Appendix A. Matrix Algebra

(b) From the spectral decomposition

\[ A = \Gamma \Lambda \Gamma', \]

we obtain

\[ \text{rank}(A) = \text{rank}(\Lambda) = \text{tr}(\Lambda) = r, \]

where \( r \) is the number of characteristic roots with value 1.

(c) Let rank\((A) = \text{rank}(\Lambda) = n\), then \( \Lambda = I_n \) and

\[ A = \Gamma \Lambda \Gamma' = I_n. \]

(a)–(c) follow from the definition of an idempotent matrix.

A.12 Generalized Inverse

**Definition A.62** Let \( A \) be an \( m \times n \)-matrix. Then a matrix \( A^{-} : n \times m \) is said to be a generalized inverse of \( A \) if

\[ AA^{-}A = A \]

holds (see Rao (1973a, p. 24)).

**Theorem A.63** A generalized inverse always exists although it is not unique in general.

**Proof:** Assume rank\((A) = r\). According to the singular-value decomposition (Theorem A.32), we have

\[ A = U \begin{pmatrix} L & X \\ Y & Z \end{pmatrix} V' \]

with \( U'U = I_r \) and \( V'V = I_r \) and

\[ L = \text{diag}(l_1, \cdots, l_r), \quad l_i > 0. \]

Then

\[ A^{-} = V \begin{pmatrix} L^{-1} & X \\ Y & Z \end{pmatrix} U' \]

\((X, Y \text{ and } Z \text{ are arbitrary matrices of suitable dimensions})\) is a \( g \)-inverse of \( A \). Using Theorem A.33, namely,

\[ A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \]

with \( X \) nonsingular, we have

\[ A^{-} = \begin{pmatrix} X^{-1} & 0 \\ 0 & 0 \end{pmatrix} \]

as a special \( g \)-inverse.
Definition A.64 (Moore-Penrose inverse) A matrix $A^+$ satisfying the following conditions is called the Moore-Penrose inverse of $A$:

(i) $AA^+A = A$,
(ii) $A^+AA^+ = A^+$,
(iii) $(A^+A)' = A^+A$,
(iv) $(AA^+)' = AA^+$.

$A^+$ is unique.

Theorem A.65 For any matrix $A : m \times n$ and any g-inverse $A^- : m \times n$, we have

(i) $A^-A$ and $AA^-$ are idempotent.
(ii) $\text{rank}(A) = \text{rank}(AA^-) = \text{rank}(A^-A)$.
(iii) $\text{rank}(A) \leq \text{rank}(A^-)$.

Proof:

(a) Using the definition of g-inverse,


(b) According to Theorem A.23 (iv), we get

$$\text{rank}(A) = \text{rank}(AA^- A) \leq \text{rank}(A^- A) \leq \text{rank}(A),$$

that is, $\text{rank}(A^- A) = \text{rank}(A)$. Analogously, we see that $\text{rank}(A) = \text{rank}(AA^-)$.

(c) $\text{rank}(A) = \text{rank}(AA^- A) \leq \text{rank}(AA^-) \leq \text{rank}(A^-)$.

Theorem A.66 Let $A$ be an $m \times n$-matrix. Then

(i) $A$ regular $\Rightarrow A^+ = A^{-1}$.
(ii) $(A^+)^+ = A$.
(iii) $(A^+)' = (A')^+$.
(iv) $\text{rank}(A) = \text{rank}(A^+) = \text{rank}(A^+A) = \text{rank}(AA^+)$.
(v) $A$ an orthogonal projector $\Rightarrow A^+ = A$.
(vi) $\text{rank}(A) : m \times n = m \Rightarrow A^+ = A'(AA')^{-1}$ and $AA^+ = I_m$.
(vii) $\text{rank}(A) : m \times n = n \Rightarrow A^+ = (A'A)^{-1}A' \text{ and } A^+A = I_n$.
(viii) If $P : m \times m$ and $Q : n \times n$ are orthogonal $\Rightarrow (PAQ)^+ = Q^{-1}A^+P^{-1}$.
(ix) $(A'A)^+ = A^+(A')^+$ and $(AA')^+ = (A')^+A^+$.
\((x)\) \(A^+ = (A'A)^+ A' = A'(AA')^+\).

For further details see Rao and Mitra (1971).

