $u(t,x) = E_x(f(I_n(t)))$ solves (10), and solves (8) for $k = 2^n$.

CAUCHY PROBLEMS ON BOUNDED DOMAINS

Since we are working on a bounded domain, the Fourier transform methods are not useful. Instead we will employ Hilbert space methods. Hence, given a complete orthonormal basis $\{\psi_n(x)\}$ on $L^2(D)$, we will call

$$\bar{u}(t,n) = \int_D \psi_n(x) u(t,x) dx;$$
$$\hat{u}(s,n) = \int_D \psi_n(x) \int_0^\infty e^{-st} u(t,x) dt dx = \int_D \psi_n(x) \tilde{u}(s,x) dx.$$

the ψ_n, and ψ_n-Laplace transforms, respectively.

Let D be bounded and every point of ∂D be regular for D^C. The corresponding Markov process is a killed Brownian motion. We denote the eigenvalues and the eigenfunctions of Δ_D by

$\{\lambda_n, \phi_n\}_{n=1}^{\infty}$, where $\phi_n \in C^{\infty}(D)$. The corresponding heat kernel is given by

$$p_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y).$$

The series converges absolutely and uniformly on $[t_0, \infty) \times D \times D$ for all $t_0 > 0$. In this case, the semigroup given by

$$\begin{aligned} T_D(t) f(x) &= E_x[f(X_t) I(t < \tau_D(X))] = \int_D p_D(t, x, y) f(y) dy \\ &= \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \bar{f}(n) \end{aligned}$$

solves the Heat equation in D with Dirichlet boundary conditions:

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x), \;\; x \in D, \; t > 0, \\ u(t,x) &= 0, \;\; x \in \partial D, \\ u(0,x) &= f(x), \;\; x \in D. \end{aligned}$$

Fractional Cauchy problems in bounded domains

Let $\beta \in (0,1)$, $D_\infty = (0,\infty) \times D$ and define

$$\mathcal{H}_\Delta(D_\infty) \equiv \Big\{ u : D_\infty \to \mathbb{R} : \ \frac{\partial}{\partial t}u, \frac{\partial^\beta}{\partial t^\beta}u, \Delta u \in C(D_\infty), \left|\frac{\partial}{\partial t}u(t,x)\right| \le g(x)t^{\beta-1}, g \in L^\infty(D),\ t > 0 \Big\}.$$

Let $0 < \gamma < 1$. Let D be a bounded domain with $\partial D \in C^{1,\gamma}$, and $T_D(t)$ be the killed semigroup of Brownian motion $\{X_t\}$ in D. Let E_t be the process inverse to a stable subordinator of index $\beta \in (0,1)$ independent of $\{X_t\}$. Let $f \in D(\Delta_D) \cap C^1(\bar{D}) \cap C^2(D)$ for which the eigenfunction expansion (of Δf) with respect to the complete orthonormal basis $\{\phi_n :\ n \in \mathbb{N}\}$

converges uniformly and absolutely. Then the unique (classical) solution of

$$\begin{aligned} u &\in \mathcal{H}_\Delta(D_\infty) \cap C_b(\bar{D}_\infty) \cap C^1(\bar{D}) \\ \frac{\partial^\beta}{\partial t^\beta} u(t,x) &= \Delta u(t,x); \;\; x \in D, \; t > 0 \\ u(t,x) &= 0, \; x \in \partial D, \; t > 0, \\ u(0,x) &= f(x), \; x \in D. \end{aligned} \tag{11}$$

is given by

$$
\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)E_\beta(-\lambda_n t^\beta) \\
&= E_x[f(X(E_t))I(\tau_D(X) > E_t)] \\
&= E_x[f(X(E_t))I(\tau_D(X(E)) > t)] \\
&= \frac{t}{\beta}\int_0^\infty T_D(l)f(x)g_\beta(tl^{-1/\beta})l^{-1/\beta-1}dl \\
&= \int_0^\infty T_D((t/l)^\beta)f(x)g_\beta(l)dl.
\end{aligned}
$$

$\boxed{\text{Joint work with Meerschaert and Vellaisamy (2008).}}$

Analytic solution in intervals $(0, M) \subset \mathbb{R}$ was obtained by Agrawal (2002).