**Theorem A.67 (Baksalary, Kala and Klaczynski (1983))** Let \(M : n \times n \geq 0\) and \(N : m \times n\) be any matrices. Then

\[
M - N'(NM + N')^+ N \geq 0
\]

if and only if

\[
\mathcal{R}(N'NM) \subset \mathcal{R}(M).
\]

**Theorem A.68** Let \(A\) be any square \(n \times n\)-matrix and \(a\) be an \(n\)-vector with \(a \not\in \mathcal{R}(A)\). Then a g-inverse of \(A + aa'\) is given by

\[
(A + aa')^- = A^- - \frac{A^-aa'U'U}{a'U'Ua} - \frac{VV'aa'A^-}{a'VV'a} + \phi \frac{VV'aa'U'U}{(a'U'Ua)(a'VV'a)},
\]

with \(A^-\) any g-inverse of \(A\) and

\[
\phi = 1 + a'A^-a, \quad U = I - AA^-, \quad V = I - A^-A.
\]

**Proof:** Straightforward by checking \(AA^-A = A\).

**Theorem A.69** Let \(A\) be a square \(n \times n\)-matrix. Then we have the following results:

(i) Assume \(a, b\) are vectors with \(a, b \in \mathcal{R}(A)\), and let \(A\) be symmetric. Then the bilinear form \(a'A^-b\) is invariant to the choice of \(A^-\).

(ii) \(A(A'A)^-A'\) is invariant to the choice of \((A'A)^-\).

**Proof:**

(a) \(a, b \in \mathcal{R}(A) \implies a = Ac\) and \(b = Ad\). Using the symmetry of \(A\) gives

\[
a'A^-b = c'A'Ad = c'Ad.
\]

(b) Using the rowwise representation of \(A\) as \(A = \begin{pmatrix} a_1' \\ \vdots \\ a_n' \end{pmatrix}\) gives

\[
A(A'A)^-A' = (a_i'(A'A)^-a_j).
\]

Since \(A'A\) is symmetric, we may conclude then (i) that all bilinear forms \(a_i'(A'A)a_j\) are invariant to the choice of \((A'A)^-\), and hence (ii) is proved.
Theorem A.70 Let $A : n \times n$ be symmetric, $a \in \mathcal{R}(A)$, $b \in \mathcal{R}(A)$, and assume $1 + b' A^+ a \neq 0$. Then

$$(A + ab')^+ = A^+ - \frac{A^+ ab' A^+}{1 + b' A^+ a}. \quad \text{Proof:} \text{Straightforward, using Theorems A.68 and A.69.}$$

Theorem A.71 Let $A : n \times n$ be symmetric, $a$ be an $n$-vector, and $\alpha > 0$ be any scalar. Then the following statements are equivalent:

(i) $\alpha A - aa' \geq 0$.

(ii) $A \geq 0$, $a \in \mathcal{R}(A)$, and $a' A^{-} a \leq \alpha$, with $A^{-}$ being any $g$-inverse of $A$.

Proof:

(i) $\Rightarrow$ (ii): $\alpha A - aa' \geq 0 \Rightarrow \alpha A = (\alpha A - aa') + aa' \geq 0 \Rightarrow A \geq 0$. Using Theorem A.31 for $\alpha A - aa' \geq 0$, we have $\alpha A - aa' = BB$, and, hence, $\alpha A = BB + aa' = (B, a)(B, a)'$.

$\Rightarrow \mathcal{R}(\alpha A) = \mathcal{R}(A) = \mathcal{R}(B, a)$

$\Rightarrow a \in \mathcal{R}(A)$

$\Rightarrow a = Ac \text{ with } c \in \mathbb{R}^n$

$\Rightarrow a' A^{-} a = c' Ac$.

As $\alpha A - aa' \geq 0$ $\Rightarrow$ $x'(\alpha A - aa')x \geq 0$

for any vector $x$, choosing $x = c$, we have

$\alpha c' Ac - c' aa' c = \alpha c' Ac - (c' Ac)^2 \geq 0$,

$\Rightarrow c' Ac \leq \alpha$.

(ii) $\Rightarrow$ (i): Let $x \in \mathbb{R}^n$ be any vector. Then, using Theorem A.54,

$x'(\alpha A - aa')x = \alpha x' Ax - (x' a)^2$

$= \alpha x' Ax - (x' Ac)^2$

$\geq \alpha x' Ax - (x' Ax)(c' Ac)$

$\Rightarrow x'(\alpha A - aa')x \geq (x' Ax)(\alpha - c' Ac)$.

In (ii) we have assumed $A \geq 0$ and $c' Ac = a' A^{-} a \leq \alpha$. Hence, $\alpha A - aa' \geq 0$.

Note: This theorem is due to Baksalary and Kala (1983). The version given here and the proof are formulated by G. Trenkler.
Theorem A.72 For any matrix $A$ we have

$$A' A = 0 \text{ if and only if } A = 0.$$  

Proof:
(a) $A = 0 \Rightarrow A' A = 0$.

(b) Let $A' A = 0$, and let $A = (a_{(1)}, \ldots, a_{(n)})$ be the columnwise presentation. Then

$$A' A = (a_{(i)}' a_{(j)}) = 0,$$

so that all the elements on the diagonal are zero: $a_{(i)}' a_{(i)} = 0 \Rightarrow a_{(i)} = 0$ and $A = 0$.

Theorem A.73 Let $X \neq 0$ be an $m \times n$-matrix and $A$ an $n \times n$-matrix. Then

$$X' XAX' X = X' X \Rightarrow XAX' X = X \text{ and } X' XAX' = X'.$$

Proof: As $X \neq 0$ and $X' X \neq 0$, we have

$$X' XAX' X - X' X = (X' XA - I)X' X = 0 \Rightarrow (X' XA - I) = 0$$

$$\Rightarrow 0 = (X' XA - I)(X' XAX' X - X' X)$$

$$= (X' XAX' - X')(XAX' X - X) = Y' Y,$$

so that (by Theorem A.72) $Y = 0$, and, hence $XAX' X = X$.

Corollary: Let $X \neq 0$ be an $m \times n$-matrix and $A$ and $b$ $n \times n$-matrices. Then

$$AX' X = BX' X \iff AX' = BX'.$$

Theorem A.74 (Albert’s theorem)

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be symmetric. Then

(i) $A \geq 0$ if and only if

(a) $A_{22} \geq 0$,
(b) $A_{21} = A_{22} A_{22}^{-1} A_{21}$,
(c) $A_{11} \geq A_{12} A_{22}^{-1} A_{21},$

($(b)$ and $(c)$ are invariant of the choice of $A_{22}$).

(ii) $A > 0$ if and only if

(a) $A_{22} > 0$,
(b) $A_{11} > A_{12} A_{22}^{-1} A_{21}$. 

Proof: Bekker and Neudecker (1989):

(i) Assume $A \geq 0$.

(a) $A \geq 0 \Rightarrow x'Ax \geq 0$ for any $x$. Choosing $x' = (0', x_2')$
$
\Rightarrow x'Ax = x_2'A_2x_2 \geq 0$ for any $x_2 \Rightarrow A_{22} \geq 0$.

(b) Let $B' = (0, I - A_{22}A_{22}^{-1}) \Rightarrow$

$$B'A = ((I - A_{22}A_{22}^{-1})A_{21}, A_{22} - A_{22}A_{22}^{-1}A_{22})$$
$= ((I - A_{22}A_{22}^{-1})A_{21}, 0)$

and $B'AB = B'A\frac{1}{2}A\frac{1}{2}B = 0$. Hence, by Theorem A.72 we get

$B'A\frac{1}{2} = 0$.

This proves (b).

(c) Let $C' = (I, -(A_{22}^{-1}A_{21}))'$. $A \geq 0 \Rightarrow$

$$0 \leq C'AC = A_{11} - A_{12}(A_{22}^{-1})'A_{21} - A_{12}A_{22}^{-1}A_{21}$$
$+ A_{12}(A_{22}^{-1})'A_{22}A_{22}^{-1}A_{21}$

$= A_{11} - A_{12}A_{22}^{-1}A_{21}$.

(Since $A_{22}$ is symmetric, we have $(A_{22}^{-1})' = A_{22}$.)

Now assume (a), (b), and (c). Then

$$D = \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{pmatrix} \geq 0,$$

as the submatrices are n.n.d. by (a) and (b). Hence,

$$A = \begin{pmatrix} I & A_{12}(A_{22}^{-1}) \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ A_{22}A_{21} & I \end{pmatrix} \geq 0.$$

(ii) Proof as in (i) if $A_{22}$ is replaced by $A_{22}^{-1}$.

Theorem A.75 If $A : n \times n$ and $B : n \times n$ are symmetric, then

(i) $0 \leq B \leq A$ if and only if

(a) $A \geq 0$,

(b) $B = AA^{-1}B$,

(c) $B \geq BA^{-1}B$.

(ii) $0 < B < A$ if and only if $0 < A^{-1} < B^{-1}$.

Proof: Apply Theorem A.74 to the matrix $\begin{pmatrix} B & B \\ B & A \end{pmatrix}$. 
Theorem A.76 Let \( A \) be symmetric and \( c \in \mathcal{R}(A) \). Then the following statements are equivalent:

(i) \( \text{rank}(A + cc') = \text{rank}(A) \).

(ii) \( \mathcal{R}(A + cc') = \mathcal{R}(A) \).

(iii) \( 1 + c'A^{-1}c \neq 0 \).