Assume that $u(t,x)$ solves (11). Taking ϕ_n- transforms in (11) we obtain

$$\frac{\partial^\beta}{\partial t^\beta}\bar{u}(t,n) = -\lambda_n \bar{u}(t,n). \tag{12}$$

taking Laplace transforms on both sides of (12), we get

$$s^\beta \hat{u}(s,n) - s^{\beta-1}\bar{u}(0,n) = -\lambda_n \hat{u}(s,n) \tag{13}$$

which leads to

$$\hat{u}(s,n) = \frac{\bar{f}(n)s^{\beta-1}}{s^\beta + \lambda_n}. \tag{14}$$

By inverting the above Laplace transform, we obtain

$$\bar{u}(t,n) = \bar{f}(n)E_\beta(-\lambda_n t^\beta)$$

in terms of the Mittag-Leffler function defined by

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}.$$

Inverting now the ϕ_n-transform, we get an L^2-convergent solution of Equation (11) as (for each $t \geq 0$)

$$u(t,x) = \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)E_\beta(-\lambda_n t^\beta) \tag{15}$$

IBM in bounded domains

Let $Z_t = X(|Y_t|)$ be the iterated Brownian motion, $D_\infty = (0,\infty) \times D$ and define

$$\mathcal{H}_{\Delta^2}(D_\infty) \equiv \left\{ u : D_\infty \to \mathbb{R} : \ \frac{\partial}{\partial t} u, \Delta^2 u \in C(D_\infty), \Delta u \in C^1(\bar{D}) \right., \\ \left. \left| \frac{\partial}{\partial t} u(t,x) \right| \leq g(x) t^{-1/2}, g \in L^\infty(D), \ t > 0 \right\}.$$

Let D be a domain with $\partial D \in C^{1,\gamma}, 0 < \gamma < 1$. Let $\{X_t\}$ be Brownian motion in $\mathbb{R}^d$, and $\{Y_t\}$ be an independent Brownian motion in $\mathbb{R}$. Let $\{E_t\}$ be the process inverse to a stable subordinator of index $\beta = 1/2$ independent of $\{X_t\}$. Let $f \in D(\Delta_D) \cap C^1(\bar{D}) \cap C^2(D) (\subset L^2(D))$ be such that the eigenfunction expansion of Δf with respect to $\{\phi_n : \ n \geq 1\}$ converges absolutely and uniformly. Then the (classical) solution of

$$\begin{aligned} u &\in \mathcal{H}_{\Delta^2}(D_\infty) \cap C_b(\bar{D}_\infty) \cap C^1(\bar{D}); \\ \frac{\partial}{\partial t} u(t,x) &= \frac{\Delta f(x)}{\sqrt{\pi t}} + \Delta^2 u(t,x), \ \ x \in D, \ t > 0; \\ u(t,x) &= \Delta u(t,x) = 0, \ t \geq 0, \ x \in \partial D; \\ u(0,x) &= f(x), \ x \in D \end{aligned} \tag{16}$$

is given by

$$\begin{aligned} u(t,x) &= E_x[f(Z_t)I(\tau_D(X) > |Y_t|)] \\ &= E_x[f(X(E_t))I(\tau_D(X) > E_t)] \\ &= E_x[f(X(E_t))I(\tau_D(X(E)) > t)] \\ &= 2\int_0^\infty T_D(l)f(x)p(t,l)dl, \end{aligned} \quad (17)$$

where $T_D(l)$ is the heat semigroup in D, and $p(t,l)$ is the transition density of one-dimensional Brownian motion $\{Y_t\}$.

Proof uses again equivalence with fractional Cauchy problem for $\beta = 1/2$.

OTHER SUBORDINATORS

A Lévy process $S = \{S_t, t \geq 0\}$ with values in $\mathbb{R}$ is called *strictly stable of index* $\alpha \in (0, 2]$ if its characteristic function is given by

$$\mathbb{E}\big[\exp(i\xi S_t)\big] = \exp\left(-t|\xi|^{\alpha}\frac{1 + i\nu\operatorname{sgn}(\xi)\tan(\frac{\pi\alpha}{2})}{\chi}\right), \quad (18)$$

where $-1 \leq \nu \leq 1$ and $\chi > 0$ are constants. When $\alpha = 2$ and $\chi = 2$, S is Brownian motion.