Corollary 1: Assume (i) or (ii) or (iii) holds; then

\[
(A + cc')^{-} = A^{-} \frac{A^{-}cc'A^{-}}{1 + c'A^{-}c}
\]

for any choice of \( A^{-} \).

Corollary 2: Assume (i) or (ii) or (iii) holds; then

\[
c'(A + cc')^{-}c = c'A^{-}c - \frac{(c'A^{-}c)^2}{1 + c'A^{-}c} = 1 - \frac{1}{1 + c'A^{-}c}.
\]

Moreover, as \( c \in \mathcal{R}(A + cc') \), the results are invariant for any special choices of the \( g \)-inverses involved.

Proof: \( c \in \mathcal{R}(A) \Leftrightarrow AA^{-}c = c \Rightarrow \mathcal{R}(A + cc') = \mathcal{R}(AA^{-}(A + cc')) \subset \mathcal{R}(A) \).

Hence, (i) and (ii) become equivalent. Proof of (iii): Consider the following product of matrices:

\[
\begin{pmatrix} 1 & 0 \\ c & A + cc' \end{pmatrix} \begin{pmatrix} 1 & -c \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-}c & I \end{pmatrix} = \begin{pmatrix} 1 + c'A^{-}c & -c \\ 0 & A \end{pmatrix}.
\]

The left-hand side has the rank

\( 1 + \text{rank}(A + cc') = 1 + \text{rank}(A) \)

(see (i) or (ii)). The right-hand side has the rank \( 1 + \text{rank}(A) \) if and only if \( 1 + c'A^{-}c \neq 0 \).

Theorem A.77 Let \( A : n \times n \) be a symmetric and nonsingular matrix and \( c \notin \mathcal{R}(A) \). Then we have

(i) \( c \in \mathcal{R}(A + cc') \).

(ii) \( \mathcal{R}(A) \subset \mathcal{R}(A + cc') \).

(iii) \( c'(A + cc')^{-}c = 1 \).

(iv) \( A(A + cc')^{-}A = A \).

(v) \( A(A + cc')^{-}c = 0 \).
Proof: As $A$ is assumed to be nonsingular, the equation $Al = 0$ has a nontrivial solution $l \neq 0$, which may be standardized as $(c'l)^{-1}l$ such that $c'l = 1$. Then we have $c = (A + cc')l \in \mathcal{R}(A + cc')$, and hence (i) is proved. Relation (ii) holds as $c \not\in \mathcal{R}(A)$. Relation (i) is seen to be equivalent to

$$(A + cc')(A + cc')^{-1}c = c.$$ 

Then (iii) follows:

$$c'(A + cc')^{-1}c = l'(A + cc')(A + cc')^{-1}c = l'c = 1,$$

which proves (iii). From

$$c = (A + cc')(A + cc')^{-1}c = A(A + cc')^{-1}c + cc'(A + cc')^{-1}c = A(A + cc')^{-1}c + c,$$

we have (v).

(iv) is a consequence of the general definition of a $g$-inverse and of (iii) and (iv):

$$A + cc' = (A + cc')(A + cc')^{-1}(A + cc') = A(A + cc')^{-1}A$$

$$+ cc'(A + cc')^{-1}cc' [= cc' using (iii)]$$

$$+ A(A + cc')^{-1}cc' [= 0 using (v)]$$

$$+ cc'(A + cc')^{-1}A [= 0 using (v)].$$

Theorem A.78 We have $A \geq 0$ if and only if

(i) $A + cc' \geq 0$.

(ii) $(A + cc')(A + cc')^{-1}c = c$.

(iii) $c'(A + cc')^{-1}c \leq 1$.

Assume $A \geq 0$; then

(a) $c = 0 \Leftrightarrow c'(A + cc')^{-1}c = 0$.

(b) $c \in \mathcal{R}(A) \Leftrightarrow c'(A + cc')^{-1}c < 1$.

(c) $c \not\in \mathcal{R}(A) \Leftrightarrow c'(A + cc')^{-1}c = 1$.

Proof: $A \geq 0$ is equivalent to

$$0 \leq cc' \leq A + cc'.$$

Straightforward application of Theorem A.75 gives (i)–(iii).

Proof of (a): $A \geq 0 \Rightarrow A + cc' \geq 0$. Assume

$$c'(A + cc')^{-1}c = 0,$$
and replace $c$ by (ii) ⇒
\[ c'(A + cc')(A + cc')^{-1}c = 0 \Rightarrow (A + cc')(A + cc')^{-1}c = 0 \]
as $(A + cc') \geq 0$. Assuming $c = 0 \Rightarrow c'(A + cc')c = 0$.