For any Borel set $I \subseteq \mathbb{R}$, the occupation measure of S on I is defined by

$$\mu_I(A) = \lambda_1\{t \in I : \;\; S_t \in A\} \tag{19}$$

for all Borel sets $A \subseteq \mathbb{R}$, where λ_1 is the one-dimensional Lebesgue measure. If μ_I is absolutely continuous with respect to the Lebesgue measure λ_1 on $\mathbb{R}$, we say that S has a local time on I and define its *local time* $L(x, I)$ to be the Radon-Nikodým derivative of μ_I with respect to λ_1, i.e.,

$$L(x, I) = \frac{d\mu_I}{d\lambda_1}(x), \qquad \forall x \in \mathbb{R}.$$

In the above, x is the so-called *space variable*, and I is the *time variable* of the local time. If $I = [0, t]$, we will write $L(x, I)$ as $L(x, t)$. Moreover, if $x = 0$ then we will simply write $L(0, t)$ as L_t.

Local time as a subordinator

It is well-known (see, e.g. Bertoin (1996)) that a strictly stable Lévy process S has a local time if and only if $\alpha \in (1, 2]$.

It is well-known (see, e.g. Bertoin (1996)) that the inverse of a local time L_t of S is a stable subordinator:

$G_t = \inf\{u : \ L_u > t\}$, then $G_t = \rho D_t$, where D_t is a stable subordinator of index $\beta = 1 - 1/\alpha$.

$\rho = \pi^{-1}\Gamma(1+1/\alpha)\Gamma(1-1/\alpha)\chi^{1/\alpha}Re\{[1+i\nu\tan(\pi\alpha/2)]^{-1/\alpha}\}$.

Hence $L_t = E_{t/\rho}$ where $\beta = 1 - 1/\alpha$ for some $c > 0$.

For $\beta = 1 - 1/\alpha$, $c > 0$, $u(t, x) = E_x[f(X(L_t))]$ solves

$$\frac{\partial^\beta}{\partial t^\beta}u(t, x) = cL_x u(t, x); \quad u(0, x) = f(x) \qquad (20)$$

Symmetric stable subordinators

α-time process is a Markov process subordinated to the absolute value of an independent one-dimensional symmetric α-stable process:

$Z_t = X(|S_t|)$, where X_t is a Markov process and S_t is an independent symmetric α-stable process both started at 0.

This process is self similar with index $1/2\alpha$ when the outer process X is a Brownian motion. In this case Nane (2006) defined the Local time of this process and obtained Laws of the iterated logarithm for the local time for large time.

PDE-connection:

Theorem 3 [Nane (2008)]

Let $T(s)f(x) = E_x[f(X(s))]$ be the semigroup of the continuous Markov process $X(t)$ and let L_x be its generator. Let $\alpha = 1$. Let f be a bounded measurable function in the domain of L_x, with $D_{ij}f$ bounded and Hölder continuous for all $1 \le i,\ j \le n$. Then $u(t,x) = E_x[f(Z_t)]$ solves

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(t,x) &= -\frac{2L_x f(x)}{\pi t} - L_x^2 u(t,x); \\ u(0,x) &= f(x). \end{aligned}$$

For $\alpha = l/m \neq 1$ rational: the PDE is more complicated since kernels of symmetric α-stable processes satisfy a higher order PDE:

$$\left[\frac{\partial^{2l}}{\partial s^{2l}} + (-1)^{l+1}\frac{\partial^{2m}}{\partial t^{2m}}\right] p_t^{\alpha}(0, s) = 0.$$

We also have to assume that we can integrate under the integral as much as we need in the case where the outer process is BM (or in general we can take the operator out of the integral). This is valid for $\alpha = 1/m$, $m = 2, 3, \cdots$ by a Lemma in Nane (2008).

Theorem 4 [Nane (2008)]

Let $\alpha \in (0,2)$ be rational $\alpha = l/m$, where l and m are relatively prime. Let $T(s)f(x) = E_x[f(X(s))]$ be the semigroup of the continuous Markov process $X(t)$ and let L_x be its generator. Let f be a bounded measurable function in the domain of L_x, with $D^\gamma f$ bounded and Hölder continuous for all multi index γ such that $|\gamma| = 2l$. Then $u(t,x) = E_x[f(Z_t)]$ solves

$$\begin{aligned}
(-1)^{l+1}\frac{\partial^{2m}}{\partial t^{2m}}u(t,x) &= -2\sum_{i=1}^{l}\left(\frac{\partial^{2l-2i}}{\partial s^{2l-2i}}p_t^\alpha(0,s)|_{s=0}\right)L_x^{2i-1}f(x) \\
&\quad - L_x^{2l}u(t,x); \\
u(0,x) &= f(x).
\end{aligned}$$

SCALING LIMITS

If $S_n = X_1 + X_2 + \cdots + X_n$ is particle location at time n then the scaling limit is $r^{-1/\alpha} S_{[rt]} \implies X_t$.