Proof of (b): Assume $A \geq 0$ and $c \in \mathcal{R}(A)$, and use Theorem A.76 (Corollary 2) ⇒
\[ c'(A + cc')^{-1}c = 1 - \frac{1}{1 + c'A^{-1}c} < 1. \]
The opposite direction of (b) is a consequence of (c).

Proof of (c): Assume $A \geq 0$ and $c \not\in \mathcal{R}(A)$, and use Theorem A.77 (iii) ⇒
\[ c'(A + cc')^{-1}c = 1. \]
The opposite direction of (c) is a consequence of (b).

Note: The proofs of Theorems A.74–A.78 are given in Bekker and Neudecker (1989).

Theorem A.79 The linear equation $Ax = a$ has a solution if and only if
\[ a \in \mathcal{R}(A) \text{ or } AA^{-1}a = a \]
for any $g$-inverse $A$.

If this condition holds, then all solutions are given by
\[ x = A^{-1}a + (I - A^{-1}A)w, \]
where $w$ is an arbitrary $m$-vector. Further, $q'x$ has a unique value for all solutions of $Ax = a$ if and only if $q'A^{-1}A = q'$, or $q \in \mathcal{R}(A')$.

For a proof, see Rao (1973a, p. 25).

A.13 Projectors

Consider the range space $\mathcal{R}(A)$ of the matrix $A : m \times n$ with rank $r$. Then there exists $\mathcal{R}(A)\perp$, which is the orthogonal complement of $\mathcal{R}(A)$ with dimension $m - r$. Any vector $x \in \mathbb{R}^m$ has the unique decomposition
\[ x = x_1 + x_2, \quad x_1 \in \mathcal{R}(A), \text{ and } x_2 \in \mathcal{R}(A)\perp, \]
of which the component $x_1$ is called the orthogonal projection of $x$ on $\mathcal{R}(A)$. The component $x_1$ can be computed as $Px$, where
\[ P = A(A'A)^{-1}A', \]
which is called the projection operator on $\mathcal{R}(A)$. Note that $P$ is unique for any choice of the $g$-inverse $(A'A)^-$. 

Theorem A.80 For any $P : n \times n$, the following statements are equivalent:

(i) $P$ is an orthogonal projection operator.

(ii) $P$ is symmetric and idempotent.

For proofs and other details, the reader is referred to Rao (1973a) and Rao and Mitra (1971).

Theorem A.81 Let $X$ be a matrix of order $T \times K$ with rank $r < K$, and $U : (K - r) \times K$ be such that $\mathcal{R}(X') \cap \mathcal{R}(U') = \{0\}$. Then

(i) $X(X'X + U'U)^{-1}U' = 0$.

(ii) $X'X(X'X + U'U)^{-1}X'X = X'X$; that is, $(X'X + U'U)^{-1}$ is a $g$-inverse of $X'X$.

(iii) $U'U(X'X + U'U)^{-1}U'U = U'U$; that is, $(X'X + U'U)^{-1}$ is also a $g$-inverse of $U'U$.

(iv) $U(X'X + U'U)^{-1}U'u = u$ if $u \in \mathcal{R}(U)$.

**Proof:** Since $X'X + U'U$ is of full rank, there exists a matrix $A$ such that

$$(X'X + U'U)A = U' \Rightarrow X'XA = U' - U'UA \Rightarrow XA = 0 \text{ and } U' = U'UA$$

since $\mathcal{R}(X')$ and $\mathcal{R}(U')$ are disjoint.

Proof of (i):

$$X(X'X + U'U)^{-1}U' = X(X'X + U'U)^{-1}(X'X + U'U)A = XA = 0.$$

Proof of (ii):

$$X'X(X'X + U'U)^{-1}(X'X + U'U - U'U) = X'X - X'X(X'X + U'U)^{-1}U'U = X'X.$$

Result (iii) follows on the same lines as result (ii).

Proof of (iv):

$$U(X'X + U'U)^{-1}U'u = U(X'X + U'U)^{-1}U'Ua = Ua = u$$

since $u \in \mathcal{R}(U)$.

A.14 Functions of Normally Distributed Variables

Let $x' = (x_1, \cdots, x_p)$ be a $p$-dimensional random vector. Then $x$ is said to have a $p$-dimensional normal distribution with expectation vector $\mu$ and covariance matrix $\Sigma > 0$ if the joint density is

$$f(x; \mu, \Sigma) = \{(2\pi)^p|\Sigma|\}^{-\frac{1}{2}} \exp \left\{-\frac{1}{2}(x - \mu)'\Sigma^{-1}(x - \mu)\right\}.$$