The limit process X_t is called an α-stable Lévy motion.

Another random walk $J_n = T_1 + T_2 + \cdots + T_n$ records the jump times.

If $P[T_n > t] \approx t^{-\beta}$ for $\boxed{0 < \beta < 1}$ then $r^{-1/\beta} J_{[rt]} \implies D_t$.

D_t is an increasing β-stable Lévy motion(a stable subordinator).

Subordinator can give random time change: $X(t) \to X(D_t)$.

Inverse Stable subordinator

Waiting time random walk has scaling limit $r^{-1/\beta}J_{[rt]} \implies D_t$.

The number of jumps by time t is $N_t = \max\{n > 0 : J_n \le t\}$.

Renewal process inverse to random walk $\{N_t \ge n\} = \{J_n \le t\}$

Inverse process has inverse scaling limit $r^{-\beta}N_{[rt]} \implies E_t$.

Inverse stable process $\{E_t \le u\} = \{D_u \ge t\}$ yields the density formula:

$$p(u,t) = \frac{d}{du}P[E_t \le u] = \frac{d}{du}P[D_u \ge t]$$

CTRW scaling limits

Particle jump random walk has scaling limit $r^{-1/\alpha} S_{[rt]} \implies X_t$.

Number of jumps has scaling limit $r^{-\beta} N_{[rt]} \implies E_t$.

CTRW is a random walk subordinated to a renewal process

$$S_{N_t} = X_1 + X_2 + \cdots + X_{N_t}$$

CTRW scaling limit is a subordinated process:

$$r^{-\beta/\alpha} S_{N_{ct}} = (r^\beta)^{-1/\alpha} S_{r^\beta \cdot r^{-\beta} N_{ct}} \approx (r^\beta)^{-1/\alpha} S_{r^\beta E_t} \implies X_{E_t}$$

CTRW scaling limit is not Markov, increments are not stationary.

Meerschaert and collaborators (2004).

OPEN PROBLEMS

Question 1. For $\beta = 1/2$, the inverse stable subordinator process E_t and the process $|Y_t|$, where Y_t is a one-dimensional Brownian motion have the same transition density. Is there a similar correspondence between E_t for $\beta \neq 1/2$ and other symmetric α-stable process Y_t for $1 < \alpha < 2$.

Question 2. Looking at the governing PDE for subordinators other than Brownian motion, are there any fractional in time PDE which has the same solution as the higher order pde?

Question 3. Are there PDE connections of the iterated processes in bounded domain as the PDE connection of Brownian motion in bounded domains?

THANK YOU

REFERENCES

O. P. Agrawal, Solution for a fractional diffusion-wave equation defined in a bounded domain. Fractional order calculus and its applications. Nonlinear Dynam. 29 (2002), no. 1-4, 145–155.

H. Allouba and W. Zheng,Brownian-time processes: The pde connection and the half-derivative generator, *Ann. Prob.* 29 (2001), no. 2, 1780-1795.

B. Baeumer, M. M. Meerschaert and E. Nane, Brownian subordinators and fractional Cauchy problems, *Trans. Amer. Math. Soc.* (to appear).

B. Baeumer and M.M. Meerschaert, Stochastic solutions for fractional Cauchy problems, *Fractional Calculus Appl. Anal.* 4 (2001), 481-500.

C. Bandle, Isoperimetric Inequalities and Applications, *Monog. Stud. Math.* 7, Pitnam, Boston, 1980.

P. Becker-Kern, M. M. Meerschaert and H. P. Scheffler, Limit theorems for coupled continuous time random walks, *Ann. Probab.* **32** (2004), 730–756.

K. Burdzy, Some path properties of iterated Brownian motion, In: *Seminar on Stochastic Processes* (E. Çinlar, K.L. Chung and M.J. Sharpe, eds.), pp. 67–87, Birkhäuser, Boston, 1993.

K. Burdzy, Variation of iterated Brownian motion, In: *Workshops and Conference on Measure-valued Processes, Stochastic*

Partial Differential Equations and Interacting Particle Systems (D.A. Dawson, ed.), pp. 35–53, Amer. Math. Soc. Providence, RI, 1994.

K. Burdzy and D. Khoshnevisan, The level set of iterated Brownian motion, *Séminarie de probabilités XXIX* (Eds.: J Azéma, M. Emery, P.-A. Meyer and M. Yor), pp. 231-236, Lecture Notes in Mathematics, 1613, Springer, Berlin, 1995.

K. Burdzy and D. Khoshnevisan, Brownian motion in a Brownian crack, *Ann. Appl. Probab.* **8** (1998), 708–748.

E. Csáki, A. Földes and P. Révész, Strassen theorems for a class of iterated processes, *Trans. Amer. Math. Soc.* **349** (1997), 1153–1167.

E. Csáki, M. Csörgö, A. Földes, and P. Révész, The local time of iterated Brownian motion, *J. Theoret. Probab.* **9** (1996), 717–743.

R. D. DeBlassie, Iterated Brownian motion in an open set, *Ann. Appl. Prob.* 14 (2004), no. 3, 1529-1558.

Y. Hu, Hausdorff and packing measures of the level sets of iterated Brownian motion, *J. Theoret. Probab.* **12** (1999), 313–346.

Y. Hu, D. Pierre-Loti-Viaud, and Z. Shi, Laws of iterated logarithm for iterated Wiener proceses, *J. Theoret. Probab.* **8** (1995), 303–319.

D. Khoshnevisan, and T.M. Lewis, The uniform modulus of continuity of iterated Brownian motion. *J. Theoret. Probab.* 9 (1996), no. 2, 317–333.

D. Khoshnevisan and T. M. Lewis, Chung's law of the iterated logarithm for iterated Brownian motion, *Ann. Inst. H. Poincaré Probab. Statist.* **32** (1996), 349–359.

D. Khoshnevisan and T. M. Lewis, Stochastic calculus for Brownian motion in a Brownian fracture, *Ann. Appl. Probab.* **9** (1999), 629–667.

M.M. Meerschaert and H.P. Scheffler (2001) *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*. Wiley Interscience, New York.

M. M. Meerschaert and H. P. Scheffler, Limit theorems for continuous time random walks with infinite mean waiting times, *J. Appl. Probab.* **41** (2004), 623–638.

M.M. Meerschaert, E. Nane and P. Vellaisamy (2008), Fractional Cauchy problems on bounded domains. Submitted

E. Nane, Iterated Brownian motion in parabola-shaped domains, *Potential Anal.* **24** (2006), 105–123.

E. Nane, Iterated Brownian motion in bounded domains in $\mathbb{R}^n$, *Stochastic Process. Appl.* **116** (2006), 905–916.

E. Nane, Higher order PDEŠs and iterated processes, *Trans. Amer. Math. Soc.* 360 (2008), 2681-2692.

E. Nane, Laws of the iterated logarithm for α-time Brownian motion, *Electron. J. Probab.* **11** (2006), 434–459 (electronic).

E. Nane, Isoperimetric-type inequalities for iterated Brownian motion in $\mathbb{R}^n$, *Statist. Probab. Lett.* **78** (2008), 90–95.

E. Nane, Lifetime asymptotics of iterated Brownian motion in $\mathbb{R}^n$, *Esaim:PS*, **11** (2007), 147–160.

R. R. Nigmatullin, The realization of the generalized transfer in a medium with fractal geometry. *Phys. Status Solidi B,* 133 (1986), 425-430.

E. Orsingher and L. Beghin (2004), Time-fractional telegraph equations and telegraph processes with Brownian time. *Prob. Theory Rel. Fields* **128**, 141–160.

E. Orsingher and L. Beghin (2008), Fractional diffusion equations and processes with randomly-varying time. *Annal. Probab.* (to appear).

Z. Shi and M. Yor, Integrability and lower limits of the local time of iterated Brownian motion, *Studia Sci. Math. Hungar.* **33** (1997), 279–298.

Y. Xiao, Local times and related properties of multi-dimensional iterated Brownian motion, *J. Theoret. Probab.* **11** (1998), 383–408.

G. Zaslavsky, Fractional kinetic equation for Hamiltonian chaos. Chaotic advection, tracer dynamics and turbulent dispersion. *Phys. D* 76 (1994), 110-122.